Iterations of the Terdragon Curve

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Abstract

Various results on the terdragon curve, including coordinates, area, boundary, enclosure sequence, convex hull, centroid, moment of inertia, some trees, fractionals, and some results on the alternate terdragon curve.

Contents

Notation

Various coordinates and other expressions use complex 3rd, 6th and 12th roots of unity, usually to express directions.

$$\omega_3 = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i \;=\; e^{2\pi i/3} \qquad\qquad \text{3rd root of unity, } 120°$$

$$\omega_6 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i \;=\; e^{2\pi i/6} = \omega_3 + 1 \qquad \text{6th root of unity, } 60°$$

$$\omega_{12} = \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i \;=\; e^{2\pi i/12} \qquad\qquad \text{12th root of unity, } 30°$$
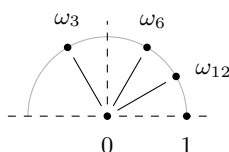
A few formulas have terms going in a repeating pattern of say 4 values according as an index $k \equiv 0\,\text{to}\,3 \bmod 4$. It's convenient to write them as for example

$$[5,\ 8,\ -5,\ 9] \qquad \text{values according as } k \bmod 4$$

meaning 5 when $k \equiv 0 \bmod 4$, or 8 when $k \equiv 1 \bmod 4$, etc. Likewise periodic patterns of other lengths, usually at most 8.

Periodic patterns like this can also be expressed using powers of -1 or i or other roots of unity, but except in simple cases that tends to be less clear than the values.

1 Terdragon Curve

The terdragon curve by Davis and Knuth[3] is defined recursively as a repeated replacement of each line segment by 3 segments in an "S" shape
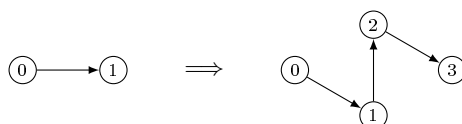
The curve touches at vertices. The following diagram has the vertices chamfered off to better see the turns and joins.
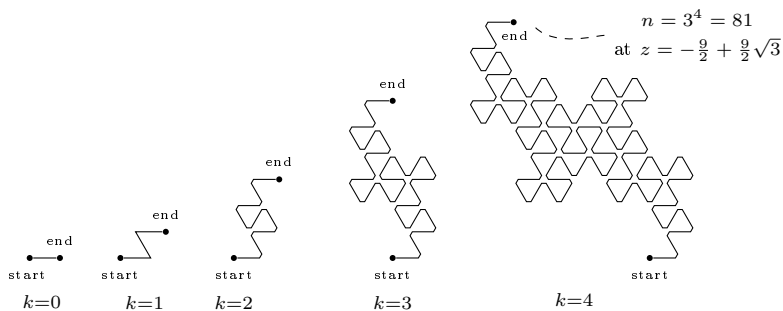
1.1 Plane Filling

Davis and Knuth show the terdragon is non-crossing and plane filling from the revolving cubic representations of its vertices. This can also be seen geometrically.

Theorem 1 (Davis and Knuth). *The terdragon curve touches at vertices but does not cross itself.*

Proof. Consider an infinite triangular grid with unit line segments connecting the points. Each line segment expands to the base pattern as follows. The corners of the new line segments are chamfered off here to show how they meet the expansions from other lines but do not cross.
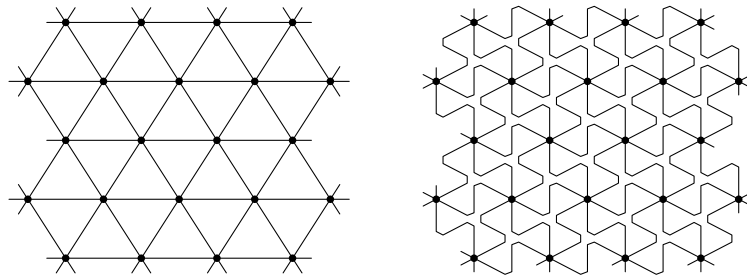
Figure 1: segment expansions

The expanded segments are the same grid pattern rotated by 30°.

Any subset of the full grid expands to a new bigger set with the number of crossings unchanged. The terdragon curve begins with a single line segment which is such a subset with no crossings and so on repeated expansions has no crossings. □

The expansion replaces each line segment with a rhombus shaped three segments. This is a classical tiling pattern[9].
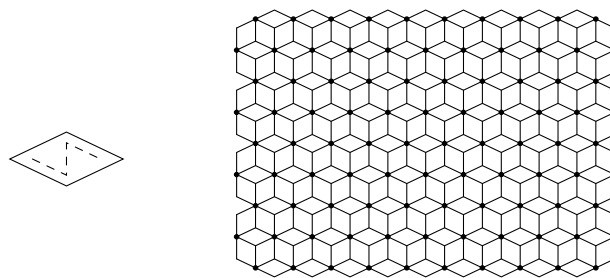
Theorem 2 (Davis and Knuth). *Six copies of the terdragon curve arranged at 60° angles fill the plane.*

Proof. The initial 6 line segments expand
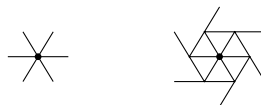
Take the central 2×2 hexagon. With two expansions it grows
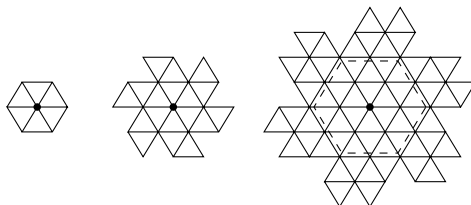
The dashed outline is a 4×4 hexagon at the origin. Each 2×2 hexagon (possibly overlapping) grows to at least 4×4. By repeated expansion they grow to an arbitrarily large hexagon at the origin. □

See end of section 10.1 for the actual diameter of 6 arm filling.

1.2 Turn Sequence

Number points of the terdragon curve starting $n = 0$ at the origin. Per Davis and Knuth, the replications give a turn sequence which is 120° turns according to the lowest non-0 digit of n in ternary,

$$turn(n) = \begin{cases} +1 & \text{if } LowestNonZero(n) = 1 \\ -1 & \text{if } LowestNonZero(n) = 2 \end{cases} \quad n \geq 1 \tag{1}$$

$$= -(-1)^{LowestNonZero(n)}$$

$$= + - ++ -- + - ++ - ++ -- + -- + - ++ -- + - ++ - \ldots$$

$$turn(3n) = turn(n), \quad turn(3n{+}1) = 1, \quad turn(3n{+}2) = -1 \tag{2}$$

$$LowestNonZero(n) = 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, \ldots \quad n \geq 1 \qquad \text{A060236}$$

Or next turn,

$$turn(n{+}1) = \begin{cases} +1 & \text{if } LowestNonTwo(n) = 0 \\ -1 & \text{if } LowestNonTwo(n) = 1 \end{cases} \quad n \geq 0 \tag{3}$$

$$= (-1)^{LowestNonTwo(n)}$$

$$LowestNonTwo(n) = 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, \ldots \quad n \geq 0 \qquad \text{offset A080846}$$

$3^k - 1$ consists entirely of 2-digits and is taken to have a 0 above the highest so $LowestNonTwo(3^k - 1) = 0$.

$turn(n)$ and $turn(n{+}1)$ are related simply by $n{+}1$ changing low 2s into low 0s and increment the digit above.

n	$\cdots$	d	$2 \cdots 2$	ternary digits, $d \neq 2$

$n{+}1$	$\cdots$	$d{+}1$	$0 \cdots 0$

$$LowestNonTwo(n) = LowestNonZero(n{+}1) - 1$$

On a binary computer, it can be convenient to represent ternary digits in 2 bits each. Arndt[1] gives an example iterating *turn* like this with bits $00, 01, 10$ to represent $0, 1, 2$ respectively and a loop for carry propagation.

Another possibility is bits $00, 01, 11$. This allows a binary increment to propagate a carry through 2s. If it increments 01 to 10 then a normalize from 10 up to 11 is necessary. Representing ternary 1 by bits 01 (rather than 10) allows the lowest non-0 digit to be determined by bit above lowest 1-bit, which can be found by bit-twiddling.

$nbits$ has bits $00, 01, 11$ representing ternary digits $0, 1, 2$ A023713

$$turn(nbits) = \begin{cases} +1 & \text{if } BitAboveLowestOne(nbits) = 0 \\ -1 & \text{if } BitAboveLowestOne(nbits) = 1 \end{cases}$$

$$increment(nbits) = PostIncFix(nbits + 1) \tag{4}$$

$$PostIncFix(n) = \text{BITOR}(n, \text{BITAND}(1010...101_2, \text{RIGHTSHIFT}(n)))$$

$$BitAboveLowestOne(n) = \begin{cases} 0 & \text{if } \text{BITAND}(n, MaskAboveLowestOne(n)) = 0 \\ 1 & \text{if } \text{BITAND}(n, MaskAboveLowestOne(n)) \neq 0 \end{cases}$$

$$= 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, \ldots \qquad \text{A038189}$$

$$MaskAboveLowestOne(n) = \text{BITXOR}(n, n{-}1) + 1 \qquad n \geq 1$$

$$= 2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 32, \ldots \qquad \text{A171977}$$

These bit operations are best suited to n in a single machine word. In a big number, masks etc on the whole number would act on a lot of unchanged bits (since carry rarely propagates very far). For a big number represented in words, fixing can stop where the carry stops.

Predicates for left and right turns are

$$TurnLpred(n) = \begin{cases} 1 & \text{if } n{\geq}1 \text{ and } LowestNonZero(n) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= 1,0,1,1,0,0,1,0,1,1,0,1,1,0,0,1,0,\ldots \quad n{\geq}1 \qquad \text{A137893}$$

$$TurnRpred(n) = \begin{cases} 1 & \text{if } n{\geq}1 \text{ and } LowestNonZero(n) = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= 0,1,0,0,1,1,0,1,0,0,1,0,0,1,1,0,1,\ldots \quad n{\geq}1 \qquad \text{A080846}$$

Generating functions for these sequences follow by considering the ternary digits of those n which are a left or right turn. A left turn is k low zeros then digit 1 so $n = 3^k + m.3^{k+1}$ for integer m. Generating function $1/(1{-}x^{3^{k+1}})$ is 1 at $m.3^{k+1}$ then multiply x^{3^k} to add 3^k. Similarly a right turn is k low zeros then digit 2 so $n = 2.3^k + m.3^{k+1}$ which is multiply by $x^{2.3^k}$ to add 2.3^k,

$$gTurnLpred(x) = \sum_{k=0}^{\infty} \frac{x^{3^k}}{1 - x^{3^{k+1}}} \qquad gTurnRpred(x) = \sum_{k=0}^{\infty} \frac{x^{2.3^k}}{1 - x^{3^{k+1}}} \tag{5}$$

With $turn(n) = TurnLpred(n) - TurnRpred(n)$, a generating function for $turn$ is their difference. Factor $1 - x^{3^k}$ cancels from numerator and denominator, though replications in 3^{k+1} blocks are then less clear.

$$gturn(x) = \sum_{k=0}^{\infty} \frac{x^{3^k} - x^{2.3^k}}{1 - x^{3^{k+1}}} = \sum_{k=0}^{\infty} \frac{x^{3^k}}{1 + x^{3^k} + x^{2.3^k}} \tag{6}$$

Paul D. Hanna in OEIS A080846 gives a generating function for *TurnRpred* based on a generating function for net directions (*dir* ahead in section 1.3). Shifting it to the numbering here so first turn n=1 term x^1 is
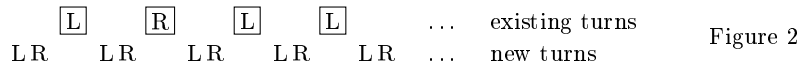
$$gTurnRpred(x) = \tfrac{1}{2}\frac{x}{1-x} - \tfrac{1}{2}\sum_{k\geq 0} \frac{x^{3^k}}{1 + x^{3^k} + x^{2.3^k}}$$

This can be thought of as changing turn form (6) values from ± 1 to 0,1 by

$$TurnRpred(n) = \tfrac{1}{2}\big(1 - turn(n)\big)$$

If a generating function for just an initial part of the sequence is required then stopping the sum (either form) at k suffices for $n < 3^{k+1}$ where the next term would begin (a left turn at k+1 low zeros and digit 1 above).

On expanding the curve, 2 new turns LR are inserted into each segment. A segment is before each existing turn.

$$\boxed{L} \qquad \boxed{R} \qquad \boxed{L} \qquad \boxed{L} \qquad \ldots \quad \text{existing turns}$$
$$\text{L R} \qquad \text{L R} \qquad \text{L R} \qquad \text{L R} \qquad \text{L R} \quad \ldots \quad \text{new turns} \qquad \text{Figure 2}$$

The new R and L each side of an existing turn make a run either RR or LL with the existing turn according to whether it is R or L. So run lengths in the turn sequence are an initial 1 then pairs either 1,2 or 2,1 according as $turn = +1$ or -1 respectively. With an index m starting m=0 for the first run,

$$TurnRun(m) = \begin{cases} 1 & \text{if } m=0 \text{ (left turns)} \\ \tfrac{3}{2} + \tfrac{1}{2}\,turn(\tfrac{m}{2}) & \text{if } m \text{ even } \geq 2 \text{ (left turns)} \\ \tfrac{3}{2} - \tfrac{1}{2}\,turn(\tfrac{m+1}{2}) & \text{if } m \text{ odd (right turns)} \end{cases}$$

$$= \begin{cases} 1 & \text{if } m=0 \\ \tfrac{3}{2} + \tfrac{1}{2}(-1)^m\,turn(\lceil \tfrac{m}{2}\rceil) & \text{if } m \geq 1 \end{cases}$$

$$= 1, \ 1,2, \ 2,1, \ 1,2, \ 1,2, \ 2,1, \ 2,1, \ 1,2,\ldots$$
$$turn = \quad +1, \ -1, \ +1, \ +1, \ -1, \ -1, \ +1,$$

$$gTurnRun(x) = -\tfrac{1}{2} + \tfrac{3}{2}\frac{1}{1-x} + \tfrac{1}{2}\left(1 - \tfrac{1}{x}\right) gturn(x^2)$$

For finite curve k, the run lengths end with a final 1 which is per the initial 1. The curve is the same in 180° rotation, so the run length sequence for finite k is a palindrome.

The n which is the start of a run follows from figure 2 turns too. In each LR, the left $n \equiv 1 \bmod 3$ is the start of a run unless preceded by an existing turn L. The right at $n \equiv 2 \bmod 3$ is always the start of a run. With index m starting m=0 for the first run again,

$$TurnRunStart(m) = 1 + \sum_{j=0}^{m-1} TurnRun(j)$$

$$= \tfrac{3}{2}m + \begin{cases} 1 - TurnLpred(\tfrac{3}{2}m) & \text{if } m \text{ even} \\ \tfrac{1}{2} & \text{if } m \text{ odd} \end{cases}$$

$$= \lceil \tfrac{3}{2}m \rceil + \begin{cases} 1 & \text{if } m{=}0 \\ TurnLpred(m) & \text{if } m \text{ even} \geq 2 \end{cases} \qquad (7)$$

$$= 1, 2, 3, 5, 7, 8, 9, 11, 12, 14, 16, 17, 19, \ldots$$

Form (7) eliminates factor $\tfrac{3}{2}$ on even m by firstly factor of 3 is no change since $turn(3n) = turn(n)$. Then $2.\tfrac{1}{2}m$ instead of $\tfrac{1}{2}m$ is $turn(2n) = -turn(n)$ since factor 2 flips the lowest non-zero $1 \leftrightarrow 2$. This swaps to $TurnRpred$, and then $1 - TurnRpred$ is back to $TurnLpred$, for $m \neq 0$.

Theorem 3. *The offset $dNextL(n)$ from n to the next n which is a left turn is given by the low ternary digits of n in the following patterns, where $\underline{2}$ means zero or more 2-digits. High 0 digits are understood on n as necessary.*

$$dNextL(n) = \begin{cases} 1 & \text{if } n = \ldots \ 0\underline{2} \\ 2 & \text{if } n = \ldots 0\underline{2}1 \ \text{or} \ \ldots 1\underline{2}2 \\ 3 & \text{if } n = \ldots 1\underline{2}1 \end{cases} \qquad (8)$$

$$= \begin{cases} 1 & \text{if } n \equiv 0 \bmod 3 \\ 2 + TurnRpred(n{+}2) & \text{if } n \equiv 1 \bmod 3 \\ 1 + TurnRpred(n{+}1) & \text{if } n \equiv 2 \bmod 3 \end{cases} \qquad (9)$$

$$= \begin{cases} 2 + TurnRpred(n{+}2) & \text{if } n \equiv 1 \bmod 3 \\ 1 + TurnRpred(n{+}1) & \text{if } n \not\equiv 1 \bmod 3 \end{cases} \qquad (10)$$

And $dNextR$ offset to the next right turn,

$$dNextR(n) = \begin{cases} 1 & \text{if } n = \ldots \ 1\underline{2} \\ 2 & \text{if } n = \ldots \ \ \ 0 \\ 3 & \text{if } n = \ldots 0\underline{2}2 \end{cases}$$

$$= \begin{cases} 2 & \text{if } n \equiv 0 \bmod 3 \\ 1 & \text{if } n \equiv 1 \bmod 3 \\ 2 + turn(n{+}1) & \text{if } n \equiv 2 \bmod 3 \end{cases} \qquad (11)$$

$$= \begin{cases} 2 & \text{if } n \equiv 0 \bmod 3 \\ 2 + turn(n{+}1) & \text{if } n \not\equiv 0 \bmod 3 \end{cases} \qquad (12)$$

$$dNextL(n) = 1, 2, 1, 1, 3, 2, 1, 2, 1, 1, 2, 1, 1, 3, 2, 1, \ldots$$
$$dNextR(n) = 2, 1, 3, 2, 1, 1, 2, 1, 3, 2, 1, 3, 2, 1, 1, 2, \ldots$$

Proof. Segment expansion inserts a new pair of turns LR after each existing point $n{=}0$ onwards. With existing points X and Y, $dNextL$ steps are

$dNextL$

```
         +1   +2 or +3
        X        Y              existing turns, X,Y each either L or R
          L  R     L  R         new turns
                 +1 or +2
n ≡    0   1   2   0   1   2    mod 3
```
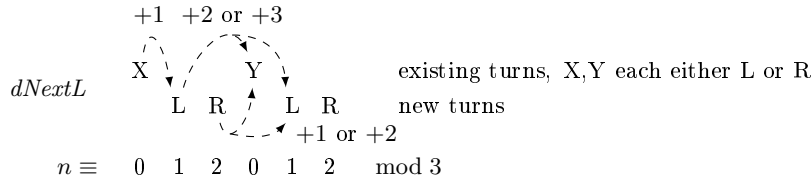
X always steps +1 to the first L. That L steps to Y if Y is an L or to the second L otherwise. $TurnRpred(n+2)$ is the possible extra 1 to add for this. Similarly R to Y or L according to $TurnRpred(n+1)$, and hence (9).

For the two case (10), $n{\equiv}0$ has $TurnRpred(n+1) = 0$ always, allowing it to combine with $n{\equiv}2$.

$dNextL(X)$ before expansion determines whether Y is L or R. If +1 then it is step +1 to L at Y, and otherwise a bigger step because Y not L. So a morphism expansion can be written as follows with 1 or not 1 determining the new steps at $n \equiv 1, 2$ mod 3 (steps either 2, 1 or 3, 2).

$$dNextL = \quad 1 \to 1,2,1 \quad 2 \to 1,3,2 \quad 3 \to 1,3,2 \quad \text{starting } 1$$

Such expansions are a state machine by ternary digits of n from high to low. Some state machine manipulations to reverse gives low to high and which are the digit patterns at (8). Those patterns can also be seen just by considering possible combinations of low digits of n and what increment is needed to reach $LowestNonZero = 1$.
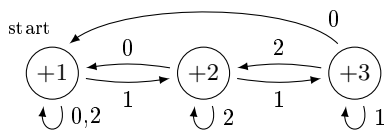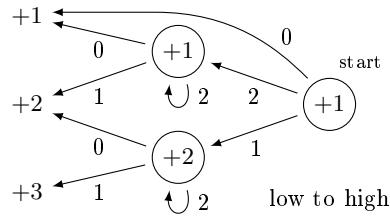

Figure 3: $dNextL(n)$ state machine, ternary high to low

For $dNextR$, similar X and Y existing points and steps are

$dNextR$

```
         +2      +1 or +3
        X       Y               existing turns, X,Y each either L or R
          L  R     L  R         new turns
                +1
n ≡    0   1   2   0   1   2    mod 3
```
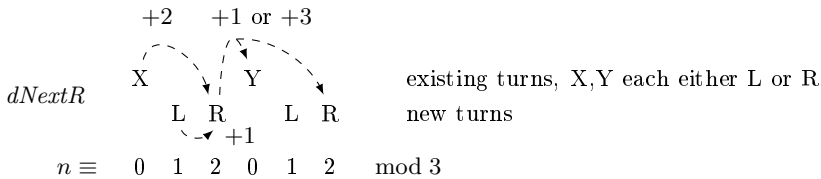
X always steps +2 to the R for the first LR pair. The L of that pair always steps +1 likewise. R steps to Y if that is an R or to the R of the secont LR pair otherwise, so +1 or +3 depending on the turn at Y, and hence (11).

For the two case (12), $n \equiv 1$ mod 3 has $turn(n+1) = -1$ always, allowing it to combine with $n \equiv 2$.

Again a morphism expansion can be written based on X having been 1 or not 1 and hence Y being R or not. Then state machine reversal low to high for the digit patterns.

$$dNextR = \quad 1 \to 2,1,1 \quad 2 \to 2,1,3 \quad 3 \to 2,1,3 \quad \text{starting } 2$$

Figure 4: *dNextR(n)* state machine,
ternary high to low
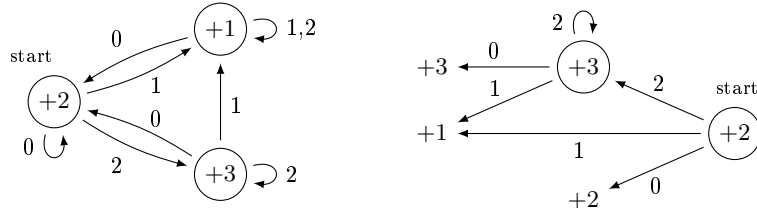
low to high

□

Second Proof of Theorem 3. A mechanical approach can be made using state machines for *TurnLpred* and *TurnRpred*. *dNextL(n)* = 1 is at those n where n+1 is L. This is next turn left per (3), and is a pair "any,L" at n.

dNextL(n) = 2 is a triplet "any,R,L" at n, so an R at n+1 and make some state machine manipulations for a test of L at n+2, which is the digit strings of left turns all subtract 2. The intersection of R next and L second next is all the *dNextL* = 2.

Similarly *dNextL(n)* = 3 is four "any,R,R,L" at n.

dNextR is similarly any,R, any,L,R, any,L,L,R. □

Arithmetically, the cases of one or two following opposite R or L can be written out

$$dNextL(n) = 1 + TurnRpred(n{+}1) + TurnRpred(n{+}1).TurnRpred(n{+}2)$$
$$= 1 + TurnRpred(n{+}1).\big(1 + TurnRpred(n{+}2)\big)$$
$$dNextR(n) = 1 + TurnLpred(n{+}1).\big(1 + TurnLpred(n{+}2)\big)$$

Exactly one of *dNextL(n)* = 1 or *dNextR(n)* = 1, and which one goes according to whether $turn(n{+}1)$ is L or R. The other is then 2 or 3 according to $turn(n{+}2)$.

When n is known to be a left turn, its lowest non-zero digit is 1, so in figure 3 low to high, digit 2 at the start state does not occur. So arithmetically just two cases in (9). Effectively this is simply $n \equiv 1$ always step +2 to the following $n \equiv 0$ and there *TurnRpred* to see if it's not L and so step +1 more.

$$dNextL(n) = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 3 \\ 2 + TurnRpred(n{+}2) & \text{if } n \equiv 1 \bmod 3 \end{cases} \quad \text{for left turn } n$$

Similar applies in *dNextR*, in that it never has lowest digit 1, but the states and cases are not reduced.

Generating functions for *dNextL* and *dNextR* follow from (9) and (11).

$$gdNextL(x) = \frac{1 + 2x + x^2}{1 - x^3} + \left(\frac{1}{x} + \frac{1}{x^2}\right) gTurnRpred(x^3) \tag{13}$$

$$= (1 + x)\left(\frac{1 + x}{1 - x^3} + \frac{1}{x^2}\sum_{k=1}^{\infty}\frac{x^{2.3^k}}{1 - x^{3^{k+1}}}\right) \tag{14}$$

$$gdNextR(x) = \frac{2 + x + 2x^2}{1 - x^3} + \frac{1}{x} gturn(x^3) \tag{15}$$

$$= \frac{2 + x + 2x^2}{1 - x^3} + \sum_{k=1}^{\infty} \frac{x^{3^k - 1}}{1 + x^{3^k} + x^{2.3^k}} \tag{16}$$

The offsets $n+2$ and $n+1$ used in the *TurnRpred* and *turn* cases each go to the same following 0 mod 3 and that 0 mod 3 is a factor of 3 which is no change to the turn sequence. So the whole turn sequence can be spread (by x^3) and replicated for (13),(15). In (14),(16), that spread is by starting the sums at $k=1$.

Factor $1+x$ on the whole of (14) is, in the usual way for a generating function, sum pairs of terms of the rest. That rest is

$$slnz(n) = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 3 \\ LowestNonZero(n+2) & \text{if } n \equiv 1 \bmod 3 \\ 0 & \text{if } n \equiv 2 \bmod 3 \end{cases} \tag{17}$$

$$= 1, 1, 0, \ 1, 2, 0, \ 1, 1, 0, \ 1, 1, 0, \ 1, 2, 0, \ 1, 2, 0, \ldots$$

$$dNextL(n) = slnz(n) + slnz(n-1) \tag{18}$$

$$gdNextL(x) = (1+x)\,gslnz(x)$$

The cases at (17) are *LowestNonZero*$(n+2)$ except that $n+2 \equiv 1,2 \bmod 3$ are result $0, 1$ instead. The two *slnz* at (18) give cases equivalent to *dNextL* form (9). For $n=0$, $slnz(-1) = 0$ is per its $n \equiv 2$ case.

The converse *slnz* in terms of *dNextL* is an alternating sign sum, either by expanding repeatedly or the usual way for generating function factor $1/(1+x)$.

$$slnz(n) = dNextL(n) - slnz(n-1)$$

$$= dNextL(n) - dNextL(n-1) + dNextL(n-2) - \cdots \pm dNextL(0)$$

Consecutive $n = 3h+1, 3h+2$ in *dNextL* have their *TurnRpred* testing the same $3h+3$ which then cancel due to the alternating signs. The constant parts of a block of 3 in *dNextL* similarly cancel $1 - 2 + 1 = 0$, leaving just the top-most one or two terms,

$$slnz(n) = \begin{cases} dNextL(n) & \text{if } n \equiv 0 \bmod 3 \\ dNextL(n) - dNextL(n-1) & \text{if } n \equiv 1 \bmod 3 \\ 0 & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

Theorem 4. *The m'th left or right turn point n is given by the following recurrences, for turns indexed by m and first turn $m=0$,*

$$TurnLeft(m) = \begin{cases} 1 & \text{if } m=0 \\ 3^k + TurnLeft\big(m - \frac{1}{2}(3^k + 1)\big) & \text{if } m < 3^k, \ m \neq 0 \\ 2.3^k + TurnLeft\big(m - 3^k\big) & \text{if } m \geq 3^k \end{cases} \tag{19}$$

where for $m \geq 1$ have biggest k with $\frac{1}{2}(3^k + 1) \leq m$

$$= 1, 3, 4, 7, 9, 10, 12, 13, 16, 19, 21, 22, \ldots \qquad \text{A026225}$$

$$TurnRight(m) = \begin{cases} 3^k + TurnRight\big(m - \tfrac{1}{2}(3^k - 1)\big) & \text{if } m < 3^k - 1 \\ 2.3^k & \text{if } m = 3^k - 1 \\ 2.3^k + TurnRight\big(m - 3^k\big) & \text{if } m \geq 3^k \end{cases} \qquad (20)$$

$$\text{where } k \text{ biggest } \tfrac{1}{2}(3^k - 1) \leq m$$

$$= 2, 5, 6, 8, 11, 14, 15, 17, 18, 20, 23, 24, \dots \qquad \text{A026179}$$

Proof. In an expansion level k, there are 3^k segments and $3^k - 1$ turns between them. Since the curve is symmetric in $180°$ rotation, there are half lefts and half rights $\tfrac{1}{2}(3^k - 1)$ each.

The recurrences follow from how many turns of each direction in each sub-part and between. Expansion level $k+1$ comprises the following sub-parts for level $k \geq 1$.

TurnLeft parts
$k+1$, sub-parts $k \geq 1$
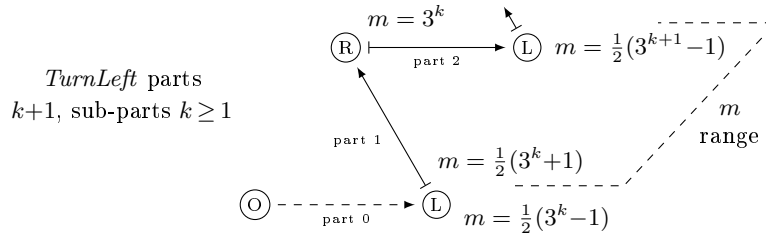
Part 0 has $\tfrac{1}{2}(3^k - 1)$ left turns so that the L after it is $m = \tfrac{1}{2}(3^k - 1)$ and the first m within part 1 is $m = \tfrac{1}{2}(3^k + 1)$. Taking k as the biggest with $\tfrac{1}{2}(3^k + 1) \leq m$ is then m ranging from the first turn in part 1 to the L after part 2, inclusive.

The m which is the first L in part 2 is the number of L preceding there, which is $2.\tfrac{1}{2}(3^k - 1) + 1 = 3^k$. Comparing m to 3^k thus determines whether it is in part 1, or after.

For part 1, subtracting its start $\tfrac{1}{2}(3^k + 1)$ reduces to an m within part 0.

$$\tfrac{1}{2}(3^k + 1) \leq m \qquad\qquad \leq 3^k - 1 \qquad \text{part 1}$$
$$0 \leq m - \tfrac{1}{2}(3^k + 1) \leq \tfrac{1}{2}(3^k - 3)$$

For part 2 and the L following it, subtracting the first $m = 3^k$ in part 2 reduces to an m which is in part 0 or the L following it.

$$3^k \leq m \qquad \leq \tfrac{1}{2}(3^{k+1} - 1) \qquad \text{part 2}$$
$$0 \leq m - 3^k \leq \tfrac{1}{2}(3^k - 1)$$

These reductions reach case $m=0$ eventually, which is the first L at $n=1$.
Similarly *TurnRight*, but its m range does not take in either of the L.

TurnRight parts
$k+1$, sub-parts $k \geq 1$
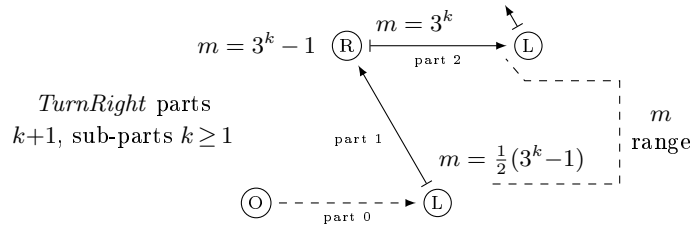
The R between parts 1 and 2 is $m=3^k-1$ and is an exception in the cases since part 0 ends with an L not the desired R. $\qquad\square$

As noted, *TurnLeft* always ends in its $m=0$ case $TurnLeft(0) = 1$. If that is made 0 instead then the result is $TurnLeft(m) - 1$ which is the segment number whose end is the m'th turn left, or equivalently the point number where the next point $n+1$ is the m'th left.

Theorem 5. *$n = TurnLeft(m)$ can be calculated by the following digit procedure*

$$n \leftarrow 2m$$
$$\text{for each ternary digit position high to low in } n$$
$$\quad \text{if digit} = 1 \text{ then } n \leftarrow n - 1$$
$$n \leftarrow n + 1 \tag{21}$$

And $n = TurnRight(m)$ can be calculated by the following digit procedure

$$n \leftarrow 2m + 2$$
$$\text{for each ternary digit position high to low in } n$$
$$\quad \text{if digit} = 1 \text{ then } n \leftarrow n + 1$$

The digit tested at each digit position is in the successively modified n, not just the original $2m$ or $2m+2$.

Proof. These procedures are implicit in the recurrences (19). For *TurnLeft*, consider the recurrence acting on a $p = 2m$ instead of m. Such doubling is

$$TurnLeft2\,(p) = TurnLeft(m) \qquad \text{where } p = 2m$$
$$= \begin{cases} 1 & \text{if } p=0 \\ 3^k + TurnLeft2\,(p - 3^k - 1) & \text{if } 3^k \le p < 2.3^k \\ 2.3^k + TurnLeft2\,(p - 2.3^k) & \text{if } p \ge 2.3^k \end{cases}$$
$$\text{where } k \text{ biggest } 3^k + 1 \le p \tag{22}$$

The effect of the procedure is to hold the *TurnLeft* result so far in the high ternary digits of n, and p in the low digits.

$$n = \boxed{\quad \text{result} \quad | \quad p \quad} \qquad \text{ternary} \qquad \text{Figure 5}$$

p is always even so condition (22) is the same as $3^k \le p$ so the *TurnLeft2* cases are on high ternary digit 1 or 2 in p.

Case $p \ge 2.3^k$ is ternary digit 2 subtracted from p and added to the result, so no change to the combined n.

Case $3^k \le p < 2.3^k$ is ternary digit 1 subtracted from p and added to the result, and an additional -1 on p, so decrease to $n - 1$. This decrement does not modify the 1 digit moved to the result since p is even so it has at least two 1 digits and any borrow stops at the lowest of them. The smaller new p is even again since it is an even amount 3^k+1 subtracted altogether from p.

Case $p=0$ result 1 is the final $n+1$ in the procedure. The procedure works through all 0 digits of p, leaving them unchanged, and applies this $+1$ last.

For *TurnRight*, consider its recurrence (20) acting on a $p = 2m+2$,

$$TurnRight2\,(p) = TurnRight(m) \qquad \text{where } p = 2m+2$$

$$= \begin{cases} 3^k + TurnRight(p - 3^k + 1) & \text{if } p < 2.3^k \\ 2.3^k & \text{if } p = 2.3^k \\ 2.3^k + TurnRight(p - 2.3^k) & \text{if } p > 2.3^k \end{cases}$$

$$\text{where } k \text{ biggest } 3^k + 1 \le p$$

Again n is high result digits and low p as in figure 5, and p is even so the cases are ternary digit 1 or 2 of p.

Case $p < 2.3^k$ is ternary digit 1 and it adds additional $+1$ to p. This increment does not modify the 1 digit moved to the result since again p is even so has at least two 1 digits and any carry will stop at the lowest of them.

Cases $p \ge 2.3^k$ are ternary digit 2 put to the result unchanged. For $p = 2.3^k$, the recurrence stops. The procedure continues but the low digits of $p = 2.3^k$ are 0s which the procedure leaves unchanged. □

In a ternary computer, testing for digit 1 is simple. In a binary computer, it may be desirable to use a vector of ternary digits and apply increments or decrements there, either with an explicit carry loop or bit twiddling. The increment and bit twiddling at (4) suits the *TurnRight* procedure. A similar decrement and bit twiddling *PostDecFix* would suit the *TurnLeft* procedure. In both cases the increment and decrement can be applied to ternary digits for the procedure tests to examine, and simultaneously to an ordinary n which will be the result.

Both *TurnLeft* and *TurnRight* are close to $2m$, roughly speaking since the number of each is the same at the end of an expansion level. Or algebraically in (19),(20) a $\frac{1}{2}(3^k \pm 1)$ subtracted from m is 3^k added to n, and in part 2 similarly 3^k subtracted from m is 2.3^k added to n. Offsets from $2m$ can be expressed

$$TurnLeftOff\,(m) = 2m - TurnLeft(m) \tag{23}$$
$$= -1, -1, 0, -1, -1, 0, 0, 1, 0, -1, -1, 0, -1, -1, \ldots$$
$$TurnRightOff\,(m) = TurnRight(m) - 2m$$
$$= 2, 3, 2, 2, 3, 4, 3, 3, 2, 2, 3, 2, 2, 3, \ldots$$

Substituting into (19),(20) gives recurrences

$$TurnLeftOff\,(m) = \begin{cases} -1 & \text{if } m=0 \\ TurnLeftOff\left(m - \frac{1}{2}(3^k+1)\right) + 1 & \text{if } m < 3^k \\ TurnLeftOff\left(m - 3^k\right) & \text{if } m \ge 3^k \end{cases}$$

$$\text{where } k \text{ biggest } \frac{1}{2}(3^k + 1) \le m$$

$$TurnRightOff\,(m) = \begin{cases} TurnRightOff\left(m - \frac{1}{2}(3^k-1)\right) + 1 & \text{if } m < 3^k-1 \\ 2 & \text{if } m = 3^k-1 \\ TurnRightOff\left(m - 3^k\right) & \text{if } m > 3^k-1 \end{cases}$$

$$\text{where } k \text{ biggest } \frac{1}{2}(3^k - 1) \le m$$

In part 2, the L and R turns between parts 0,1 and 1,2 balance, so offsets are unchanged on descending. In part 1 the preceding L is an extra, making

smaller *TurnLeft*. The offsets thus grow according to how many middle parts, and in particular

$$TurnLeftOff(m) \geq -1$$
$$TurnRightOff(m) \geq 2$$

See ahead at (40),(41) for new highs in the offsets.

The increments between successive turns L or R are

$$dTurnLeft(m) = TurnLeft(m{+}1) - TurnLeft(m)$$
$$= 2 - \big(TurnLeftOff(m{+}1) - TurnLeftOff(m)\big)$$
$$= 2, 1, 3, 2, 1, 2, 1, 3, 3, 2, 1, 3, 2, 1, 2, 1, 3, \ldots \qquad \text{A026141}$$
$$dTurnRight(m) = TurnRight(m{+}1) - TurnRight(m)$$
$$= TurnRightOff(m{+}1) - TurnRightOff(m) + 2$$
$$= 3, 1, 2, 3, 3, 1, 2, 1, 2, 3, 1, 2, 3, 3, 1, 2, 3, \ldots \qquad \text{A026181}$$

The expansions in figure 2 show these increments are always 1, 2 or 3. The m'th such increment can be expressed by recurrences by expanding *TurnLeft* or *TurnLeftOff* etc.

$$dTurnLeft(m) = \begin{cases} 2, 1 & \text{if } m = 0, 1 \\ dTurnLeft\big(m - \tfrac{1}{2}(3^k{+}1)\big) & \text{if } m < 3^k{-}1 \\ 3 & \text{if } m = 3^k{-}1 \\ dTurnLeft\big(m - 3^k\big) & \text{if } m \geq 3^k \end{cases}$$
$$\text{where } k \text{ biggest } \tfrac{1}{2}(3^k + 1) \leq m \text{ and } k \geq 1$$

$$dTurnRight(m) = \begin{cases} 3 & \text{if } m = 0 \\ dTurnRight\big(m - \tfrac{1}{2}(3^k{-}1)\big) & \text{if } m < 3^k{-}2 \\ 1 & \text{if } m = 3^k{-}2 \\ 2 & \text{if } m = 3^k{-}1 \\ dTurnRight\big(m - 3^k\big) & \text{if } m \geq 3^k \end{cases} \qquad (24)$$
$$\text{where } k \text{ biggest } \tfrac{1}{2}(3^k - 1) \leq m \text{ and } k \geq 1$$

In these recurrences, nothing is accumulated, just descend down m by parts until reaching one of the 1, 2 or 3 cases.

For *dTurnLeft*, case $m{=}3^k{-}1$ is the L of the last LR pair in part 1. It must step across the R between parts 1 and 2, so $dTurnLeft = 3$ there.

For *dTurnRight*, case $m{=}3^k{-}1$ is the R between parts 1 and 2, and $m{=}3^k{-}2$ preceding that is R of the last LR pair in part 1.

The cases at (24) correspond to the recurrence given by Neil Sloane in A131989 (indexed there starting from 1). That sequence is defined by run lengths in a symbol substitution

$$* \to * * \,|\, * \qquad \text{starting from } * * \,|\, *$$
$$\text{run lengths} = 2, 3, 1, 2, 3, 3, 1, 2, 1, 2, \ldots \qquad \text{A131989}$$

This is the terdragon curve expansion with $*$ as a segment and $|$ as the R turn between parts 1 and 2. The sequence values are the lengths of runs of $*$

separated by |, and thus steps between successive R turns. The run lengths include the initial two $**$ at the start of the symbol sequence as a run 2. That would be a step from the origin $n{=}0$ to the first R at $n{=}2$, which *dTurnRight* here does not include. (The $*$ and | symbols as integers 1, 2 are A133162.)

Sloane also in A131989 gives an expansion where copies of the sequence are concatenated and the terms each side of the first join are added together. That first join is a new left turn so sum the distances each side to be between right turns.

$$2\,3\,1\,2\,\underline{1\quad 2}\,3\,1\,2\,1\quad 2\,3\,1\,2\,\underline{1}\ \ldots \qquad\qquad \text{\textit{dTurnRight} three copies,}$$
$$\underset{\substack{\text{first join}\\ \text{sum}}}{}\qquad\qquad\qquad\qquad\qquad \text{extra initial 2 final 1}$$

Dekking gives a morphism for the sequence by repeated expansions,

$$\text{A131989} = \quad 1 \to 2,1 \quad 2 \to 2,3,1 \quad 3 \to 2,3,3,1 \quad \text{starting from 2} \qquad (25)$$

reached by considering return words in the symbol sequence. In the curve these correspond to runs of 1, 2 or 3 segments between the L turns. The expansions of the return words correspond to inserting LR into each segment and consequent new runs. Notice in (25) each term expands to 2, $\underline{3}$, 1 where $\underline{3}$ means 0, 1 or 2 many 3s.

Another approach can be made for *dTurnLeft* and *dTurnRight* (the latter without the initial 2 which A131989 has) by considering LR each side of an existing turn.
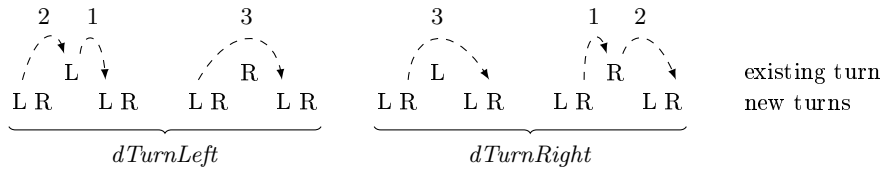
Theorem 6. *The dTurnLeft sequence is the turn sequence mapped*

$$L \to 2,1 \qquad R \to 3 \qquad\qquad\qquad\qquad (26)$$

and the dTurnRight sequence is the turn sequence mapped

$$L \to 3 \qquad R \to 1,2 \qquad\qquad\qquad\qquad (27)$$

Proof. On expansion, each existing turn in the curve gains a new pair LR before, per figure 2. The steps between successive lefts or rights each side of an existing turn are

For *dTurnLeft*, in the second-last expansion level identify a 2,1 pair with each L and a 3 with each R.

For *dTurnRight*, similarly in the second-last expansion level identify 3 with each L and a 1,2 pair with each R. □

The effect of (26) is to insert an extra symbol 1 after each L, or the effect of (27) is to insert an extra symbol 2 after each R (or in both cases a symbol before if preferred).

The turn sequence expansion per figure 2 is

$$turn = \quad \text{L} \rightarrow \text{L,R, L} \qquad \text{R} \rightarrow \text{L,R, R} \qquad \text{starting from L}$$

Substituting (26) and (27) into this is

$$dTurnLeft = \quad 2,1 \rightarrow 2,1,3,2,1 \quad 3 \rightarrow 2,1,3,3 \quad \text{starting from } 2,1$$
$$dTurnRight = \quad 3 \rightarrow 3,1,2,3 \quad 1,2 \rightarrow 3,1,2,1,2 \quad \text{starting from } 3$$

The expanding pairs 2,1 or 1,2 can be split at any point to become expansions of individual symbols. For example,

$$dTurnLeft = \quad 1 \rightarrow 1 \quad 2 \rightarrow 2,1,3,2 \quad 3 \rightarrow 2,1,3,3 \quad \text{starting from } 2$$
$$dTurnRight = \quad 1 \rightarrow 3,1,2,1 \quad 2 \rightarrow 2 \quad 3 \rightarrow 3,1,2,3 \quad \text{starting from } 3$$

These highlight the way $dTurnLeft$ has a 1 after every 2, and vice versa in $dTurnRight$. The start for $dTurnLeft$ can be just 2 as long as the split makes 2 expand to more than just itself.

**Theorem 7.** $d = dTurnLeft(m)$ *can be calculated by the following procedure*

$$n \leftarrow 2m \text{ and } d \leftarrow 2$$
for each ternary digit position high to low in n
    if digit $= 0$ then if $d = 0$ then $d \leftarrow 3$
                                  else $d \leftarrow 2$ (28)
    if digit $= 1$ then $d \leftarrow 0$ and $n \leftarrow n - 1$
    if digit $= 2$ and $d \neq 0$ then $d \leftarrow 1$ (29)

Proof. This procedure finds $TurnLeft$ by calculating n from m as in the procedure of theorem 5, and puts the resulting digits of n through $dNextL$.

$$dTurnLeft(m) = dNextL\big(TurnLeft(m)\big)$$

The final $n \leftarrow n+1$ step (21) of the $TurnLeft$ procedure is eliminated by using the digits of $n-1$, ie. the digits before that final step, to determine $dNextL(n)$. Some state machine manipulations can apply a decrement to the digit strings of $dNextL$ in figure 3 giving

$dNextL(n)$, for $n \geq 1$, by state machine on ternary digits of $n-1$ high to low
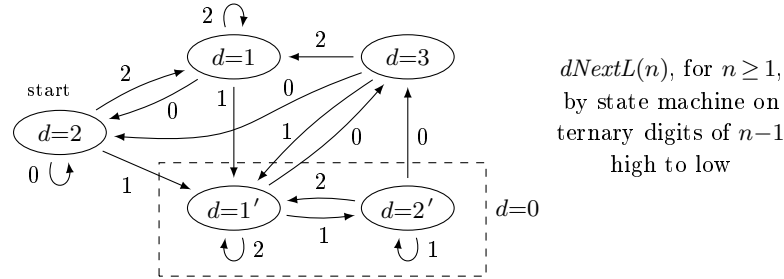
This state machine is for any n, but $TurnLeft$ is only left turn n so only $n-1$ from such an n will be a final state. State $d=2'$ is a final state for $n-1 = \ldots 1$ which is $n = \ldots 2$ not a left turn. State $d=1'$ is a final state for $n-1 = \ldots 1\underline{2}$, where the underline means zero or more repetitions, which is $n = \ldots 2\underline{0}$ and likewise not left. So their different $d=1$ or $d=2$ results do not need to be distinguished. They are combined as special $d=0$ in the procedure.

Digit 0 in the state machine goes from $d=0$ to $d=3$ or from anywhere else to $d=2$, and hence (28).

Digit 1 goes from anywhere to the special $d=0$.

Digit 2 stays in $d=0$, or anywhere else goes to $d=1$, hence (29). $\qquad\square$

As a remark, digit 1 from anywhere goes to $d=0$ and that state always eventually goes to $d=3$, but it does not suffice to have digit 1 go directly to $d=3$ because the result of a following digit 0 is different according to whether in $d=0$ or $d=3$.

The digit string cases are as follows. The $n{-}1$ column is the digits seen by the procedure,

$n{-}1$	n	$dTurnLeft(n)$	next n
$\ldots 0\underline{2}2$	$\ldots 1\underline{0}0$	$+1$	$\ldots 1\underline{0}1$
$\ldots 0\underline{2}0$	$\ldots 0\underline{2}1$	$+2$	$\ldots 1\underline{0}0$
$\ldots 1\underline{2}0$	$\ldots 1\underline{2}1$	$+3$	$\ldots 2\underline{0}1$

Theorem 8. $d = dTurnRight(m)$ *can be calculated by the following procedure*

$$n \leftarrow 2m + 2 \ \text{ and } \ d \leftarrow 2$$

for each ternary digit position high to low in n

$\qquad$ if digit $= 0$ then $d \leftarrow 2$

$\qquad$ if digit $= 1$ then $d \leftarrow 1$ and $n \leftarrow n + 1$

$\qquad$ if digit $= 2$ and $d = 2$ then $d \leftarrow 3$

Proof. Similar to theorem 7, this procedure finds *TurnRight* by calculating n from m as in theorem 5 and puts the digits of n through *dNextR*,

$$dTurnRight(m) = dNextR\big(TurnRight(m)\big)$$

The digit and d steps are *dNextR* figure 4 state machine high to low. $\qquad\square$

Sequences *dTurnLeft* and *dNextL* are related by inserting into *dTurnLeft* the successive steps down which are non-lefts skipped, $1 \to 1$, $2 \to 2,1$, $3 \to 3,2,1$, and an initial 1 at the start for $dNextL(0){=}1$. The same for *dTurnRight* to *dNextR*, and for it initial $2, 1$.

1.3   Direction

The total turn is a count of ternary 1 digits since each "1" sub-part is rotated $+120°$ and sub-parts "0" and "2" are unchanged.

$$dir(n) = \sum_{j=0}^{n-1} turn(j)$$

$$= \text{count ternary 1 digits in } n \qquad\qquad (30)$$

$$= 0, 1, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 2, 3, 2, 1, 2, 1, \ldots \qquad \text{\scriptsize A062756}$$

$$gdir(x) = \frac{1}{1 - x}\, gturn(x) \qquad\qquad (31)$$

$$= \sum_{k=0}^{\infty} \frac{x^{3^k} - x^{2.3^k}}{(1-x)\left(1 - x^{3^{k+1}}\right)} \tag{32}$$

$$= \sum_{k=0}^{\infty} \frac{x^{3^k}}{(1-x)\left(1 + x^{3^{k+1}} + x^{2.3^{k+1}}\right)} \tag{33}$$

Generating function (31) is the usual factor $1/(1-x)$ for cumulative turns. In (32), each term is a generating function which has coefficient 1 where n has a 1-digit at position k in n, so summing to count 1-digits. (33) has a factor $1 - x^{3^k}$ cancelled between numerator and denominator.

Some of the structure of dir can be illustrated in a plot.
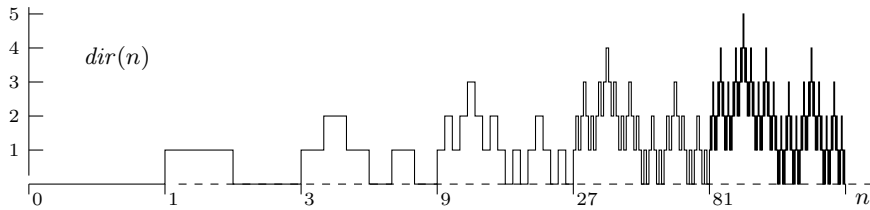
Blocks of $n = 3^k$ to $3.3^k - 1$ are shown scaled to the same width (and linear within them) in order to see successive refinements. The next block is the same overall shape of its predecessor but adds a middle-third excursion up in each n (being those n with a new low 1-digit).

The successive new highs are where n is entirely 1 digits.

$$DirMaxN_k = AllOnes_k = \tfrac{1}{2}\left(3^k - 1\right)$$
$$= \text{ternary } 11...11 \text{ of } k \text{ many digits}$$
$$= 0, 1, 4, 13, 40, 121, 364, 1093, \dots \qquad \text{\small A003462}$$

The number of left and right turns from 1 to n inclusive are

$$TurnsL(n) = \sum_{j=1}^{n} TurnLpred(n)$$
$$= 0, 1, 1, 2, 3, 3, 3, 4, 4, 5, 6, 6, 7, 8, \dots \qquad \text{\small A189674}$$
$$TurnsR(n) = \sum_{j=1}^{n} TurnRpred(n)$$
$$= 0, 0, 1, 1, 1, 2, 3, 3, 4, 4, 4, 5, 5, 5, \dots \qquad \text{\small A189672}$$

Generating functions are again factor $1/(1-x)$ for cumulative left or right predicate,

$$gTurnsL(x) = \frac{1}{1-x}\, gTurnLpred(x) = \sum_{k=0}^{\infty} \frac{x^{3^k}}{(1-x)\left(1 - x^{3^{k+1}}\right)}$$

$$gTurnsR(x) = \frac{1}{1-x}\, gTurnRpred(x) = \sum_{k=0}^{\infty} \frac{x^{2.3^k}}{(1-x)\left(1 - x^{3^{k+1}}\right)}$$

All turns are left or right so total lefts plus rights is simply n. The difference lefts minus rights is net dir for (35). In the generating functions, this difference is $gdir$ form (32).

$$TurnsL(n) + TurnsR(n) = n \tag{34}$$

$$TurnsL(n) - TurnsR(n) = dir(n) \tag{35}$$

Sum and difference of (34),(35) are

$$TurnsL(n) = \tfrac{1}{2}\big(n + dir(n)\big)$$
$$TurnsR(n) = \tfrac{1}{2}\big(n - dir(n)\big)$$

Clark Kimberling in OEIS A189674 and A189672 gives the following recurrences, with the first adapted here to $TurnsL$ numbered first turn at $n=1$,

$$TurnsL(n) = TurnsL\big(\big\lfloor \tfrac{n}{3}\big\rfloor\big) + \big\lfloor \tfrac{n+2}{3}\big\rfloor$$
$$TurnsR(n) = TurnsR\big(\big\lfloor \tfrac{n}{3}\big\rfloor\big) + \big\lfloor \tfrac{n+1}{3}\big\rfloor \tag{36}$$

These forms can be seen from the turn expansions in figure 2 (and general morphism expansions like A189674, A189672). $TurnsL(\lfloor n/3\rfloor)$ counts lefts in the "existing turns". Each point is preceded by a new pair LR so $+\lfloor n/3\rfloor$ further lefts. When $n \equiv 1,2 \bmod 3$ the new L following the last "existing" is to be included too, so total $+\lfloor (n+2)/3\rfloor$. Similarly $TurnsR$, but for it the following new R is only when $n \equiv 2 \bmod 3$, so $+\lfloor (n+1)/3\rfloor$.

$TurnLeft$ from theorem 4 and $TurnsL$ are inverses in the sense that

$$TurnsL\big(TurnLeft(m)\big) = m + 1$$

The left turn at $n = TurnLeft(m)$ increments $TurnsL$ so this n is the smallest for which $TurnsL(n) = m+1$, or equivalently $n-1$ is the greatest for which $TurnsL(n-1) = m$. Similarly $TurnRight$ and $TurnsR$.

The $TurnLeft$ procedure in theorem 5 finds, for given m, the least solution n to

$$TurnsL(n) = m + 1$$
$$\tfrac{1}{2}\big(n + dir(n)\big) = m + 1 \tag{37}$$
$$n = 2m + 2 - dir(n) \tag{38}$$

The successive decrements in the procedure effectively adjust for count of 1-digits, which is dir, in order to satisfy (38). Of course the correctness of the procedure depends on a decrement for a given 1-digit not upsetting higher 1-digits already considered.

The procedure for $TurnRight$ finds in a similar way

$$n = 2m + 2 + dir(n) \tag{39}$$

These relations show too $TurnLeftOff$ and $TurnRightOff$ from (23) are

$$TurnLeftOff(m) = dir(n) - 2 \qquad \text{where } n = TurnLeft(m)$$
$$TurnLeftRight(m) = dir(n) + 2 \qquad \text{where } n = TurnRight(m)$$

New highs in *TurnLeftOff* are new highs in *dir* among left turn n. *dir* is a maximum when $n = 11...11$ ternary and this is a left turn. For k many ternary digits, its index m per (37) is

$$m = \tfrac{1}{2}\left(AllOnes_k + k\right) - 1 \qquad \text{for } k \geq 1 \tag{40}$$
$$= \tfrac{1}{4}\left(3^k + 2k - 5\right)$$
$$= 0, 2, 7, 21, 62, 184, 549, \ldots \qquad\qquad {\scriptstyle -1 + \text{A047926}}$$

New highs in *TurnRightOff* are new highs in *dir* among right turn n. To be a right turn is lowest non-zero digit 2, so take low digit 2 so $n = 11...112$ ternary. For k many ternary digits, its index m per (39) is

$$m = \tfrac{1}{2}\left(AllOnes_k + 1 - (k{-}1)\right) - 1 \qquad \text{for } k \geq 1 \tag{41}$$
$$= \tfrac{1}{4}\left(3^k - 2k - 1\right)$$
$$= 0, 1, 5, 18, 58, 179, 543, \ldots \qquad\qquad {\scriptstyle \text{A000340}}$$

These offsets are the maximum number of decrements or increments made by the *TurnLeft* and *TurnRight* procedures in theorem 5. In both cases for n of k digits they make at most $k{-}1$ decrements or increments in their respective loops, and the n and m where that maximum occurs is unique.

$\mathrm{dir}(n) \bmod 3$ is a net direction East, North West or South West. This net angle suffices for drawing etc.

$$dir(n) \bmod 3$$
$$= 0, 1, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 2, 0, 2, 1, \ldots$$
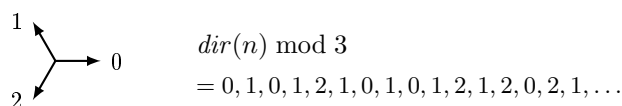
$$= 0 \text{ at } n = 0, \ 2, \ 6, \ 8, 13, 18, 20, 24, 26, \ldots$$
$$= 1 \text{ at } n = 1, \ 3, \ 5, \ 7, \ 9, 11, 15, 17, 19, \ldots$$
$$= 2 \text{ at } n = 4, 10, 12, 14, 16, 22, 28, 30, 32, \ldots$$

On expansion, each middle part is 1 greater direction, with wrap-around. (Philippe Deléham has this in OEIS A062756 for the full *dir*, no wrap-around.)

$$dir \bmod 3 = \quad 0 \to 0, 1, 0 \quad 1 \to 1, 2, 1 \quad 2 \to 2, 0, 2 \quad \text{starting from } 0$$

1.4   Coordinates

It's convenient to calculate terdragon curve coordinates in complex numbers using ω_3 or ω_6 roots of unity and a base b which is the end of a 3-segment unit expansion. The roots of unity act as rotations by 120° or 60°.

$$b = \omega_3 + 2 \ = \ \omega_6 + 1 \qquad \text{base}$$
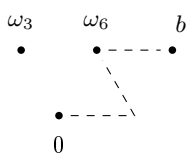
Per Davis and Knuth, and counting vertices starting n=0 at the origin, point number n is given by ternary digits of $n = a_{k-1} \ldots a_2 a_1 a_0$.

$digit(a) = 0, 1, \omega_6 \quad$ for $a = 0, 1, 2$

$$point(n) = b^{k-1}\, digit(a_{k-1}) \qquad\qquad\qquad\text{high digit} \qquad\qquad (42)$$
$$+\, b^{k-2}\, digit(a_{k-2})\, \omega_3^{dir(a_{k-1})}$$
$$+\, b^{k-3}\, digit(a_{k-2})\, \omega_3^{dir(a_{k-1}\, a_{k-2})}$$
$$\ldots$$
$$+\, b^1 \quad digit(a_1) \quad \omega_3^{dir(a_{k-1}\, a_{k-2} \ldots a_2)}$$
$$+\, b^0 \quad digit(a_0) \quad \omega_3^{dir(a_{k-1}\, a_{k-2} \ldots a_2\, a_1)} \qquad \text{low digit}$$

$= 0,\ 1,\ \omega_6,\ 1{+}\omega_6,\ 2\omega_6,\ \omega_6,\ \omega_6,\ -1 + 2\omega_6,\ 2\omega_6,\ -1 + 3\omega_6,\ \ldots$

$= 0,\ 1,\ \frac{1}{2}{+}\frac{1}{2}\sqrt{3}i,\ \frac{3}{2}{+}\frac{1}{2}\sqrt{3}i,\ 1{+}\sqrt{3}i,\ \frac{1}{2}{+}\frac{1}{2}\sqrt{3}i,\ \sqrt{3}i,\ 1{+}\sqrt{3}i,\ \frac{1}{2}{+}\frac{3}{2}\sqrt{3}i,\ \ldots$

Digits can be taken high to low as

$$point(3^k a_k + n_{k-1}) \;=\; b^k\, digit(a_k) \;+\; point(n_{k-1}).\omega_3^{dir(a_k)}$$

a_k is the highest digit and is located per the base pattern scaled by b^k. The n_{k-1} digits below it go in direction $dir(a_k)$ by multiplying ω_3. Repeated expansion is

$$point(n) = b^k\, digit(a_k) \qquad\qquad\qquad\qquad\qquad (43)$$
$$+\, \omega_3^{dir(a_k)}\Big(b^{k-1}\, digit(a_{k-1})$$
$$\ldots$$
$$+\, \omega_3^{dir(a_2)}\Big(b^1\, digit(a_1)$$
$$+\, \omega_3^{dir(a_1)}\Big(b^0\, digit(a_0) \Big)\Big)\Big)$$

Digits can be taken low to high by segment replacement,

$$point(3n_1 + a_0) \;=\; point(n_1).b \;+\; \omega_3^{dir(n_1)}.digit(a_0) \qquad (44)$$

a_0 is the low ternary digit and n_1 the digits above it. $dir(n_1)$ is the segment direction before expansion, so rotating the new base figure. This direction depends on all of n_1. Evaluating the nested (43) from innermost to outermost builds it successively by multiplying each direction onto all below.

For computer calculation, integer coordinates x, y representing $x + y\omega_3$ can be maintained. Or $x + y\omega_6$ if preferred. Multiplication by ω_3, ω_6 or b are then various integer additions or subtractions of x, y.

It's also possible to calculate with an x, y representing $\frac{1}{2}(x + y\sqrt{3}i)$ so that y is a purely imaginary term (vertical). In this case x, y are integers $x \equiv y \bmod 2$, ie. both even or both odd. The effect of plotting those x, y directly on an integer grid, without $\frac{1}{2}$ or $\sqrt{3}$ factors, is to flatten to right triangles height 1 base 2 (instead of equilateral triangles).

$$x, y \text{ as } \tfrac{1}{2}\left(x + y\sqrt{3}i\right) \qquad (45)$$
$$\text{for terdragon } k=2$$
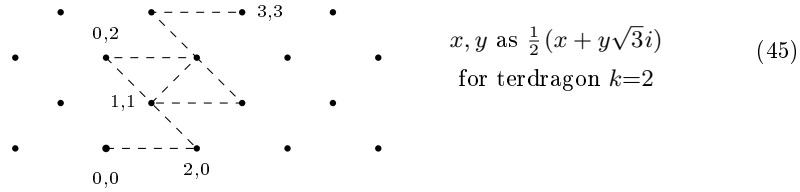
This form can be useful for a graphics display using every second pixel of a square grid. It avoids uneven spacing at small scales. If a factor $\sqrt{3}$ for equilateral triangles is used then it's necessary to round to an integer pixel and at resolutions near a few pixels this rounding becomes noticeable.

A grid of every second integer position is the same as a square grid rotated $45°$. A further possible integer coordinate system is to take triangles on a $45°$ angle. This corresponds to integers x, y representing points $x\omega_6 + y\omega_3$.

$$x, y \text{ as } x\omega_6 + y\omega_3$$
$$\text{for terdragon } k=2$$
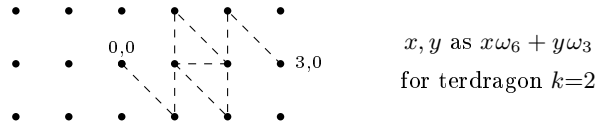
The low to high *point* formula (44) can be reversed to calculate n for a given segment. Suppose a segment is at $z = point(n)$ in direction $d = 0, 1, 2 \equiv dir(n)$ mod 3.

$unpoint(z, d) \qquad\qquad d = 0, 1, 2$
    loop
        if $z{=}0$ **then** $arm = 2d$ **end loop**
        if $z{=}\omega_6,\ d{\equiv}2$ **then** $arm = 1$ **end loop**
        if $z{=}{-}1,\ d{\equiv}0$ **then** $arm = 3$ **end loop**
        if $z{=}\overline{\omega_6},\ d{\equiv}1$ **then** $arm = 5$ **end loop**

$$a = \begin{cases} 0 & \text{if } z \equiv 0 \mod b \\ 1 & \text{if } z \equiv 1 \mod b \\ 2 & \text{if } z \equiv \omega_6 \mod b \end{cases} \qquad \text{ternary digit } a$$

        $d \leftarrow d - dir(a) \mod 3$
        $z \leftarrow \left(z - digit(a).\omega_3^d\right)/b$
        n digits low to high $\leftarrow a$
    end loop
    if arm even **then** n
    if arm odd **then** $3^k{-}n$
    where k is the number of digits of n generated

$z \bmod b$ determines the low ternary digit a of n since all terms of $point(n)$ except the lowest are multiples of b, and in that low term $\omega_3 \equiv 1 \bmod b$ so

$$z \equiv digit(a_0) \mod b$$

The direction factor in (44) is all digits except a_0,

$$dir(a_k...a_1) = d - dir(a_0)$$

Then the low digit is subtracted, b divided out, and the procedure repeated for the second lowest digit a_1, etc.

For segments in the terdragon curve starting in direction $d=0$ this ends with location $z=0$ and direction $d=0$.

For segments in a 120° rotated curve $z.\omega_3$, the procedure also ends with $z=0$ but direction $d=1$. This is since $\omega_3 \equiv 1 \bmod b$ so that factor ω_3 does not change digits generated from z, and the initial d includes $+1$ for the rotation. Similarly segments in a 240° rotated curve $z.\omega_3^2$ reach $z=0$ and direction $d=2$.

For segments in a 60° rotated curve,

$$point(n).\omega_6 \;=\; b^k.\omega_6 + point(3^k{-}n).\omega_3^2$$

Geometrically this is starting at a 60° endpoint $b^k.\omega_6$ and going in direction $d=2$.
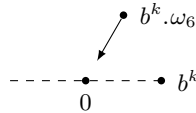
So the procedure gives digits of a 240° curve $point(3^k{-}n).\omega_3^2$, and loop ending $z=\omega_6$. Similarly for 180° and 300° rotated curves as arms 3 and 5. Notice these odd arms all take segment direction d as $0, 120, 240$, the same as the even arms. For the odd arms this is reverse along those arms, but the arm is not known until the end of the procedure.

If calculations are made in coordinates $x+y\omega_3$ then low digit a is simply

$$a = 0, 1, 2 \;\equiv x{+}y \bmod 3$$

If using $x+y\omega_6$ then a similar $x{-}y \bmod 3$. Or every second point coordinates of (45) is $-x \bmod 3$

The geometric interpretation of the procedure is to find which rhombus shaped expansion from figure 1 contains the segment, then step back to the multiple of b which is its start. The rhombus tiling and directions are a repeating pattern and, depending on the x, y coordinate style used, can also be done in a $12{\times}12$ table lookup.
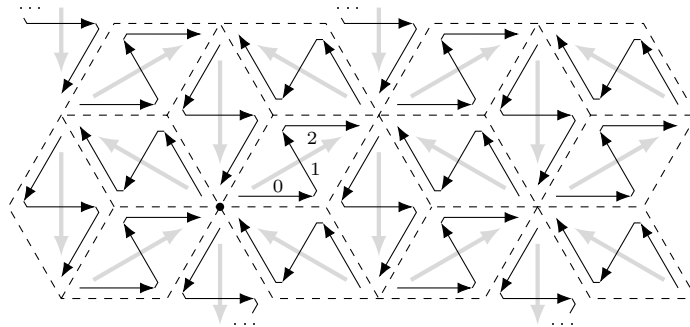
1.5   Other N

Each curve location z is visited 1, 2 or 3 times. Applying the *unpoint* procedure above for $d=0, 1, 2$ gives the n which are those visits. For a given n, the other

n_1, n_2 at the same location can be calculated from the ternary digits of n without going via the location.

Theorem 9. *For $n \geq 1$, the other n_1 and n_2 at the same location are given by the ternary digits of n put low to high though the following state machine.*
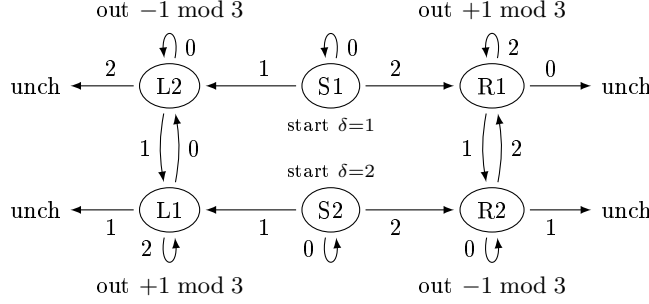

Figure 6: $other(n, \delta)$ by ternary digits low to high

$$other(n, \delta) = \ start\ in\ S\delta,\ output\ digits\ \pm 1\ mod\ 3\ in\ L,R$$

$$
\begin{array}{rrrrrrrrrrl}
other(n, 1) = & 0, & 2, & 5, & 6, & 17, & 1, & 15, & 4, & 11, & 18, \ldots \\
arm\ = & 0, & -1, & 0, & -1, & -1, & 1, & 0, & 0, & 0, & -1, \ldots \\
other(n, 2) = & 0, & 5, & 1, & 15, & 7, & 2, & 3, & 17, & 14, & 45, \ldots \\
arm\ = & 0, & -1, & 1, & -1, & 0, & 0, & 1, & -1, & 0, & -1, \ldots
\end{array}
$$

The start state is S1 or S2 for $\delta = 1, 2$ respectively for other direction $dir(n_\delta) \equiv dir(n) + \delta \mod 3$. In states L1,R1 the output digit is the n digit $+1 \mod 3$. In states L2,R2 the output digit is the n digit $-1 \mod 3$. In S states the output digits are n digits unchanged, as are all further digits after reaching "unch".

One additional high 0 is reckoned on n. The final state is L2, R2, or unch.

If final L2 then this is a left turn on the right boundary and the further visit is in arm -1. The output is reversed $n_\delta = 3^k - output$ to count from the origin, where k is the number of digits.

If final R2 then this is a right turn on the left boundary and further visit in arm 1. The output is again reversed $n_\delta = 3^k - output$ to count from the origin.

Proof. Suppose m is the same location as n but direction $+\delta$, and a certain dz offset away from n,

$$dir(m) \equiv dir(n) + \delta \mod 3 \tag{46}$$

$$point(m) = point(n) + \omega_3^{dir(n)}.dz$$

Factor $\omega_3^{dir(n)}$ on dz makes dz relative to the direction of segment n, like a low term of *point* formula (42). This allows step (48) to require only the low digit of n.

The digits of m are to be determined from δ, dz and the digits of n. Let a be the low digit of n and c be the low digit of m so that

$$n = 3n' + a \qquad m = 3m' + c$$

From the low digit *point* formula (44), a and c are related by

$$\omega_3^{dir(m')}.\, digit(c) \ \equiv \ \omega_3^{dir(n')}.\, digit(a) + \omega_3^{dir(n)}.\, dz \mod b \tag{47}$$

$\omega_3 \equiv 1 \bmod b$ so all factors of ω_3 can be ignored, leaving c determined by a and dz. New direction difference δ' is those two low digits dropped from (46)

$$\delta' = \delta - dir(c) + dir(a)$$

New location offset is the low digits taken off (44). The whole m' is not known yet, but $dir(m') \equiv dir(n') + \delta' \bmod 3$ is enough for its ω_3 power.

$$
\begin{aligned}
dz'.\omega_3^{dir(n')} &= point(m') - point(n') \\
&= \left(point(m) - \omega_3^{dir(m')}.digit(c) \right)/b \\
&\quad - \left(point(n) - \omega_3^{dir(n')}.digit(a) \right)/b \\
&= \left(dz.\omega_3^{dir(n)} - \omega_3^{dir(n')+\delta'}.digit(c) + \omega_3^{dir(n')}.digit(a) \right)/b
\end{aligned}
$$

$$dz' = \left(dz.\omega_3^{dir(a)} - \omega_3^{\delta'}.digit(c) + digit(a) \right)/b \tag{48}$$

From (47), the bracketed part of (48) is a multiple of b.

These steps begin from $dz = 0$ initially so m and n are the same location, and $\delta = 1 \,\text{or}\, 2$ other direction. The possible digits $a = 0, 1, 2$ from n then give the following transitions between δ, dz combinations, and output digit c related to a. These are per figure 6.
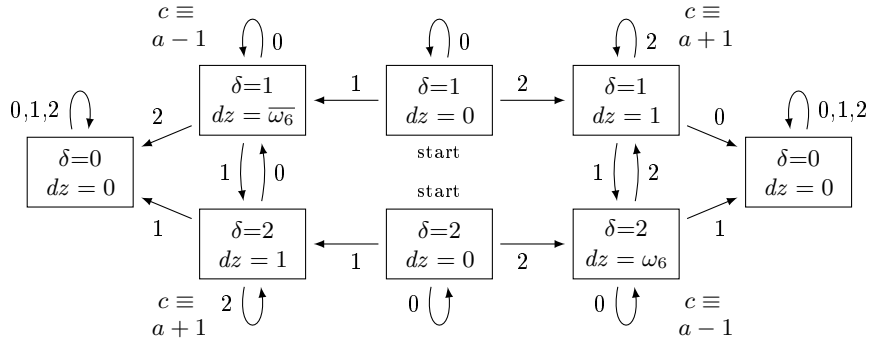
$\delta{=}0$, $dz{=}0$ gives $c = a$ unchanged from there onwards.

A high 0 digit on n goes to state L2 or R2, or from R1 it goes to unchanged. The latter is when m is bigger than n, representing a further visit to the same location in a higher curve level.

In states L2 or R2, high 0 digits on n loop. To see the rule for these as adjacent arms, first for L2 suppose n had an extra high digit 2, so it goes to "unch", with new high $c = a{-}1 = 1$ on m.
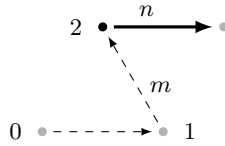
So the other visit to n is at m along a curve directed from 1. Taking 2 as the origin means it is $3^k - m$ along a curve directed away from that 2, in arm -1 at $-60°$.

For R2 suppose n has an extra high digit 1, so it goes to "unch", with new high $c = a-1 = 0$. Taking 1 as the origin, this is m in the 0 curve which is $3^k - m$ away from 1 in arm 1 at $+60°$. $\qquad\square$

L states are for n a left turn and R for n a right turn. They are reached from the S starts by lowest non-zero digit 1 or 2 respectively as per *turn* at (1).

Right boundary single-visited points are always left turns, otherwise non-overlapping plane filling would not be possible. So arm -1 is from R when high 0s on n don't reach "unch". Conversely left boundary points are right turns and arm $+1$ is from L. So the arm is either 0 when within the curve or $-turn(n)$ when adjacent arm.

The states of figure 6 and outputs can be expressed arithmetically using δ and the lowest non-zero digit of n,

$$other(n, \delta) \qquad\qquad \text{for } \delta = 1 \text{ or } 2$$

> digits $n = a_k a_{k-1}...a_0$ and extra high $a_{k+1} = 0$
>
> output digits $c_{k+1} c_k c_{k-1}...c_0$
>
> $a_t = $ lowest non-zero of n
>
> $c_t...c_0 \leftarrow a_t...a_0 \qquad$ unchanged
>
> **loop** $j = t+1$ to $k+1$
>
> > $c_j = 0, 1, 2 \equiv (a_j - \delta.a_t) \bmod 3$ $\qquad\qquad$ (49)
> >
> > $\delta \leftarrow \delta + dir(a_j) - dir(c_j)$ $\qquad\qquad$ (50)
>
> **end loop**
>
> **if** $\delta \equiv 0 \bmod 3$ **then** $n_\delta = \qquad\qquad c_{k+1}...c_0$, same arm
>
> **if** $\delta \equiv 1 \bmod 3$ **then** $n_\delta = 3^{k+1} - c_{k+1}...c_0$, arm -1
>
> **if** $\delta \equiv 2 \bmod 3$ **then** $n_\delta = 3^{k+1} - c_{k+1}...c_0$, arm $+1$

For δ at (50), taking *dir* of a single digit is simply 1 or 0 according as digit 1 or not. δ can be kept mod 3 at all stages.

a_t is the transition digit out of S states. Its use as $\delta.a_t$ at (49) flips the sense of δ for the R states. For example from S1 which is $\delta=1$, an $a_t=1$ goes to L2 and $a_t=2$ goes to R1. Multiplying a_t gives $-\delta.a_t \equiv 2,1$ to add for the output digit in those respective states.

The new n_δ can have up to 1 extra ternary digit over what n has. This is output digit c_{k+1} and the input a_{k+1} taken as 0.

If $\delta=0$ is reached in the loop then all further digits are unchanged $c_j = a_j$. $\delta=0$ means $c_j = a_j$ at (49) so $dir(c_j) - dir(a_j) = 0$ at (50), maintaining $\delta=0$. If $\delta=0$ initially then is no change $other(n, 0) = n$.

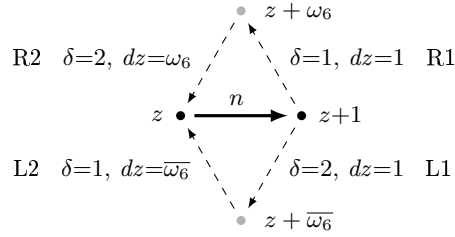The L and R state δ, dz segments are located

Figure 7: adjacent segments

Starting from these states gives, from n, the segment numbers of those segments. If in an adjacent arm then the reversal is $3^k - 1 - output$ for segment rather than point.

Similar initial δ, dz can be used for other segments or points at further locations relative to n. Bigger dz may extend further than just one adjacent arm, going into other of the 6 arms which fill the plane.

Adjacent segment numbers as in figure 7 can be found by digits high to low (instead of low to high). Suppose a segment n has segment numbers s and e on its right. Expansion is a new low digit on n, and on the other segments, is
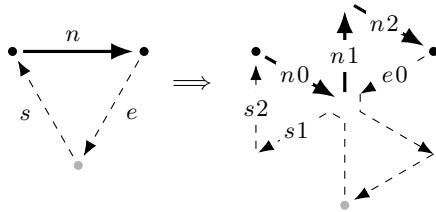
Figure 8: right side segment expansion

For example new low 0 on n means new adjacent segments are s with new low 2 or 1. The new segments for a given low digit of n are

n digit	s'	e'
0	$s2$	$s1$
1	$e0$	$n2$
2	$n1$	$e0$

$$(51)$$

Initial $n=0$ is no digits yet,
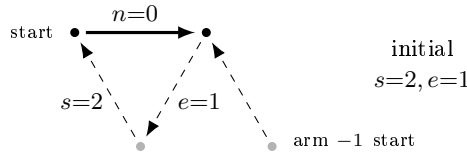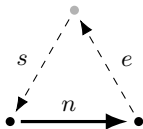
initial
$s=2, e=1$

Initial $s=2, e=1$ are segments in arm -1, on the right, directed towards the origin. Or instead start $s=0$ and an extra high 0 on n to step in (51) to $2,1$ (initial e being unused by this). A segment in arm -1 directed away from the origin is reversal $3^k - 1 - output$. After all digits of n are processed, an adjacent arm is identified by having high initial 1 or 2, above the digits of n.

other visits at the point of a left-turn n are given by one further low digit expansion. A further low 1 digit or 100...00 sequence on all of n, s, e are their middle common point. e is in direction $\delta=1$ and s in direction $\delta=2$. So for

other(n) go high to low, not including the 1 which is lowest non-zero, and copy that 1 and low 0s to s and e.

Similar high to low holds for left side segments, and from them *other* of right turn n. The pattern of new low digits is the same as in (51), but which of n, s, e they take differs.

$$
\begin{array}{l}
\text{left} \\
\text{side} \\
\text{segments}
\end{array}
\qquad
\begin{array}{c|cc}
n \text{ digit} & s' & e' \\
\hline
0 & s2 & n1 \\
1 & n0 & s2 \\
2 & e1 & e0
\end{array}
\tag{52}
$$

In tables (51),(52), some entries copy n for the new s' or e'. This is where the output digits are to be n unchanged. This is somewhere at or above where the low to high of theorem 9 would be in "unch".

Theorem 10. *Differences* $\bigl|n - other(n,\delta)\bigr|$ *which occur are sums of distinct powers of* 3 *with alternating signs,*

$$
\bigl|n - other(n,\delta)\bigr| = 3^{k_0} - 3^{k_1} + 3^{k_2} - \cdots + (-1)^t 3^{k_t} \tag{53}
$$
$$
\text{where } k_0 > k_1 > k_2 > \cdots > k_t \geq 1
$$
$$
= 3, 6, 9, 18, 21, 24, 27, 54, 57, \ldots \qquad {\scriptstyle 3\times \text{A306556}}
$$

Proof. Consider the segment number next around a unit triangle,

$$
\begin{array}{l}
\text{left side} \\[2em]
\text{right side}
\end{array}
\qquad m, n, m
\qquad
\begin{array}{l}
\text{segment } m \\
\text{next around a unit triangle} \\
\text{either left side or right side}
\end{array}
$$
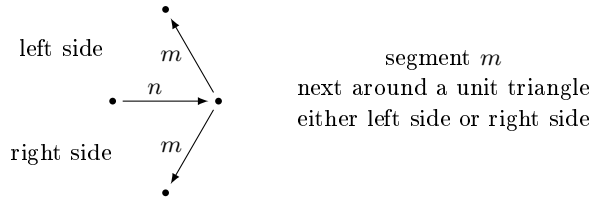
Here "next" around the triangle means n in its segment direction, and the following m either $+120°$ on the left side unit triangle or $-120°$ on the right side. One of these sides is where the curve turns, so that one of them is $m=n+1$. The other side, when there is a segment there, can be a bigger or smaller segment number.

The claim will be that difference $m-n$ is a sum of the following form, and that all such sums with $p_0 < k$ occur in curve k.

$$
m - n = (-1)^t 3^{p_0} \cdots + 3^{p_{t-1}} - 3^{p_{t-1}} + 3^{p_t} \tag{54}
$$
$$
\text{powers } p_0 > \cdots > p_{t-1} > p_t = 0
$$
$$
\text{low term } +3^0 = +1, \text{ signs alternating above there}
$$
$$
\begin{aligned}
= \text{ positives } & 1, 7, 19, 25, 55, 61, 73, 79, \ldots \qquad {\scriptstyle \text{A055246}} \\
\text{negatives } & -2, -8, -20, -26, -56, -62, -74, -80, \ldots \quad {\scriptstyle -\text{A190640}}
\end{aligned}
$$

In curve $k=1$, the only $m-n$ difference occurring is $+1$, which is of this form. Segments n and m expand

Figure 9
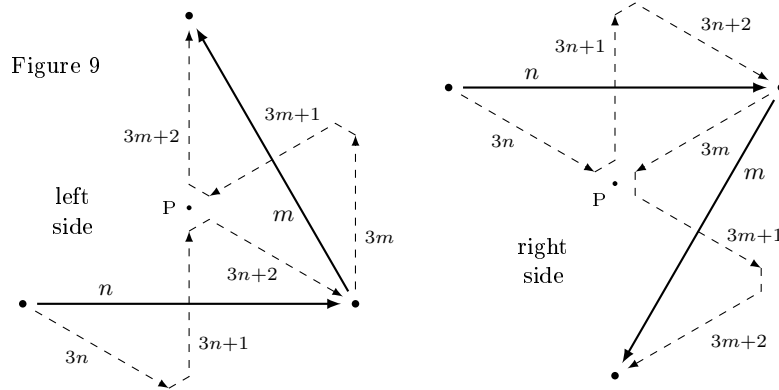
New sides $3n, 3n{+}1$ are difference $+1$, as are $3n{+}1, 3n{+}2$, and likewise $3m, 3m{+}1$ and $3m{+}1, 3m{+}2$.

On the left side, the remaining new difference is $3n{+}2$ to $3m$,

$$3m - (3n{+}2) = 3\,(m{-}n - 1) + 1 \qquad\qquad (55)$$

$3\,(m{-}n - 1)$ is sum (54) with its low $+1$ term removed and the rest raised by a factor of 3 so all p powers increment. Final $+1$ in (55) restores the low $+1$ term.

On the right side, the remaining new difference is

$$(3n{+}2) - (3m{+}1) = -3\,(m{-}n) + 1$$

Factor -3 increments each p power and flips their sign, then $+1$ introduces a new low $+1$ term. Notice this introduces a $p_{t-1}{=}1$ term in the sum, whereas the left side (55) skips such a term.

These power increments either making or skipping $p_{t-1}{=}1$ build all forms (54) in curve $k{+}1$.

For point differences, and in the manner of figure 8, or the new point P here in figure 9, expansion of sides of a unit triangle gives a new double or triple visited point. For segment n, the middle point on the right is $3n{+}1$. Two segment sides expanding to there are point difference

$$(3m{+}1) - (3n{+}1) = 3(m - n)$$

and thus the difference form (54). On further expansion, the point visits are $3\times$ each, so give any low k_t in (53). $\qquad\square$

Second Proof of Theorem 10. Differences can also be calculated from the *other* digit transformation of theorem 9. This shows where the difference powers fall in the *other* digit transformation.

The states of figure 6 loop on digit 0 or digit 2. For $\delta{=}1$ the digit runs which loop and their resulting outputs are net ± 1,

$$(56)$$

	or 1 go to unch	or 2 go to unch				or 1 go to unch	or 0 go to unch	
high	L1	L2	low		high	R2	R1	low
n ...	$0\,22...22$	$100...00$	$10...0$		n ...	$200...00$	$1\,22...22$	$20...0$
n_1 ...	$100...00$	$0\,22...22$	$10...0$		n_1 ...	$1\,22...22$	$200...00$	$20...0$
	$+1$	-1	$+1$			$+1$	-1	$+1$

For $\delta=2$ the runs are the same, but starting opposite lowest L1 and R2. The states alternate and hence the signs for the increment. □

$\delta=1$ can go to low run either R1 or L2, giving it either $+1$ or -1 lowest term. $\delta=2$ low run R2 or L1 likewise. So $\delta=1$ and $\delta=2$ give the same set of differences.

$turn=1$ goes to low R1,L1 always, but with an odd number of runs its highest can be -1 too and the absolute value flips all signs so that again $turn=1$ or $turn=-1$ are the same set of differences.

Triangle side differences (54) in ternary are

$$m-n = \text{ternary} \quad \boxed{\underbrace{\,...\,0\,\text{or}\,2\,...\,}_{\geq 0 \text{ digits}} \mid 1\,} \qquad \text{triangle next side difference}$$

This is each pair of terms $3^x - 3^y$ giving a run of ternary $022...22$, and final $+1$ term a single 1 digit. Negatives $m-n < 0$ this way too, with infinite high 2s for a "3's complement" negative.

Or negatives written with a $-$ sign are a low 2 digit instead

$$m-n = \text{ternary} \quad -\boxed{\underbrace{\,...\,0\,\text{or}\,2\,...\,}_{\geq 0 \text{ digits}} \mid 2\,} \qquad \text{triangle next side difference}$$

other differences (53) in ternary are at least one low 0 digit, then an arbitrary digit, then digits 0 or 2 above. This is again since each pair $3^{k_0} - 3^{k_1}$ is a run $022...22$ and if the lowest term is t even then it is an unpaired $+3^{k_t}$ so lowest non-zero digit can be 1.

$$Opred(p) = \begin{cases} 1 & \text{if } p = \big|n - other(n,\delta)\big| \text{ for some } n \\ 0 & \text{if not} \end{cases}$$

$$= p \text{ ternary digits } 0,2 \text{ with low } 0, \text{ and lowest non-zero can be 1}$$
$$\text{ternary } 10, 20, 100, 200, 210, 220, 1000, 2000, 2010, ...$$

$$\boxed{\underbrace{\,...\,0\,\text{or}\,2\,...\,}_{\geq 0 \text{ digits}} \mid \text{any} \mid \underbrace{0...0}_{\geq 1 \text{ digits}}} \tag{57}$$

The unpaired $+3^{k_t}$ can be taken as $+1$ of a 2220 low run, so ternary 0,2s, or 0,2s $+$ 1.

$$Opred(p) = p \geq 3 \text{ and } p \equiv 0 \bmod 3 \text{ and } \big(Cpred(p/3) \text{ or } Cpred(p/3-1)\big)$$

$$Cpred(n) = \text{ternary digits } 0,2 \text{ only} \tag{58}$$
$$= 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, \ldots \qquad \text{\scriptsize A088917}$$
$$= 1 \text{ at } \quad n = 0, 2, 6, 8, 18, 20, 24, 26, 54, 56, \ldots \qquad \text{\scriptsize A005823}$$
$$= 0 \text{ at } \quad n = 1, 3, 4, 5, 7, 9, 10, 11, 12, 13, \ldots \qquad \text{\scriptsize A081606}$$

1.6 Segments in Direction

Theorem 11. *With the curve starting in direction $d=0$, the number of segments of terdragon k in each direction $dir(n) \equiv d = 0, 1, 2 \bmod 3$ is*

$$S(k,d) = 3^{k-1} + s(k-4d).3^{\left\lfloor \frac{k-1}{2} \right\rfloor} \tag{59}$$

$$= \tfrac{1}{3}\left(3^k + \overline{\omega_3}^d b^k + \omega_3^d \overline{b}^k \right) \tag{60}$$

$$= \tfrac{1}{3}\left(\left| b^k + \omega_3^d \right|^2 - 1 \right)$$

$$s(j) = [2, 1, 1, 0, -1, -1, -2, -1, -1, 0, 1, 1] \qquad \text{\scriptsize } s(j-1) = \text{A214438}$$

$$S(k,0) = 1, 2, 4, \ \ 9, 24, 72, 225, 702, 2160, 6561, \ldots \qquad \text{\scriptsize A092236}$$
$$S(k,1) = 0, 1, 4, 12, 33, 90, 252, 729, 2160, 6480, \ldots \qquad \text{\scriptsize A135254}$$
$$S(k,2) = 0, 0, 1, \ \ 6, 24, 81, 252, 756, 2241, 6642, \ldots \qquad \text{\scriptsize A133474}$$

$$S(k,0) = 4 \quad \longrightarrow \qquad k=2$$
$$S(k,1) = 4 \qquad \qquad \text{segments in direction}$$
$$S(k,2) = 1 \qquad \qquad \text{relative to start}$$
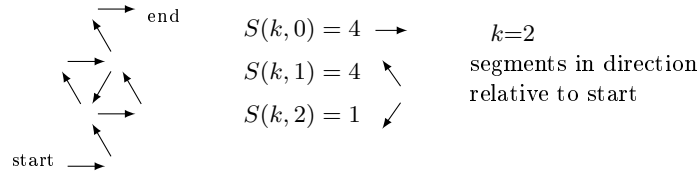
Proof. When the curve replicates the new sub-part 2 is in the same direction as the preceding level, so the segment counts double. The new sub-part 1 rotates $+120°$. The rotation means those segments in direction $d=2$ move to direction $d=0$. Similarly the other directions. So mutual recurrences

$$S(k+1, 0) = 2\,S(k,0) + S(k,2) \tag{61}$$
$$S(k+1, 1) = 2\,S(k,1) + S(k,0) \tag{62}$$
$$S(k+1, 2) = 2\,S(k,2) + S(k,1) \tag{63}$$

Using (63) for $S(k,1)$ and substituting into (62) then using (61) for $S(k,2)$ and substituting again gives the following recurrence for $d=0$. By symmetry the same for $d=1$ and $d=2$.

$$S(k+3, d) = 6\,S(k+2,d) - 12\,S(k+1,d) + 9\,S(k,d)$$

The characteristic polynomial is

$$x^3 - 6x^2 + 12x - 9 = (x-3)(x-b)(x-\overline{b})$$

So $S(k,d)$ has a power form $X.3^k + Yb^k + Z\overline{b}^k$. From the initial values the coefficients are per (60).

The imaginary parts of the conjugate powers cancel out. Their real part gives factor $s(j)$ on the half power $3^{\lfloor (k-1)/2 \rfloor}$ for (59). $\qquad\square$

There are 3^k segments in total. The selector function s has

$$s(j) + s(j{+}4) + s(j{+}8) \;=\; 0 \qquad \text{for all } j$$

so the half powers cancel out leaving

$$S(k,0) + S(k,1) + S(k,2) \;=\; 3^k$$

$S(k,d)$ can also be calculated by dir from (30). The segments in direction $d{=}0$ are those n which have $\mathrm{dir}(n) = 0, 3, 6,$ etc. This means count $0, 3, 6,$ etc many 1-digits among k ternary digits of n. The number of arrangements of those 1-digit positions is a binomial coefficient in k and then the remaining digits are each 0 or 2. So

$$S(k,0) = \quad 2^k \binom{k}{0} + 2^{k-3}\binom{k}{3} + 2^{k-6}\binom{k}{6} + \cdots$$
$$S(k,1) = 2^{k-1}\binom{k}{1} + 2^{k-4}\binom{k}{4} + 2^{k-7}\binom{k}{7} + \cdots$$
$$S(k,2) = 2^{k-2}\binom{k}{2} + 2^{k-5}\binom{k}{5} + 2^{k-8}\binom{k}{8} + \cdots$$
$$S(k,d) = \sum_{j=d,d+3,\dots} 2^{k-j}\binom{k}{j}$$

These forms are among the power-weighted binomial sums considered by Justus[7] as a generalization of the binomial sums of Cournot and Ramus (see *Lines* ahead in section 5.1 for the latter).

$S(k,0)$ was also a proposed International Mathematical Olympiad problem [6]. In that problem dividing out factors of 3 is ternary lowest non-0 which is the terdragon turn sequence. Summing is the direction $dir(n)$. Counting sums divisible by 3 is segments in direction $d{=}0$.

Theorem 12. *The number of segments in each direction* $dir(n) \equiv d = 0, 1, 2$ $\bmod\ 3$ *relative to the middle segment are*

$$SM(k,d) = S(k, d{+}k)$$
$$= 3^{k-1} + sm(k,d).3^{\left\lfloor \frac{k-1}{2} \right\rfloor}$$
$$= \tfrac{1}{3}\left(3^k + \omega_3^d (i\sqrt{3})^k + \overline{\omega_3^d (i\sqrt{3})^k} \right)$$
$$= \tfrac{1}{3}\left(\left| (i\sqrt{3})^k + \overline{\omega_3}^d \right|^2 - 1 \right)$$
$$sm(k,0) = [\ \ 2, \ \ 0, \ -2, \ \ \ 0]$$
$$sm(k,1) = [-1, -1, \ \ 1, \ \ \ 1]$$
$$sm(k,2) = [-1, \ \ 1, \ \ 1, -1] \quad = sm(k{+}1,1)$$
$$SM(k,0) = 1,1,1,\ \ 9, 33, 81, 225, 729, 2241, 6561, \dots \qquad \text{\small A101990}$$
$$SM(k,1) = 0,0,4, 12, 24, 72, 252, 756, 2160, 6480, \dots \qquad \text{\small A318610}$$
$$SM(k,2) = 0,2,4, \ \ 6, 24, 90, 252, 702, 2160, 6642, \dots \qquad \text{\small A318609}$$

$$SM(k,0) = 1 \longrightarrow$$

$$SM(k,1) = 4 \quad \searangle$$

$$SM(k,2) = 4 \quad \nearrow$$

$k{=}2$

segments in direction
relative to middle
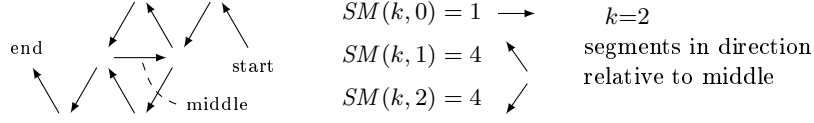
Proof. The middle segment is in direction $k \bmod 3$ so $SM(k,d) = S(k,d+k)$. In $S(k,d+k)$ the factor $s(k - 4(d+k)) = s(-3k - 4d)$ gives $sm(k,d)$. The $-3k \bmod 12$ becomes $k \bmod 4$ for $sm(k,d)$. $\qquad \square$

The periodic factors $sm(k,d)$ can be expressed variously as powers of -1. For example $sm(k,2) = (-1)^{\lfloor (k-1)/2 \rfloor}$ gives

$$SM(k,2) = 3^{k-1} + (-3)^{\left\lfloor \frac{k-1}{2} \right\rfloor}$$

Theorem 13. *With the curve starting in direction d=0, the number of the first n segments of the terdragon curve in each direction $d = 0, 1, 2$ is*

$$SN(n,d) = \tfrac{1}{3}\Big(n + 2\operatorname{Re}\overline{\omega_3}^d point(n)\Big) \tag{64}$$

$$SN(n,0) = 0, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 4, 5, \ldots$$
$$SN(n,1) = 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 6, \ldots$$
$$SN(n,2) = 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3, \ldots$$

Proof. There are total n segments,

$$SN(n,0) + SN(n,1) + SN(n,2) = n \tag{65}$$

The real part of segments in direction 0 is $+1$ each. The real part of segments in directions 1 and 2 are $-\tfrac{1}{2}$ each. The total of these is net horizontal position *point*,

$$SN(n,0) - \tfrac{1}{2}SN(n,1) - \tfrac{1}{2}SN(n,2) = \operatorname{Re} point(n) \tag{66}$$

(65)+2×(66) cancels the direction 1 and 2 terms, giving the theorem for $d{=}0$. The other directions have corresponding forms after rotating by $\overline{\omega_3}$ or $\overline{\omega_3}^2$ so the desired d is the real part,

$$SN(n,1) - \tfrac{1}{2}SN(n,0) - \tfrac{1}{2}SN(n,2) = \operatorname{Re} \overline{\omega_3} \, point(n)$$
$$SN(n,2) - \tfrac{1}{2}SN(n,0) - \tfrac{1}{2}SN(n,1) = \operatorname{Re} \overline{\omega_3}^2 \, point(n)$$

Each combined with (65) gives the general case (64). $\qquad \square$

2 Boundary

2.1 Boundary Triangles

A unit triangle can be placed on each boundary segment of the curve.

$k=2$
boundary triangles
$BT_2 = 2^{2+1} = 8$
$RT_2 = BT_2/2 = 2^2 = 4$

These boundary triangles are similar in style to the boundary squares which Daykin and Tucker[5] count on the Heighway/Harter dragon curve.

Theorem 14. *The number of triangles on the boundary of terdragon curve k is*

$$BT_k = 2^{k+1} \qquad \text{boundary triangles}$$

The curve is symmetric on each side so half on one side

$$RT_k = BT_k/2 = 2^k \qquad \text{one-side boundary triangles} \qquad (67)$$

The number of triangles in a "V" pair of curves is the same as "R"

$$VT_k = RT_k \qquad \text{"V" part boundary triangles}$$

Proof. The "V" part boundary is between two level k curves at a 60° angle as in the following diagram. A level k curve can be drawn across the V endpoints to make a triangle.
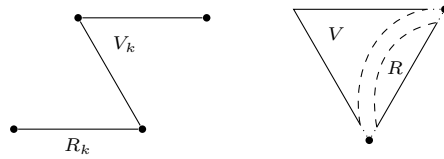
Figure 10:
R,V boundary parts
and triangle

Per plane filling theorem 2, all segments within the triangle are traversed precisely once so the unit triangles on the R boundary and those on the V boundary are identical $VT_k = RT_k$.

The left diagram shows that R_{k+1} comprises an R_k and a V_k. They meet as the outside of a 60° angle so do not have any boundary triangles in common.

$$RT_{k+1} = RT_k + VT_k = 2\,RT_k \qquad (68)$$

Starting from $RT_0 = 1$ gives $RT_k = 2^k$. $\qquad \square$

Each boundary triangle touches either 1 or 2 boundary segments. The two can be counted separately. The total is BT_k,

$$BT_k = BT1_k + BT2_k$$

Theorem 15. *The triangles on the terdragon boundary touch alternately 1 and 2 sides. For $k \geq 1$ there are half 1-side and half 2-side.*

$$BT1_k = \begin{cases} 2 & \text{if } k = 0 \\ BT_k/2 = 2^k & \text{if } k \geq 1 \end{cases} \qquad \text{1-side triangles} \qquad (69)$$

$$= 2, 2, 4, 8, 16, \ldots$$

$$BT2_k = \begin{cases} 0 & \text{if } k = 0 \\ BT_k/2 = 2^k & \text{if } k \geq 1 \end{cases} \qquad \textit{2-side triangles} \qquad (70)$$

$$= 0, 2, 4, 8, 16, 32, \ldots \qquad \text{A155559}$$

The curve is symmetric on each side so one side

$$RT1_k = \tfrac{1}{2} BT1_k = 1, 1, 2, 4, 8, 16, \ldots \qquad \text{A011782}$$

$$RT2_k = \tfrac{1}{2} BT2_k = 0, 1, 2, 4, 8, 16, \ldots \qquad \text{A131577}$$

The 1s and 2s in a "V" part are opposite to an "R"

$$VT1_k = RT2_k \qquad \textit{opposites } 1 \leftrightarrow 2$$
$$VT2_k = RT1_k$$

Proof. For k=1 the R boundary is two triangles, a 1-side and a 2-side, so they alternate.

Per the triangle of figure 10, the V boundary is the opposite side of an R, so each 1-side triangle of R is a 2-side triangle of V and vice-versa. These V triangles are in reverse order to R, so they are 1-side and 2-side alternately the same as R.

Level $k+1$ is an R_k followed by V_k and so alternates. $\qquad \square$

2.2 Boundary Segments

The boundary of the curve can be measured by unit line segments around the outside of the curve.

$k=2$ boundary
$B_2 = 12$
$R_2 = B_2/2 = 6$
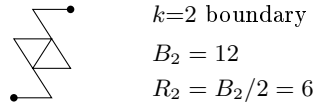
The boundary on one side is counted from start to end. The full boundary is counted by continuing around to the origin again.

The ends of the curve are isolated line segments (see theorem 21 for more on this). For the full boundary both the left and right sides of those ends are counted.

Theorem 16. *The boundary length of the terdragon curve after k iterations is*

$$B_k = \begin{cases} 2 & \text{if } k = 0 \\ 3.2^k & \text{if } k \geq 1 \end{cases} \qquad \textit{boundary} \qquad (71)$$

$$= 2, 6, 12, 24, 48, 96, \ldots$$

The curve is symmetric on its two sides so one side

$$R_k = B_k/2 = \begin{cases} 1 & \text{if } k = 0 \\ 3.2^{k-1} & \text{if } k \geq 1 \end{cases} \qquad \textit{right boundary} \qquad (72)$$

$$= 1, 3, 6, 12, 24, 48, \ldots \qquad \text{A003945}$$

The length in a "V" part is

$$V_k = \begin{cases} 2 & \text{if } k = 0 \\ 3.2^{k-1} & \text{if } k \geq 1 \end{cases} \qquad \text{``V'' boundary} \qquad (73)$$

$$= 2, 3, 6, 12, 24, 48, \ldots \qquad \text{\small A042950}$$

Proof. The boundary segments are found by counting the sides of the 1-side and 2-side boundary triangles (69),(70)

$$B_k = BT1_k + 2\,BT2_k$$
$$R_k = RT1_k + 2\,RT2_k$$
$$V_k = VT1_k + 2\,VT2_k \qquad \square$$

Second Proof of Theorem 16. R and V parts expand as
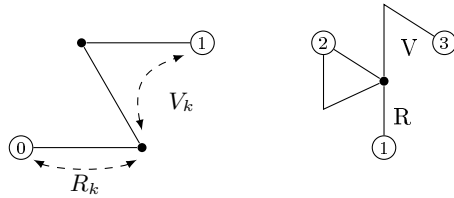
Figure 11:
R and V expansion,
initial segments
$R_0 = 1$
$V_0 = 2$

giving mutual recurrences

$$R_{k+1} = R_k + V_k \qquad\qquad (74)$$
$$V_{k+1} = R_k + V_k \qquad\qquad (75)$$

which are the same right-hand sides so $R_{k+1} = V_{k+1}$ and hence

$$R_{k+2} = 2R_{k+1} \qquad\qquad k \geq 0$$
$$V_{k+2} = 2V_{k+1} \qquad\qquad k \geq 0 \qquad \square$$

Recurrence (74) is the equivalent of (68) for the boundary triangles. (75) also holds for the boundary triangles per the expansion in figure 11, but doesn't show as clearly that the shape is opposite to R the way the triangle in figure 10 does.

$$VT_{k+1} = RT_k + VT_k$$

2.3   Boundary Segment Numbers

right boundary
segment numbers
$0, 1, 2, 3, 7, 8, \ldots$
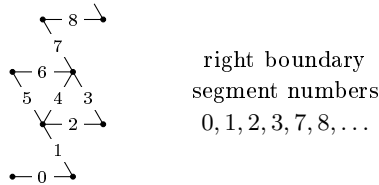
Theorem 17. *Number the segments of the terdragon curve starting n=0 for the first segment. The right boundary segments are characterized by*

$$Rpred(n) = \begin{cases} 1 & \textit{if } n \textit{ in ternary has no digit pair } 11, 12 \textit{ or } 20 \\ 0 & \textit{if } n \textit{ in ternary does have} \end{cases} \tag{76}$$

$$= 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, \ldots$$

Proof. Take the boundary in three types of part
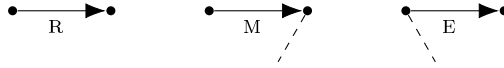
R has both endpoints on the boundary and is the right side of the full curve. M has an adjacent sub-curve at its end and so only some segments at its start are on the boundary. E has a sub-curve at its start.

Let R_k, M_k, E_k be the segment numbers which are on the boundary in the respective configurations at level k. These numbers are in the range 0 to $3^k - 1$ and hence can be written with k many ternary digits. The initial sets are a single 0 in each so $R_0 = M_0 = E_0 = 0$ corresponding to a single line segment. These zeros are understood as 0 many digits.

The curve expands as
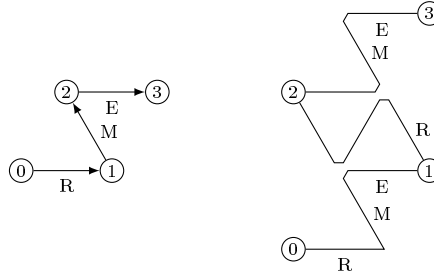
The R segment 0–1 expands to sub-parts 0.R, 1.M, 2.E. The number 0, 1, 2 is the high ternary digit on top of the digits of the subsection. Treating each section this way gives

$$\begin{aligned} R_k &= \quad 0.R_{k-1}, \quad 1.M_{k-1}, \quad 2.E_{k-1} \\ M_k &= \quad 0.R_{k-1} \\ E_k &= \qquad\qquad\quad\;\; 1.M_{k-1}, \quad 2.E_{k-1} \end{aligned} \tag{77}$$

Taking ternary digits from high to low, this expansion is a state machine. In state R, any digit is permitted and switch to state R, M, E according to that digit. In state M, only 0 is allowed and switch to state R. In state E, either 1 or 2 is allowed and switch to state M or E.
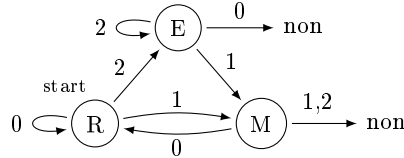
Figure 12:
$Rpred(n)$ state machine,
ternary high to low

Digit 0, when permitted, always goes to state R. Digit 1 always goes to state M. Digit 2 always goes to state E. This means the state at any position is given by the preceding higher digit. A state transition permitted or not is therefore a

digit pair permitted or not. So 11, 12, 20 not permitted. Possible runs of digits in n follow from this. □

The lengths of sub-parts M and E are

$$M_k = \begin{cases} 1 & \text{if } k = 0, 1 \\ 3.2^{k-2} & \text{if } k \geq 2 \end{cases} \qquad \text{"M" part boundary length}$$

$$= 1, 1, 3, 6, 12, 24, 48, 96, \ldots$$

$$E_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ 3.2^{k-2} & \text{if } k \geq 2 \end{cases} \qquad \text{"E" part boundary length}$$

$$= 1, 2, 3, 6, 12, 24, 48, 96, \ldots$$

This is by writing the expansions as recurrences, $M_0 = E_0 = 1$, and substituting

$$R_{k+1} = R_k + M_k + E_k$$
$$M_{k+1} = R_k$$
$$E_{k+1} = \qquad M_k + E_k$$

M and E together are the V part $M_k + E_k = V_k$.

The states also give a count of how many sides the triangle on the right of segment n has. This is 1 or 2 for a boundary segment, or 3 for a non-boundary.

$$Rsides(n) = \begin{cases} 1 & \text{if } Rpred \text{ state R} \\ 2 & \text{if } Rpred \text{ state M or E} \\ 3 & \text{if } Rpred \text{ state "non"} \end{cases} \qquad \text{right triangle sides} \qquad (78)$$

$$= 3 - [2, 1, 1].Rpred(n) \qquad (79)$$

$$= 1, 2, 2, 1, 3, 3, 3, 2, 2, 1, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 1, \ldots$$

For (79), low 0 on n goes to state R and low 1 or 2 to states M,E, so $n \bmod 3$ determines respective factor 2 or 1 on $Rpred$ to reduce from 3 sides.

M,E always occur in pairs, since the expansions at (77) always produce them in pairs. The 0-digit there is an R or non which separate such pairs. So each pair $2, 2$ in $Rsides$ is M, E and $Rsides$ can be written as a morphism expansion

$$Rsides \quad 1 \to 1, 2, 2 \quad 2, 2 \to 1, 3, 3, 3, 2, 2 \quad 3 \to 3, 3, 3 \qquad \text{starting 1}$$

Total $Rsides$ in a level is 1 for each $RT1$ triangle, 2 for each of the 2 segments of $RT2$, and 3 for each of the 3 segments of enclosed AR (ahead in section 3),

$$\sum_{n=0}^{3^k - 1} Rsides(n) = RT1_k + 2.2.RT2_k + 3.3.AR_k = AR_{k+2} \qquad (80)$$

The geometric interpretation of total AR_{k+2} is that each respective $1, 2, 3$ side triangle after 2 expansions has $1, 4, 9$ unit triangles enclosed on the right, which are the coefficients 1, 2.2, 3.3 in (80).

A state machine for *Rpred* on ternary digits low to high follows by usual state machine manipulations reversing the high to low form, or just from the allowed and disallowed digit pairs. State s0 is when the digit immediately below is a 0. State s12 is when the digit immediately below is 1 or 2.

Rpred(n) state machine, ternary low to high
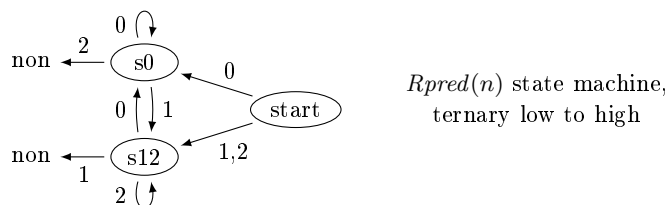
Theorem 18. *Let* $Rn(m)$ *be the m'th right boundary segment number, for* $m \geq 0$.

$$Rn(m) = 0, 1, 2, \quad 3, 7, 8, \quad 9, 10, 11, \quad 21, 25, 26, \quad 27, 28, \ldots$$

$$\text{ternary} = 0, 1, 2, 10, 21, 22, 100, 101, 102, 210, 221, 222, 1000, 1001, \ldots$$

Rn can be calculated by writing m in mixed radix with low ternary digit and then binary above. For $m \leq 2$*, write a single ternary digit.*

$$Rn(m) = change\ each\ "1, non\text{-}zero"\ to\ "2, non\text{-}zero"$$
$$and\ interpret\ the\ result\ as\ ternary$$

The effect of the change rule is that each maximal run $1, 1, \ldots, 1, \text{NZ}$ becomes $2, 2, \ldots, 2, \text{NZ}$, where NZ is a non-zero digit. If NZ is within the binary then it is 1. If NZ is the low ternary digit then it can be 1 or 2. In both cases its value is unchanged. So ternary runs of digit 2 ending 1, except least significant digit can be either 1 or 2.

$$Rn(m) = 222\ldots221\ 00\ldots00\ 222\ldots221\ 00\ldots00\ 222\ldots222 \quad \text{ternary}$$

Proof. The allowed digit pairs in Rn are those not disallowed in theorem 17,

10	00
21	01
22	02

In a pair with a given low digit, there are two choices for its high digit. For example 0 can have above it either 1 or 0 (the first row of the table). Start from low digit any $0, 1, 2$. Above it take each of the two choices in the table, which steps through all and only allowed pairs. The highest digit must be non-zero and so the top-most pair is a single choice from the high 1-bit of the mixed base representation. $\qquad \square$

A generating function for Rn can be formed by following the mixed radix conversion. The generating function has periodic terms of the usual form for

each digit, but with longer periods so as to apply the $1{\to}2$ change rule. The low ternary digit is a repeating $0, 1, 2$,

$$g012(x) = \frac{0 + 1x + 2x^2}{1 - x^3} \quad = 0 + 1x + 2x^2 + 0x^3 + 1x^4 + 2x^5 + \cdots$$

For base conversion, start from a generating function with coefficient 1 when bit k of n is a 1. This is repeating blocks of period 2.2^k

$$gBit_k(x) = \frac{x^{2^k} + x^{2^k+1} + \cdots + x^{2.2^k-1}}{1 - x^{2.2^k}} = \frac{x^{2^k} - x^{2.2^k}}{(1-x)(1-x^{2.2^k})}$$

The change rule $1{\to}2$ in the binary part is $+1$ where a 1-bit below. This is periodic blocks for bit position $k \geq 1$ (so there is a bit below k),

$$gBit11_k(x) = \frac{x^{\frac{3}{2}.2^k} + \cdots + x^{2.2^k-1}}{1 - x^{2.2^k}} = \frac{x^{\frac{3}{2}.2^k} - x^{2.2^k}}{(1-x)(1-x^{2.2^k})}$$

These bit forms are raised above the low ternary by substituting x^3 and multiplying $1+x+x^2$ to apply at each $n \bmod 3$. Bit position $k{=}0$ has its change $1{\to}2$ when $n \equiv 4, 5 \bmod 6$, and $k \geq 1$ has it by $gBit11$.

$$gRn(x) = g012(x) + 3\left(1 + x + x^2\right) gBit_0(x^3) + \frac{3\left(x^4 + x^5\right)}{1 - x^6}$$
$$+ 3\left(1 + x + x^2\right) \sum_{k=1}^{\infty} 3^k \left(gBit_k(x^3) + gBit11_k(x^3)\right)$$

$$= \frac{x+2x^2}{1 - x^3} + 3\frac{x^3+2x^4+2x^5}{1 - x^6} + 9 \sum_{k=0}^{\infty} 3^k \frac{x^{6.2^k} + x^{9.2^k} - 2x^{12.2^k}}{(1-x)(1 - x^{12.2^k})} \qquad (81)$$

At (81), each term is a successive ternary digit (low to high) added to the coefficients. For $Rn(m) < 3^l$, which is $m < 3.2^{l-1}$, it suffices to take the first l terms (so the sum part up to $k = l{-}3$ inclusive).

In the sum numerator, $-2x^{12.2^k}$ is the top end of two ranges 6 to 12 and 9 to 12. This is 6 to 9 digit 1 and then where they overlap 9 to 12 is digit 2.

2.4 Left Boundary Segment Numbers

Some of the left boundary in level k is enclosed by level $k{+}1$ and so is no longer on the boundary. (Unlike the right boundary which is never enclosed and so its level k boundary segment numbers are a prefix of the level $k{+}1$ boundary segment numbers.)

Three forms of left boundary segment numbers can be considered

- segments on boundary for particular level k
- segments on boundary for every level, so the curve continued infinitely
- segments on boundary for some level, a union of all left boundaries

left boundary   left boundary   left boundary
level $k = 5$ infinite curve   all levels
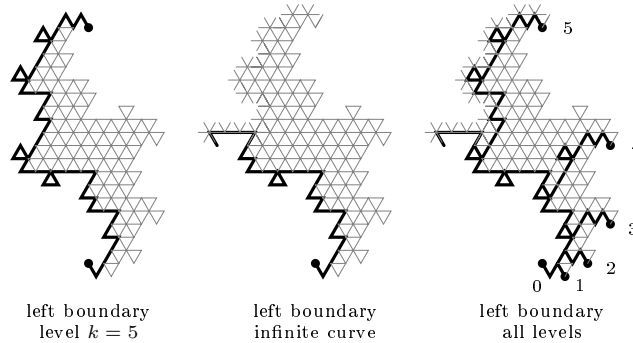
Theorem 19. *Number the segments of the terdragon curve starting from* 0. *The left boundary segments are those which written in ternary do not have any digit pair* 02, 10 *or* 11.

Within curve k, *pad to* k *many digits with high* 0 *digits as necessary. This means the highest non-zero cannot be* 2 *except when that* 2 *is the most significant digit (position* $k{-}1$).

$$Lpred_k(n) = no\ 02, 10, 11\ within\ k\ ternary\ digits\ of\ n$$
$$= Rpred(3^k{-}1 - n)$$
$$= 1\quad for\ k{=}0$$
$$1, 1, 1\quad for\ k{=}1$$
$$1, 1, 0, 0, 0, 1, 1, 1, 1\quad for\ k{=}2$$

For the curve continued infinitely, write infinitely many digits, with high 0 *digits. One high* 0 *suffices for the digits rule and means the most significant non-zero digit cannot be* 2.

$$Lpred_\infty(n) = no\ 02, 10, 11\ in\ n\ with\ high\ 0$$
$$= Lpred_k(n)\quad for\ k\ with\ 3^k > 3n$$
$$= 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, \ldots$$

$$1\ at\ n = \begin{matrix} decimal \\ ternary \end{matrix} \begin{matrix} 0,1,\ \ 5,\quad 15,16,17,\qquad\qquad 45,46,50,51,52,53,... \\ 0,1,\ \ 12,\ \ 120,121,122,\ \ 1200,1201,1212,1220,1221,1222,... \end{matrix}$$

For the union of all left boundary segments, do not write any high 0 *digits.*

$$Lpred_{all}(n) = any\ Lpred_k(n),\ least\ k\ with\ 3^k > n\ suffices$$
$$= 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, \ldots$$

$$1\ at\ n = \begin{matrix} decimal \\ ternary \end{matrix} \begin{matrix} 0,1,2,\quad 5,6,7,8,\qquad\quad 15,16,17,18,19,23,24,25,26,... \\ 0,1,2,\ \ 12,20,21,22,\ \ 120,121,122,200,201,212,220,221,222,... \end{matrix}$$

Proof. The curve is symmetric on its left and right sides, so the left boundary segment numbers are the right segment numbers but numbered in reverse $3^k{-}1{-}$ n. This means digits 0,1,2 become 2,1,0. The digit pairs to exclude are the digit reversals of those in the right boundary pairs.

For the curve to level k the reversal is from endpoint $3^k{-}1$ and therefore applied to k digits.

For the curve extended infinitely the sub-part 2 is enclosed by the continuing curve, so the high digit cannot be 2, only 1.

For the union of all levels the reversal is made from any endpoint $3^k-1 \geq n$. The endpoint giving no high 0 digits is the minimum disallowing. $\square$

The number of sides on the triangle to the left of segment n follows in a similar way as the reversal of $Rsides$ within k.

$$
\begin{aligned}
Lsides_k(n) &= Rsides(3^k-1-n) \qquad \text{left triangle sides} \\
&= 1 \quad \text{for } k{=}0 \\
&\quad 2,2,1 \quad \text{for } k{=}1 \\
&\quad 2,2,3,3,3,1,2,2,1 \quad \text{for } k{=}2 \\
Lsides_\infty(n) &= Lsides_k(n) \quad \text{for } 3^k > 3n \\
&= 2,2,3,3,3,1,3,3,3,3,3,3,3,3,3,2,2,1,3,3,3,3,\ldots
\end{aligned}
$$

Theorem 20. *Left boundary segment number $Ln(m)$ for $m \geq 0$ can be calculated as follows. Write m in mixed radix with a ternary low digit then binary above.*

For curve level k, write a total k many digits.

$$
m = \begin{array}{|c|c|c|c|c|} \hline \overset{\text{binary}}{0\,\text{or}\,1} & \overset{\text{binary}}{0\,\text{or}\,1} & \ldots & \overset{\text{binary}}{0\,\text{or}\,1} & \overset{\text{ternary}}{0,1,2} \\ \hline \underset{\text{high}}{} & & & & \underset{\text{low}}{} \end{array} \qquad k \text{ digits}
$$

For the curve continuing infinitely, write an extra 0 at the high end.

$$
m = \begin{array}{|c|c|c|c|c|c|} \hline \overset{\text{binary}}{0} & \overset{\text{binary}}{1} & 0\,\text{or}\,1 & \ldots & \overset{\text{binary}}{0\,\text{or}\,1} & \overset{\text{ternary}}{0,1,2} \\ \hline \underset{\text{high}}{} & & & & & \underset{\text{low}}{} \end{array}
$$

For the union of all levels, for $m \leq 2$ take $Ln(m){=}m$. For $m{=}3$ take $Ln(3) {=} 5$. For $m \geq 4$ write $m{+}2$ in mixed radix and then change the high two bits $10 \rightarrow 1$ (a single 1 bit) or $11 \rightarrow 01$.

$$
m{+}2 = \begin{array}{|c|c|c|c|c|} \hline \overset{\text{binary}}{1\,\text{or}\,01} & \overset{\text{binary}}{0\,\text{or}\,1} & \ldots & \overset{\text{binary}}{0\,\text{or}\,1} & \overset{\text{ternary}}{0,1,2} \\ \hline \underset{\text{high}}{} & & & & \underset{\text{low}}{} \end{array}
$$

Take each binary digit from low to high and transform according to the digit below it and the following table. The digit below is reckoned after any transformation in that lower position.

bit	digit below	change bit to
0	0	0
0	1	0
0	2	1
1	0	2
1	1	2
1	2	2

The resulting digits interpreted as ternary are $Ln(m)$.

For example for the curve continued infinitely, $m=8$ is mixed radix 0102. The low 0 bit has digit 2 below it so in the table 0,2 (third row) is bit change to a digit 1, giving 0112. Then the next higher position is a 1 bit and the digit 1 below so per table change that bit to digit 2 giving 0212. Finally the high 0 bit has a 2 below so per table change that bit to digit 1 for final ternary 1212 = decimal 50. This is the $m=8$ sample value shown in theorem 19 (the first value as $m=0$).

Proof. The allowed digit pairs for the left boundary are those not disallowed in theorem 19. The transformations give all and only these pairs.

$$
\begin{array}{cc}
20 & 00 \\
21 & 01 \\
22 & 12
\end{array}
$$

For the curve continued infinitely, the extra high 0 bit ensures the high ternary digit is not 2 since first three rows of the table map that bit to digit 0 or 1.

For the union of all levels, the mixed radix forms are to be those of all k. When there is one high 0 bit it becomes either 0 or 1 per the bit 0 column of the table. Any further high 0 bits would remain as 0, per the first two rows of the bit 0 column. Therefore the values resulting from two or more high 0s are the same as from a single high 0. So it suffices to take mixed forms with and without a single extra 0 bit. The rule in the theorem uses the second highest bit to choose with or without. The mixed radix is formed on $m+2$ since there are just 4 initial values 0,1,2,5 before beginning this mixed form. □

Theorem 21. *The only terdragon level k segments which are on both the left and right boundary are the first two and last two segments.*
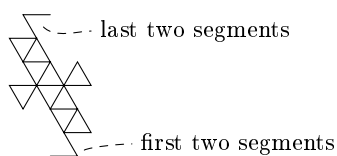
Proof. For $k=0$ the single segment is on the left and right boundary.

For $k=1$ the three segments $0, 1, 2$ are on the left and right boundary.

For $k \geq 2$, combining digit pair conditions of theorem 17 and theorem 19 gives permitted digit pairs only $00, 01, 21, 22$ for segment on both left and right boundaries. The only numbers which can be made with these pairs are

$$
\left. \begin{array}{c} 0 \\ 1 \end{array} \right\} \text{first two segments}
$$
$$
\left. \begin{array}{c} 222\ldots221 \\ 222\ldots222 \end{array} \right\} \text{last two segments}
$$

These are the first two and last two segments. For two digits they are simply the four permitted pairs. □

2.5 Boundary Turn Sequence

The right boundary of the terdragon at each point turns either $+120°$ (left), $-120°$ (right), or goes straight ahead. Number the right boundary points starting from $m=0$ so the first turn is at $m=1$.

The following diagram illustrates the first few boundary turns.

$$+1 = Rturn(6) \quad m=6, h=4$$
$$-1 = Rturn(5) \quad m=5$$
$$0 = Rturn(4) \quad m=4, h=3$$
$$+1 = Rturn(3) \quad m=3, h=2$$
$$-1 = Rturn(2) \quad m=2$$
$$+1 = Rturn(1) \quad m=1, h=1$$
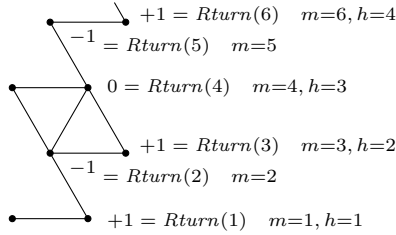
Theorem 22. *The terdragon right boundary turn sequence is the Heighway/ Harter dragon curve with -1 inserted at every third position starting from the second.*

$$Rturn(m) = \begin{cases} -1 \text{ (right)} & \text{if } m \equiv 2 \bmod 3 \\ +1 \text{ (left)} & \text{if } m \not\equiv 2 \bmod 3 \text{ and } BitAboveLowestOne(h) = 0 \\ 0 \text{ (straight)} & \text{if } m \not\equiv 2 \bmod 3 \text{ and } BitAboveLowestOne(h) = 1 \end{cases}$$

where $h = m - \lfloor m/3 \rfloor$ counts positions excluding -1 right turns,
$$= +1, -1, +1, 0, -1, +1, +1, -1, 0, 0, -1, +1, +1, -1, \ldots$$

Proof. Take the curve boundary in two parts R and V
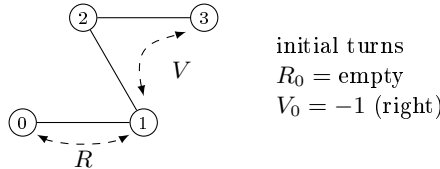
initial turns
$R_0 = $ empty
$V_0 = -1$ (right)

The turn at 1 is always left, so

$$R_{k+1} = R_k, +1, V_k$$

As per figure 11, V_{k+1} is an R and V with $0°$ turn (straight ahead) in between,

$$V_{k+1} = R_k, 0, V_k$$

These expansion rules are the dragon curve turn sequence, and per Davis and Knuth[3] those turns are bit above lowest 1-bit. The initial $R_0 = $ empty and $V_0 = -1$ mean the final V expansion adds an extra -1 at every third position starting from $m=2$. □

3   Area

The area enclosed by the curve can be counted in unit triangles. The curve does not cross itself so each enclosed triangle is either on the left or the right side of the curve.
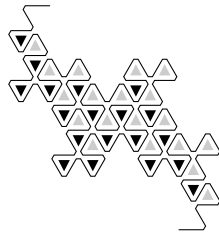
Figure 13:
$k{=}4$ enclosed area

black right of curve
grey left of curve

$AL_4 = AR_5 = 19$
total $A_4 = 38$

The left and right side triangles alternate along each row and each diagonal. The left side is all the upward pointing triangles. The right side is all the downward pointing triangles. (This arises later in theorem 26 with the Cantor dust.)

Lemma 1. *Consider line segments on a triangular grid where any enclosed unit triangle has segments on all 3 sides. The enclosed area A and boundary B are related to total line segments N by*

$$3A + B = 2N \tag{82}$$

Proof. Count the sides of the line segments. There are N segments so total $2N$ sides. Each side is either on a boundary or is inside.
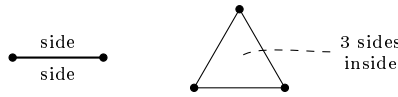
There are B outside sides on the boundary. The inside sides are all in enclosed unit triangles. Each area triangle A has 3 inside sides, so $3A$ inside sides and total $B + 3A = 2N$. $\qquad\square$

Theorem 23. *The number of unit triangles enclosed by the terdragon k is*

$$A_k = \begin{cases} 0 & \text{if } k = 0 \\ 2\left(3^{k-1} - 2^{k-1}\right) & \text{if } k \geq 1 \end{cases} \qquad area \tag{83}$$

$$= 0, 0, 2, 10, 38, 130, 422, 1330, 4118, \ldots \qquad {\scriptstyle k \geq 1 \;\; \text{A056182}}$$

Each side is symmetric so half area on each side

$$AR_k = AL_k = A_k/2$$

$$= \begin{cases} 0 & \text{if } k = 0 \\ 3^{k-1} - 2^{k-1} & \text{if } k \geq 1 \end{cases} \qquad area\ one\ side$$

$$= 0, 0, 1, 5, 19, 65, 211, 665, 2059, \ldots \qquad {\scriptstyle k \geq 1 \;\; \text{A001047}}$$

Proof. Non-crossing theorem 1 and plane filling theorem 2 mean that for all lengths every enclosed unit triangle has all three sides traversed. If this were

not so then the curve would have to cross itself, or another copy of the curve cross in, to fill that area to make 6 copies plane filling.

So lemma 1 applies with $N = 3^k$ line segments and boundary B_k from (71).

$$3A_0 + 2 = 2.3^0 \qquad\qquad \text{for } k = 0$$
$$3A_k + 3.2^k = 2.3^k \qquad\qquad \text{for } k \geq 1$$

Non-crossing means each enclosed unit triangle is either on the left or right side of the curve. By symmetry the two sides are equal so half the area each. $\square$

Second Proof of Theorem 23. When three terdragon curves are arranged in a triangle all segments inside are traversed precisely once (by non-crossing plane filling again) so the unit triangles are either enclosed by one side of the curve or are boundary triangles. The boundary triangles from the three curves overlap as in the following diagram.
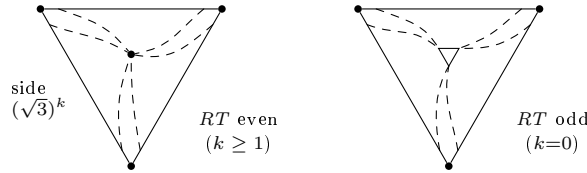
Boundary triangles of adjacent sides overlap. If RT_k is even then by symmetry there is a vertex in the middle common to all three. If RT_k is odd then there is a unit triangle in the middle which is common to all three.

The curve length end-to-end is $(\sqrt{3})^k$ and triangles of curves like this partition the plane into identical shapes so there are 3^k unit triangles inside.

$$3^k = 3\,AR_k + 3\,RT_k/2 \qquad\qquad \text{if } RT_k \text{ even} \qquad (84)$$
$$3^k = 3\,AR_k + 3\,(RT_k - 1)/2 + 1 \qquad\qquad \text{if } RT_k \text{ odd} \qquad \square$$

RT_k from (67) is odd only for $k{=}0$. When RT_k is even, (84) is equivalent to $3A + B = 2N$ from (82). The boundary triangles alternate 1-side and 2-side from theorem 15 giving $R_k = \frac{3}{2}RT_k$ for $k \geq 1$, so that (84) is $3^k = 3A_k/2 + B_k/2$.

As from *TurnRun* in section 1.2, the curve turns go in runs of either 1 or 2 consecutive left or right. A run of 2 consecutive turns encloses a unit triangle.

2 consecutive left turns, is left-side enclosed unit triangle
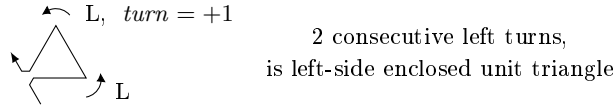
The run lengths are pairs either 1,2 or 2,1. There is one 2 for each of the $3^{k-1}{-}1$ turns of the previous expansion level. So the number of runs of 2 turns in curve k is

$$TurnRuns2_k = \begin{cases} 0 & \text{if } k{=}0 \\ 3^{k-1} - 1 & \text{if } k \geq 1 \end{cases}$$
$$= 0, 0, 2, 8, 26, 80, 242, \ldots \qquad {\scriptstyle k \geq 1 \ \ A024023}$$

$k=3$

LL squares black

RR squares grey

total

$TurnRuns2_3 = 8$

$A_3 = 10$
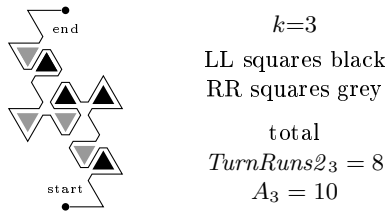
The proportion of enclosed unit triangles formed by 2-turns, out of the total area, is

$$\frac{TurnRuns2_k}{A_k} = \frac{1}{2} + \frac{2^{k-1}-1}{A_k} \ \to \frac{1}{2}$$

This limit is approached from above since $2^k - 1 > 0$ for $k \geq 2$ which is where $A_k > 0$. For example in $k=3$ the ratio is $\frac{4}{5}$,

Some segments have these triangles on both sides. Such pairs are a sequence of turns LLRR. As from the turn expansion in figure 2, such consecutive 2-runs occur only as an LR with L,R existing turns surrounding. An L,R is then only the middle of an LLRR of preceding segment expansion. So there is one LLRR for each $k-2$ segment.

There are no RRLL pairs, since the Rs could only be an LRR with existing R, but then LR follows, not LL.

$$TurnRuns2pairs_k = \begin{cases} 0 & \text{if } k = 0, 1 \\ 3^{k-2} & \text{if } k \geq 2 \end{cases}$$

3.1 Join Area

The join between two terdragon curves at 60° angle encloses new area.

$J_3 = 4$

$J_2 = 2$

$J_1 = 1$

$J_0 = 0$
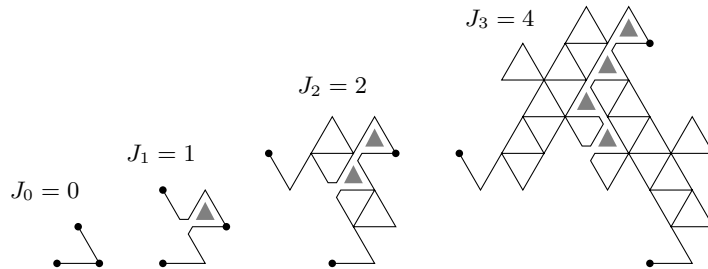
Theorem 24. *The join area between two terdragon curves k is the previous level right boundary triangles*

$$J_k = \begin{cases} 0 & \text{if } k = 0 \\ RT_{k-1} & \text{if } k \geq 1 \end{cases} \qquad \text{join area}$$

$$= 0, 1, 2, 4, 8, 16, 32, 64, \ldots$$

A131577

Proof. Two curves $k \geq 1$ have their $k-1$ sub-curves touching at point T as follows.
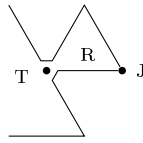
T is on the boundary since there are two absent sub-curves there (West and South-West). The join start J through to T is a curve side so the join area is its right boundary triangles RT_{k-1}. $\qquad\square$

Join area can also be calculated from the excess of area A_{k+1} over three copies of the previous A_k. This counts the join triangles but doesn't give their shape.
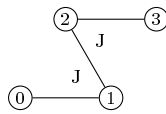
One join area is on the left side of the curve and one is on the right. The curve is symmetric left and right so the two joins are the same size.

$$A_{k+1} - 3A_k = 2J_k$$

The joins are also the shortfall of the boundary B_{k+1} over three copies of the previous B_k. Each unit triangle enclosed by the joins reduces the boundary by 3 segments,

$$3B_k - B_{k+1} = 2 \cdot 3\, J_k$$

3.2 Hanging Triangles

On the boundary of the terdragon curve there are some hanging unit triangles which touch the rest of the curve at only a single point.

$k=4$
hanging triangles
$H_4 = 4$
$HR_4 = H_4/2 = 2$
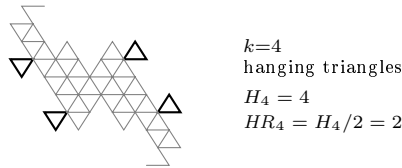
Theorem 25. *The number of hanging triangles on terdragon k is*

$$H_k = \begin{cases} 0 & \text{if } k = 0, 1, 2 \\ 2^{k-2} & \text{if } k \geq 3 \end{cases} \qquad \text{hanging triangles}$$

$$= 0, 0, 0, 2, 4, 8, 16, 32, \ldots$$

Each side is symmetric so half on one side

$$HR_k = \tfrac{1}{2}H_k = \begin{cases} 0 & \text{if } k = 0, 1, 2 \\ 2^{k-3} & \text{if } k \geq 3 \end{cases} \qquad \text{one side}$$

$$= 0, 0, 0, 1, 2, 4, 8, 16, \ldots$$

Proof. A hanging triangle is boundary turn sequence $-1, 1, 1, -1$ as from section 2.5.
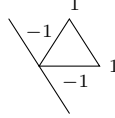
This is a pair $BitAboveLowestOne(j) = 0$ and $BitAboveLowestOne(j+1) = 0$ with j even. This requires j is binary low 0100, and possible further low 0 bits.

$$j = \boxed{\text{any} \mid 0 \mid 1 \mid 0 \mid 0 \mid 0...0} \qquad \text{total } k \text{ bits}$$

$$\underbrace{}_{\geq 0 \text{ zeros}}$$

The "any" bits at the high end can be any value of length 0 to $k-4$ bits. In addition the "1" shown can be the highest bit for value $j = 100\ldots00$ binary. The total number of such values is therefore

$$HR_k = 1 + \sum_{i=0}^{k-4} 2^i = 2^{k-3} \qquad \text{for } k \geq 3$$

For $k=3$ the sum is understood as empty so $HR_3 = 1$ which is single value $j = 100$ in binary. When $k \leq 2$ there are not enough bits to have any "100" at all and so $HR = 0$. $\square$

4 Cantor Dust

The Cantor dust fractal is formed by removing the middle third of a line segment and doing the same to each remaining line segment recursively.
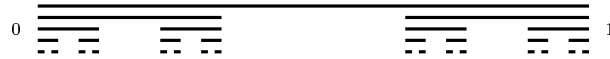
An integer version can be formed by multiplying by 3^k. The effect is to start with a unit line segment and triple out by a gap then a copy.

Counting the first segment as 0, segment number n is present when no digit 1s as per $Cpred(n)$ from (58).

Theorem 26. *The right side of the terdragon can be placed in one-to-one correspondence with the Cantor dust.*

Right-side boundary segments occur in triplets. Each unit segment of the Cantor dust corresponds to such a triplet.

Right-side non-boundary segments occur in triplets making a right-side enclosed unit triangle. Each unit gap in the Cantor dust corresponds to such a unit triangle.

Proof. Let *Tperm* change ternary digit pairs 10 to 20 and vice versa 20 to 10. This is a self-inverse permutation of the integers.

$$Tperm(n) = \text{flip ternary digit pairs } 10 \leftrightarrow 20 \text{ of } n$$
$$= 0, 1, 2, 6, 4, 5, 3, 7, 8, 18, 19, 20, 15, 13, 14, 12, 16, \ldots$$
$$\text{ternary} = 0, 1, 2, 20, 11, 12, 10, 21, 22, \; 200, 201, 202, 120, 111, \ldots$$

In *Rpred* (76), with *Tperm* applied the digit pairs 10 allowed and 20 disallowed become instead 10 disallowed and 20 allowed. So *Rpred*(*Tperm*(n)) has pairs 10, 11, 12 disallowed and hence

$$Cpred(n) = Rpred(Tperm(3n)) \tag{85}$$

The terdragon right boundary segments occur in triplets which have successively $n \equiv 0, 1, 2 \bmod 3$ (since any non-boundary excursion is a multiple of 3 length). A Cantor unit segment is identified with such a triplet.

For the enclosed unit triangles, the terdragon curve always steps in direction $0°$, $120°$ or $-120°$. Any path taking such steps has each unit triangle with segment numbers going $0, 1, 2 \bmod 3$ in the following pattern.
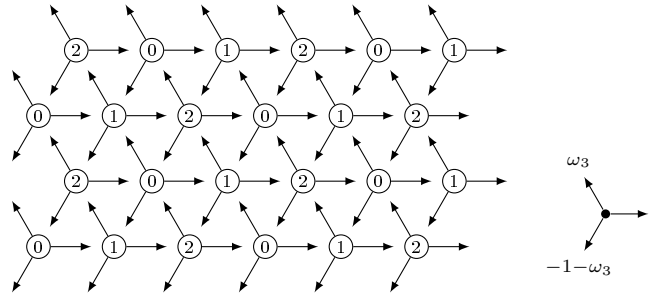
For a point at $x + y\omega_3$ the number shown is $x + y \bmod 3$. Stepping in direction 1, ω_3 or $-1 - \omega_3$ which are $0°$, $120°$ or $-120°$ change that $x + y$ index by $+1 \bmod 3$. Hence the pattern.

Each unit triangle is either on the left or right side of each segment. Those on the left have segment numbers going clockwise. Those on the right have segment numbers going anti-clockwise.

The right-side unit triangles are all the right-side non-boundary segments. Each unit triangle can be identified by its $0 \bmod 3$ segment and this corresponds to the Cantor non-segments as per (85). $\qquad\square$

5 Points

The terdragon curve touches at various vertices. Each point may be visited 1, 2 or 3 times.

Level $k=4$

○ singles S_4 $= 18$

●(gray) doubles D_4 $= 14$

● triples T_4 $= 12$

   total P_4 $= 44$
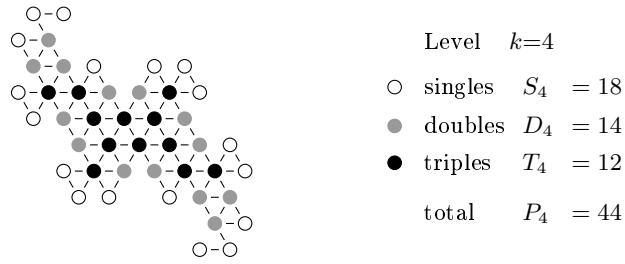
Theorem 27. *The number of single, double, and triple visited points in ter-dragon k are*

$$S_k = \begin{cases} 2 & \text{if } k = 0 \\ 2^k + 2 & \text{if } k \geq 1 \end{cases} \qquad \text{single-visited} \qquad (86)$$

$$= 2, 4, 6, 10, 18, 34, \ 66, 130, 258, \ldots \qquad \text{A133140}$$

$$D_k = \begin{cases} 0 & \text{if } k = 0 \\ 2^k - 2 & \text{if } k \geq 1 \end{cases} \qquad \text{double-visited}$$

$$= 0, 0, 2, \ 6, 14, 30, \ 62, 126, 254, \ldots$$

$$T_k = \begin{cases} 0 & \text{if } k = 0 \\ 3^{k-1} - 2.2^{k-1} + 1 & \text{if } k \geq 1 \end{cases} \qquad \text{triple-visited}$$

$$= 0, 0, 0, \ 2, 12, 50, 180, 602, 1932, \ldots \qquad \text{A028243}$$

Proof. For $k=0$ the curve is a single line segment. Each end is a single-visited point.

For $k \geq 1$, when each line segment of the previous level expands it makes a new vertex in the middle of an adjacent triangle.
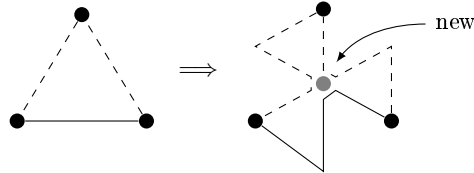
Figure 14:
new vertex
beside segment

The visits to the original vertex points are unchanged by the expansion. The visits to each new middle point are the number of sides of the triangle. Triangles with three sides are the enclosed area A_k (83). Each of them gives a new triple-visited point. Triangles with 1 or 2 sides are the boundary triangles $BT1_k$ and $BT2_k$ from (69),(70). Each of them gives a single or double visited point respectively. So the following recurrences, giving sums. The sums are taken as empty when $k=0$.

$$S_k = S_{k-1} + BT1_{k-1} \quad = 2 + \sum_{j=0}^{k-1} BT1_j$$

$$D_k = D_{k-1} + \mathit{BT2}_{k-1} \quad = \sum_{j=0}^{k-1} \mathit{BT2}_j$$

$$T_k = T_{k-1} + A_{k-1} \quad = \sum_{j=0}^{k-1} A_j \qquad \qquad \square$$

Second Proof of Theorem 27. When the curve triples to make its next level there are three copies of the points. Where they join some point visits merge.

Each sub-curve endpoint is single-visited and when they join it remains a single,

second copy

join start
single + single → single

first copy
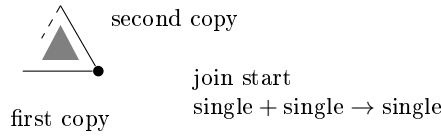
Adjacent join area triangles touch at a corner as follows.

between join triangles
single + double → triple
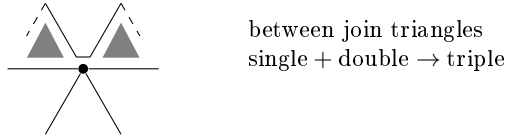
The join touches are always a single meeting a double this way, since otherwise there would be untraversed segments within the curve.

The boundary at the end of a join is always a straight line. This is so for the first join in level $k=2$ and for any subsequent level the expansion is

straight boundary
remains straight
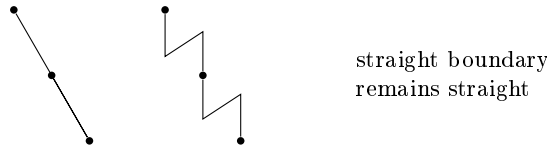
A straight line at the join end can only be formed from two single-visited points becoming double-visited.

join end triangle
single + single → double
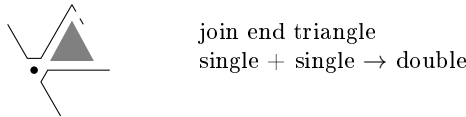
There are two identical join areas so the above merges apply twice. When there is at least one join triangle $J_{k-1} \geq 1$, which is when $k \geq 2$, the following recurrences

$$S_k = 3\,S_{k-1} + 2\left(-(J_{k-1}-1)-3\right) \qquad \text{for } k \geq 2$$
$$D_k = 3\,D_{k-1} + 2\left(-(J_{k-1}-1)+1\right)$$
$$T_k = 3\,T_{k-1} + 2\left(\quad J_{k-1}-1 \quad\right) \qquad\qquad (87)$$

There are $J_{k-1}-1$ new triple points in between join triangles. They reduce the singles and doubles and increase the triples. The singles are further -1 at

the join start and -2 at the join end. The doubles are $+1$ at the join end. With $J_{k-1} = 2^{k-2}$ and the initial S,D,T values the formulas (86) etc follow. $\qquad\square$

Per OEIS A028243, the triples T_k are twice Stirling numbers of the second kind

$$T_k = 2\,St(k,3) \qquad\qquad \text{Stirling second kind}$$

The triples recurrence in J at (87) is the usual Stirling recurrence since $J_k - 1 = 2^{k-1} - 1 = St(k,2)$ for $k \geq 1$.

$$T_k/2 = 3\,T_{k-1}/2 \quad + J_{k-1} - 1 \qquad\qquad (87)/2, \text{ for } k \geq 2$$
$$St(k,3) = 3\,St(k{-}1,3) + St(k{-}1,2) \qquad\qquad \text{Stirling recurrence}$$

On each visit to a given location, the curve turns the same way either left or right, as otherwise it would cross or overlap (see ahead section 12.1). By symmetry, the left-turn points and right-turn points are on one-to-one correspondence. So $St(k,3) = T_k/2$ is the number of right-turn triple visited points, or left turn the same.

All single and double visited points are on the boundary. Some triple visited points are on the boundary too.
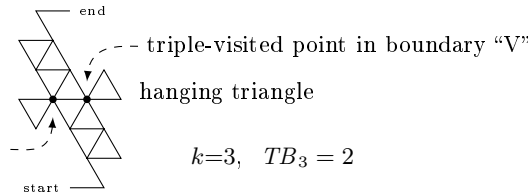
A boundary triple is in each V shape 2-side boundary triangle, except the 4 such at curve start and end are not triple visited, and at a hanging triangle the V each side is the same triple point.

$$TB_k = BT2_k - H_k - 4 \qquad\qquad \text{for } k \geq 2$$
$$= \begin{cases} 0 & \text{if } k \leq 2 \\ 3.2^{k-2} - 4 & \text{if } k \geq 3 \end{cases} \qquad \text{triple-visited on boundary}$$
$$= 0,0,0,2,8,20,44,92,188,\ldots \qquad\qquad {\scriptstyle k\geq3 \ \text{A131128}}$$

The total number of distinct visited points is

$$P_k = S_k + D_k + T_k$$
$$= \begin{cases} 2 & \text{if } k = 0 \\ 3^{k-1} + 2^k + 1 & \text{if } k \geq 1 \end{cases} \qquad \text{distinct points}$$
$$= 2,4,8,18,44,114,308,858,\ldots \qquad\qquad {\scriptstyle k\geq1 \ \ 2\times\text{A099754}}$$

It can be noticed

$$P_k + A_k = 3^k + 1$$

In general $P + A = N + 1$ for any path with N line segments on a triangular grid which is non-overlapping and each enclosed unit triangle has all three sides traversed. Such a path starts as a single point and no line segments. Then each further line segment either goes to an unvisited point which increases P, or it revisits a point and encloses a new unit triangle which increases A. So for each N either A or P increments.

Per figure 14, the number of sides of the triangle adjacent to a segment determines the number of visits to new points $n \equiv 1, 2 \bmod 3$. The number of visits is unchanged by further expansions, which are low ternary 0-digits.

$$Visits_k(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 3^k \\ Rsides(n) & \text{if } n = (3m{+}1).3^l, \; m \geq 1 \\ Lsides_{k-l-1}(n) & \text{if } n = (3m{+}2).3^l \end{cases}$$

$$= 1, 1 \quad \text{for } k{=}0$$
$$1, 1, 1, 1 \quad \text{for } k{=}1$$
$$1, 1, 2, 1, 2, 2, 1, 2, 1, 1 \quad \text{for } k{=}2$$

For the curve continued infinitely, $Lsides_\infty$ is used. Or it suffices to take 1 level bigger,

$$Visits_\infty(n) = Visits_k(n) \quad \text{for } 3^k > 3n$$
$$= 1, 1, 2, 1, 2, 2, 2, 2, 3, 1, 1, 3, 2, 3, 3, 2, 3, 1, 2, 3, \ldots$$
$$={=}1 \text{ at } n = 0, \; 1, \; 3, \; 9, 10, 17, 27, 28, 30, 51, 53, 64, \ldots$$
$$={=}2 \text{ at } n = 2, \; 4, \; 5, \; 6, \; 7, 12, 15, 18, 21, 22, 25, 31, \ldots$$
$$={=}3 \text{ at } n = 8, 11, 13, 14, 16, 19, 20, 23, 24, 26, 29, 32, \ldots$$

Visits also follow from $other(n, \delta)$ from theorem 9. The visits are all those occurring in the same curve arm and within the same k, or same arm and anywhere for the curve continued infinitely.

$$Visits_k(n) = \operatorname*{count}_{\delta = 0 \text{ to } 2} \big(other(n, \delta) \text{ same arm and} \leq 3^k \big) \qquad (88)$$
$$Visits_\infty(n) = \operatorname*{count}_{\delta = 0 \text{ to } 2} \big(other(n, \delta) \text{ same arm} \big)$$

The total of this *Visits* count within level k is 1 for each single, 2 each for the 2 visits to doubles, and 3 each for the 3 visits to triples.

$$\sum_{n=0}^{3^k} Visits_k(n) = S_k + 4D_k + 9T_k \quad = 3.3^k - 4.2^k + 3$$
$$= 2, 4, 14, 52, 182, 604, 1934, \ldots \qquad \text{\footnotesize $2 \times$ A134063}$$

5.1 Lines

Some unit segments in the terdragon are consecutive and they can be considered in runs making lines in directions $d = 0, 1, 2 \times 120°$. In the following samples, $d{=}0$ and $d{=}1$ both have lines which are co-linear but not consecutive. Those

lines are counted separately, so that number of lines is more than just curve width or height.

$$Lines_4(0) = 12 \qquad Lines_4(1) = 9 \qquad Lines_4(2) = 10$$

$$k=4, \ \text{total} \ Lines_4 = 31$$
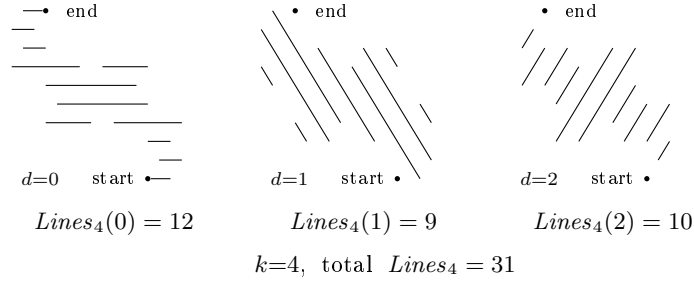
Theorem 28. *The number of lines in terdragon level k is*

$$Lines_k = 2^{k+1} - 1$$
$$= 1, 3, 7, 15, 31, 63, 127, 255, \ldots \qquad \text{\scriptsize A126646}$$

Proof. There are 3^k curve segments in level k. If none are consecutive then the segments are the lines. This occurs for $k{=}0$ and $k{=}1$ with $Lines_0 = 1$ and $Lines_1 = 3$.

At each triple-visited point, there are consecutive line segments in all 3 directions, reducing the lines by 3.

At each double-visited point, the two absent segments must be adjacent or the curve would cross or overlap when filling the plane.

double-visited point
missing segments are
adjacent around the point
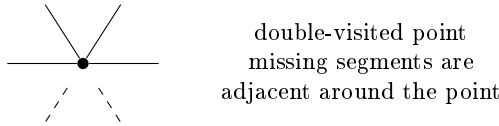
So at each double-visited point, there are consecutive segments in one direction, reducing the lines by 1.

$$Lines_k = 3^k - (3\,T_k + D_k) \tag{89}$$

$\square$

Second Proof of Theorem 28. A similar argument can be made counting line ends.

At a single visited point there are 2 line ends, except for the curve start and end where just 1 each, so $2S_k - 2$ line ends from singles.

At a double-visited point there is one line continuing across and 2 lines ending.

At a triple-visited point there are no line ends (all 3 directions continue across).

Every line has 2 ends so

$$Lines_k = \tfrac{1}{2}\,(2S_k - 2 \ + \ 2D_k) \tag{90}$$

$\square$

The visits considered in (89) and (90) are together the total $3^k + 1$ visits to all points,

$$S_k + 2\,D_k + 3\,T_k = 3^k + 1$$

Theorem 29. *The number of lines in directions $d = 0, 1, 2$ of terdragon k is*

$$Lines_k(0) = \tfrac{1}{3}\left(2^{k+1} + ld(k)\right)$$
$$Lines_k(1) = \tfrac{1}{3}\left(2^{k+1} - ld(k-1)\right)$$
$$Lines_k(2) = \tfrac{1}{3}\left(2^{k+1} - ld(k+1)\right)$$
$$ld(m) = [\,1, 2, 4, 5, 4, 2\,]_m$$

$$Lines_k(0) = 1, 2, 4, 7, 12, 22, 43, 86, 172, 343, 684, \ldots$$
$$Lines_k(1) = 0, 1, 2, 4, \ \ 9, 20, 42, 85, 170, 340, 681, \ldots$$
$$Lines_k(2) = 0, 0, 1, 4, 10, 21, 42, 84, 169, 340, 682, \ldots \qquad \text{\small A111927}$$

Lines in the three directions are each $\frac{1}{3}$ of the total except for the variation by the periodic ld, giving differences up to 3, depending on k.

Proof. Use line ends similar to the second proof above, but with ends in each direction d. Start with boundary triangles. Count 1-side boundary triangles by the direction of their segment. Count 2-side boundary triangles by the direction of their missing segment.

1-side and 2-side
right boundary triangles
in direction $d=0$
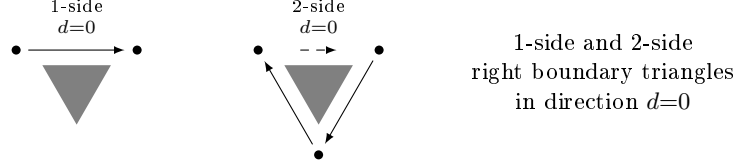
Let $RTS_k(d)$ be the number of 1-side triangles plus 2-side triangles on the right boundary and in direction d. The R,V expansion of figure 10 applies. In the "V" part triangles are swapped $1 \leftrightarrow 2$ sides but their direction is unchanged. The whole of V is turned -1 relative to the desired direction, so the count of $d+1$ there is required.

$$RTS_k(d) = RTS_{k-1}(d) + RTS_{k-1}(d+1)$$

Initial $RTS_0(0) = 1$ and $RTS_0(1) = RTS_0(2) = 0$ gives

$$RTS_k(d) = \tfrac{1}{3}\left(2^k + [2, 1, -1, -2, -1, 1]_{k+2d}\right) \qquad \text{1+2 side triangles by } d$$

$$RTS_k(0) = 1, 1, 1, 2, 5, 11, 22, 43, 85, 170, 341, \ldots \qquad \text{\small A024493}$$
$$RTS_k(1) = 0, 0, 1, 3, 6, 11, 21, 42, 85, 171, 342, \ldots \qquad \text{\small A024495}$$
$$RTS_k(2) = 0, 1, 2, 3, 5, 10, 21, 43, 86, 171, 341, \ldots \qquad \text{\small A131708}$$

The triangles on the left side of the curve are a 180° rotation. A horizontal $d=0$ remains horizontal in 180° rotation and similarly $d=1$ and $d=2$ unchanged. So total triangles $2\,RTS_k(d)$.

Count a double-visited point by the direction of its two cross segments. Count a single-visited point by the direction of its absent two cross segments.
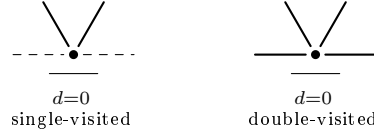
Let $SD_k(d)$ be the number of single and double points in direction d, excluding the first and last points of the curve which are singles but only one segment at each.

When the curve expands, the existing single-visited and double-visited points and their direction are unchanged. Each 1-side or 2-side boundary triangle gives a new single-visited or double-visited point respectively, per theorem 27. A new SD in direction d arises from an RTS triangle direction $d{+}1$.

$$SD_k(d) = \sum_{j=0}^{k-1} RTS_j(d{+}1) \qquad \text{single, double points by } d$$

$$= 2\,RTS(k,d)\ -(2 \text{ if } d{=}0)$$

$$SD_k(0) = 0,0,0,2,\ 8,20,42,84,168,338,\ldots \qquad 2\times\text{A111927}$$

$$SD_k(1) = 0,0,2,6,12,22,42,84,170,342,\ldots \qquad \text{A086953}$$

$$SD_k(2) = 0,2,4,6,10,20,42,86,172,342,\ldots \qquad 2\times\text{A131708}$$

Lines in a given direction have an end at a non-crossing segment of a single or double visited point. For example each SD point $d{=}0$ is the end of a line in directions $d{=}1$ and $d{=}2$. So $Lines(d)$ is SD of directions other than d. The very first and very last points of the curve are ends of a horizontal $d{=}0$.

$$Lines_k(d) = \tfrac{1}{2}\Big(SD_k(d{+}1) + SD_k(d{+}2)\ +(2 \text{ if } d{=}0)\Big) \qquad \square$$

$RTS_k(d)$ is the 3-period binomial sums of Cournot[2], but with $-d$ meaning $d{=}1$ is the 2 mod 3 binomials and $d{=}2$ is the 1 mod 3 binomials.

$$RTS_k(d) = \tbinom{k}{-d} + \tbinom{k}{-d+3} + \tbinom{k}{-d+6} + \cdots \qquad d = 0,1,2$$

The sum in $SD_k(d)$ is total of those binomials in columns down to row $k{-}1$.

$$
\begin{array}{llllll}
\binom{1}{0} & \binom{1}{1} \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\
\binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5}
\end{array}
\qquad
SD_k(d) = \sum_{j=0}^{k-1} 2\,RTS_k(d{+}1 \bmod 3)
$$

$d{=}0$, columns 2 mod 3

Then $Lines_k(d)$ is the "other" two $SD_k(d)$ which means 2 out of 3 columns down to row k.

$$
\begin{array}{ccccc}
\dbinom{1}{0} & \dbinom{1}{1} & & & \\
\dbinom{2}{0} & \dbinom{2}{1} & \dbinom{2}{2} & & \\
\dbinom{3}{0} & \dbinom{3}{1} & \dbinom{3}{2} & \dbinom{3}{3} & \\
\dbinom{4}{0} & \dbinom{4}{1} & \dbinom{4}{2} & \dbinom{4}{3} & \dbinom{4}{4} \\
\dbinom{5}{0} & \dbinom{5}{1} & \dbinom{5}{2} & \dbinom{5}{3} & \dbinom{5}{4} & \dbinom{5}{5}
\end{array}
\qquad
Lines_k(d) = \tfrac{1}{2}\left(\begin{array}{c} SD_k(d{+}1) + SD_k(d{+}2) \\ +\,2 \text{ if } d{=}0 \end{array} \right)
$$

$d{=}0$, columns $0,1 \bmod 3$

RTS_k combines 1-side and 2-side triangles, and SD_k combines 1 and 2 points, since those combinations suffice for the lines calculation. The 1s and 2s of each can be counted separately if desired and they are mod 6 columns of the binomials. When expressed as powers, they have a 12-periodic half-power term $3^{\lfloor k/2 \rfloor}$. By taking 1s and 2s together, those half-powers cancel out leaving just a 6-periodic constant term.

6 Enclosure Sequence

When a segment is appended to the curve it can be the first, second or third segment of the unit triangle on its right. Let $RsideNum(n) = 1, 2, 3$ be the side number of n on that triangle. A segment may have one or both segments s or e as follows,
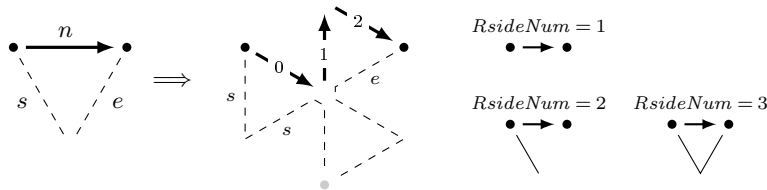
The expansion shows how a segment with s and/or e expands to a new combination. For new low digit 1 on n it can be noted that segment 2 is after n so is not yet present. This means e occurs only with s so there is only a single $RsideNum{=}2$ form.

Figure 15: n ternary high to low
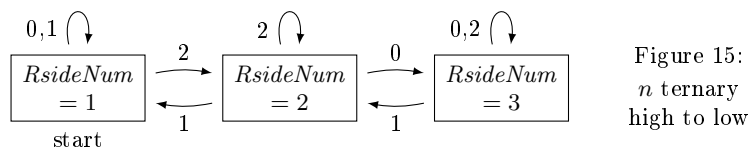
$$
\begin{aligned}
RsideNum(n) &= \text{figure 15 final state} \\
&= 1,1,2,1,1,2,3,1,2,1,1,2,1,1,2,3,1,2,\ldots \\
&= 1 \text{ at } \quad n = 0,1,3,4,7,9,10,12,13,16,\ldots \\
&= 2 \text{ at } \quad n = 2,5,8,11,14,17,19,23,26,29,\ldots \\
&= 3 \text{ at } \quad n = 6,15,18,20,24,33,42,45,47,51,\ldots
\end{aligned}
$$

Left side segments follow similarly
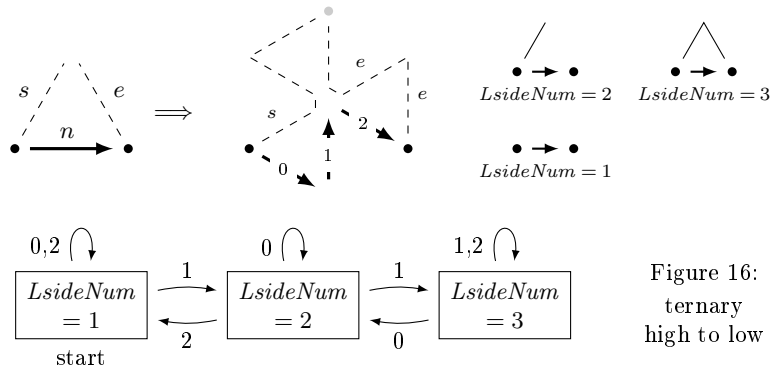
Figure 16: ternary high to low

$$LsideNum(n) = \text{figure 16 final state}$$
$$= 1, 2, 1, 2, 3, 1, 1, 2, 1, 2, 3, 1, 2, 3, 3, 1, 2, 1, \ldots$$
$$= 1 \text{ at} \quad n = 0, 2, 5, 6, 8, 11, 15, 17, 18, 20, \ldots$$
$$= 2 \text{ at} \quad n = 1, 3, 7, 9, 12, 16, 19, 21, 25, 27, \ldots$$
$$= 3 \text{ at} \quad n = 4, 10, 13, 14, 22, 28, 31, 32, 37, 40, \ldots$$

LsideNum state machine figure 16 is a reversal of *RsideNum* state machine figure 15. Digits are reversed $0 \leftrightarrow 2$ and the side number reversed $1 \leftrightarrow 3$.

Geometrically this is simply the curve being the same in $180°$ rotation, so that the left side counted from the end is the same as the right side counted forward. The reversal of the side number counts downwards from how many sides it will have, so

$$LsideNum(3^k - 1 - n) \ = \ Rsides(n) + 1 - RsideNum(n)$$

$RsideNum(n) = 3$ is where n encloses a unit triangle on the right. Similarly $LsideNum(n) = 3$ on the left.

left enclosures
$EpredL(n)$

right enclosures
$EpredR(n)$
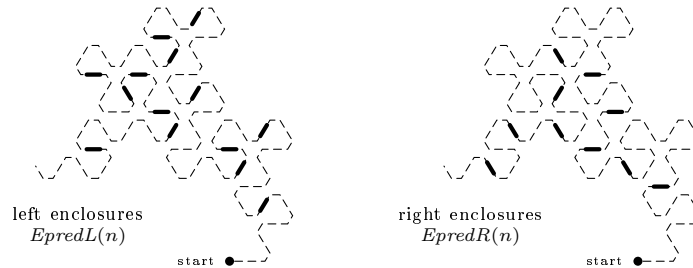
$$EpredR(n) = \begin{cases} 1 & \text{if } RsideNum(n) = 3 \\ 0 & \text{if not} \end{cases}$$
$$= \begin{cases} 1 & \text{if pair 20 and any 1s below it are in pairs 10} \\ 0 & \text{otherwise} \end{cases} \tag{91}$$
$$= 0, 0, 0, \ 0, 0, 0, \ 1, 0, 0, \ 0, 0, 0, \ 0, 0, 0, \ 1, 0, 0, \ \ldots$$
$$EpredL(n) = \begin{cases} 1 & \text{if } LsideNum(n) = 3 \\ 0 & \text{if not} \end{cases}$$

$$= 0,0,0,\ 0,1,0,\ 0,0,0,\ 0,1,0,\ 0,1,1,\ 0,0,0,\ \dots$$

Form (91) is since 20 in figure 15 goes to or stays in side 3. A 1 digit would leave there, unless it's a 10 pair which goes back. The digit 2 loop in side 2 would be another 20 if it goes back to side 3 that way.

Some usual state machine manipulations can take digits of n low to high instead. Reaching $EpredR$ is a right enclosure. Reaching "not" or ending in rm0 or rm1 is not. Similarly $EpredL$.

Figure 17:
ternary
low to high
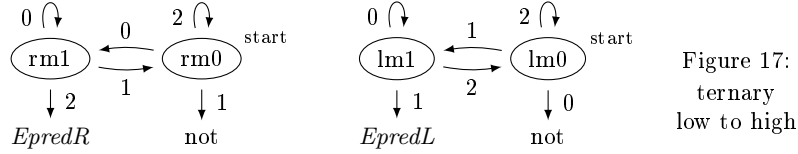
Each enclosure is an enclosed unit triangle on the respective side right or left, so totals AR and AL from theorem 23.

$$AR_k \ = \ AL_k \ = \ \sum_{n=0}^{3^k-1} EpredR(n) \ = \ \sum_{n=0}^{3^k-1} EpredL(n) \tag{92}$$

When $EpredR(n)$ encloses a unit triangle the next turn is left $turn(n+1) = +1$, since otherwise the next segment would overlap the triangle just enclosed. Conversely $EpredL$ is followed by a right turn

$turn(n+1) = 1$ left
or would overlap segment
of triangle just enclosed
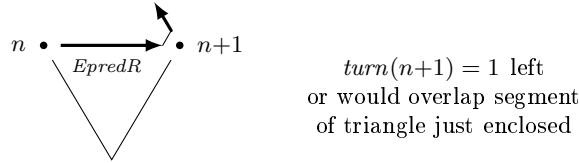
As from section 1.2, a left turn at $n+1$ is $LowestNonTwo(n) = 0$. For $EpredR$ in figure 17, low 2s loop in rm0 and then if a 1 go to "not" so never a right turn. For $EpredL$ conversely 0 goes to "not" so never left turn.

$EpredR$ can enclosure 2 triangles consecutively. This occurs first at $n=56$, 57 which are ternary 2002 and 2010. There cannot be 3 or more consecutive $EpredR$ or that would be 3 left turns and the segments would overlap. Similarly $EpredL$ pair, which first occurs at $n=13, 14$, ternary $111, 112$.

Some state machine manipulations can test whether $n+1$ is also the respective enclosure, then intersection n and $n+1$ for a pair. Taking that low to high shows enclosure pairs are the original digit forms with extra low.

The last segment of curve k is not an enclosure, since it is the first visit to its endpoint, so pairs do not cross a level. The number of pairs within a level follow from (92) and the extra digits.

$$\sum_{n=0}^{3^k-1} EpredRpair(n) \ = \sum_{h=0}^{k-2} AR_h \ = \begin{cases} 0 & \text{if } k{=}0 \\ \frac{1}{2}T_{k-1} & \text{if } k \geq 1 \end{cases} \tag{93}$$

$$= 0, 0, 0, 0, 1, 6, 25, 90, 301, 966, \ldots \qquad \text{A000392}$$

$$\sum_{n=0}^{3^k-1} EpredLpair(n) \ = AL_k$$

At (93), cumulative AR is $\frac{1}{2}T$ in the manner of theorem 27. New triples are formed when segments expand into each triangle A, here it is just AR triangles so half. The result is the Stirling numbers of the second kind.

When 2 consecutive $EpredR$ occur the next segment is always an $EpredL$ left enclosure, since it was 2 left turns. Conversely 2 consecutive $EpredL$ is always followed by $EpredR$.

2 right enclosures
are 2 left turns T
so next segment
is left enclosure
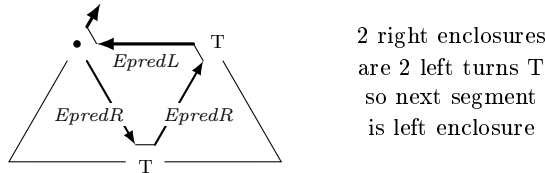
Runs of right and left enclosures can occur. For example at $n{=}373$ ternary 111211 there is a run of 12 consecutive enclosures. The following diagrams show how this run falls within its surrounding segments.

$n = 373$
ternary 111211
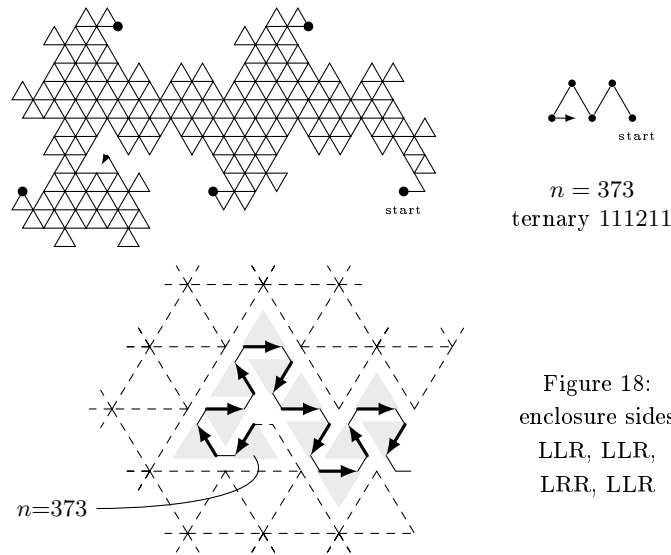
Figure 18:
enclosure sides
LLR, LLR,
LRR, LLR

There are no runs longer than 12. That can be seen by some state machine manipulations on $Epred$ left or right to ask whether $n{+}1$, $n{+}2$ etc also enclosing. The intersection of $Epred$ on 13 terms n through $n{+}12$ inclusive is empty.

State machine manipulations on the 12 intersection shows it is $EpredL$ with some extra low digits,

$$EpredTwelve = \boxed{\ \begin{array}{|c|c|c|c|} \hline EpredL & 1 & 2\ldots2 & 11 \\ \hline \end{array}\ } \qquad \text{ternary} \tag{94}$$

≥ 1 digits

The count of how many 12 runs in k is the same as *EpredRpair* in $k-2$. The digit form for *EpredTwelve* is like *EpredRpair* but with 2 extra fixed digits. The high here is *EpredL* rather than *EpredR*, but their counts are the same (92).

Runs of 12 all have the same enclosure side sequence shown in figure 18. The enclosed side is the opposite of $turn(n+1)$ and $turn(n+1)$ is *LowestNonTwo* on low digits of 1211 through 2020 of *EpredTwelve* at (94). It is the same when more 2s for 12...211 there.

6.1 Point Visit Number

Each n is visit number 1, 2 or 3 to its point. This is given by *RsideNum* or *LsideNum* when the sides of such a triangle expand to meet in the middle. $n \equiv 1 \bmod 3$ is the right side or $n \equiv 2 \bmod 3$ is the left side, and then any number of low 0s since those 0s do not change existing points.

$$VisitNum(n) = \begin{cases} 1 & \text{if } n=0 \\ RsideNum(m) & \text{if } n = (3m+1).3^l \\ LsideNum(m) & \text{if } n = (3m+2).3^l \end{cases}$$

$$= 1, 1, 1, 1, 1, 2, 1, 2, 1, 1, 1, 2, 1, 1, 3, 2, 2, 1, 1, 3, \ldots$$

high			low
RsideNum	1	0...0	

or

high			low
LsideNum	2	0...0	

≥ 0 digits $\qquad$ ≥ 0 digits

The visit number is also how many $other(n, \delta)$ are on the same arm and preceding n.

$$VisitNum(n) = 1 + \underset{\delta=1,2}{\text{count}} \Big(other(n, \delta) \text{ same arm and } < n \Big)$$

or count with $\delta=0$ to include n itself unchanged

$$VisitNum(n) = \underset{\delta=0,1,2}{\text{count}} \Big(other(n, \delta) \text{ same arm and } \le n \Big)$$

Total of *VisitNum* within level k counts 1 each single, 1+2 each double, and 1+2+3 each triple,

$$\sum_{n=0}^{3^k} VisitNum(n) = S_k + 3D_k + 6T_k \;\; = 2.3^k - 2.2^k + 2$$
$$= 2, 4, 12, 40, 132, 424, 1332, \ldots \qquad \text{\small $2\times$A083323}$$

7 Multiple Arms

Six copies of the terdragon at 60° angles mesh perfectly and fill the plane (theorem 2). The boundary of 2 to 6 such arms can be calculated simply as R_k (72) on the ends and one or more V_k (73) in between. The area follows from the boundary by (82).

Arms		Boundary	Area	
2		$\begin{cases} 4 \\ 9.2^{k-1} \end{cases}$	$\begin{cases} 0 \\ 4.3^{k-1} - 3.2^{k-1} \end{cases}$	if $k = 0$ if $k \geq 1$
3		12.2^{k-1}	$6.3^{k-1} - 4.2^{k-1}$	
4		$\begin{cases} 8 \\ 15.2^{k-1} \end{cases}$	$\begin{cases} 0 \\ 8.3^{k-1} - 5.2^{k-1} \end{cases}$	if $k = 0$ if $k \geq 1$
5		$\begin{cases} 10 \\ 18.2^{k-1} \end{cases}$	$\begin{cases} 0 \\ 10.3^{k-1} - 6.2^{k-1} \end{cases}$	if $k = 0$ if $k \geq 1$
6		$\begin{cases} 12 \\ 18.2^{k-1} \end{cases}$	$\begin{cases} 0 \\ 12.3^{k-1} - 6.2^{k-1} \end{cases}$	if $k = 0$ if $k \geq 1$

The boundary increases by an extra V_k with each extra arm. For 3 arms the $k=0$ and $k \geq 1$ cases coincide.

In 3 arms, the boundary and area are B_{k+1} and A_{k+1}, ie. the plain curve one level bigger. This is since the 3 arms are 3 sub-curves and 2 joins which is the same as the whole curve $k+1$, just the orientation of the joins changed (to the $k=1$ base shape).

In 5 arms the gap is $2R_k$ and in 6 arms the corresponding section is $2V_k$. With $R_k = V_k$ for $k \geq 1$ from (72)(73) the 5 and 6 arm curves are $B6(k) = B5(k)$ for $k \geq 1$.

8   Shortcut Boundary

The terdragon boundary has "V" notches at every third boundary position. These are the 2-side boundary triangles $BT2_k$ from theorem 15 and the -1 boundary turns from theorem 22. A variation on the curve can be made by taking shortcuts across those Vs.

k=3
shortcut boundary
(dragon curve θ=120°)
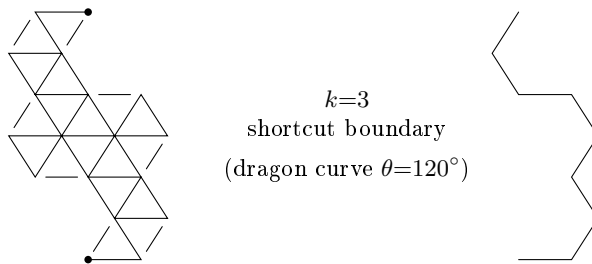
Theorem 30. *The shortcut boundary length is*

$$BSH_k = 2^{k+1} \qquad\qquad boundary$$

$$RSH_k = BSH_k/2 = 2^k \qquad\qquad one\ side$$

and the area enclosed is

$$ASH_k = \begin{cases} 0 & if\ k = 0 \\ 2.3^{k-1} & if\ k \geq 1 \end{cases} \qquad area \qquad (95)$$

Proof. The shortcuts add the 2-sided boundary triangles as additional area,

$$ASH_k = A_k + BT2_k$$

$$= \begin{cases} 0 + 0 & \text{if } k = 0 \\ 2(3^{k-1} - 2^{k-1}) + 2^k & \text{if } k \geq 1 \end{cases}$$

The shortcuts shorten the boundary by 1 side at each 2-sided boundary triangle,

$$BSH_k = B_k - BT2_k$$

$$= \begin{cases} 2 + 0 & \text{if } k = 0 \\ 3.2^k - 2^k & \text{if } k \geq 1 \end{cases} \qquad \square$$

The shortcuts maintain the three-sides-enclosed property of lemma 1 and so shortcut area and boundary are related to total line segments by

$$3\,ASH_k + BSH_k = 2\,(3^k + BT2_k)$$

Riddle[8] takes this shortcut curve form to show the terdragon as a fractal has area $1/(2\sqrt{3})$. Scaling ASH_k by the curve endpoint distance $\sqrt{3}^{\,k}$ squared gives

$$\frac{ASH_k}{(\sqrt{3})^{2k}} \;=\; \frac{2.3^{k-1}}{3^k} \;=\; \frac{2}{3} \qquad\qquad \text{of base triangle area}$$

A base equilateral triangle of unit side has height $\frac{1}{2}\sqrt{3}$ so area $\frac{1}{4}\sqrt{3}$, giving

$$\frac{2}{3} \cdot \tfrac{1}{4}\sqrt{3} \;=\; \frac{1}{2\sqrt{3}} \;=\; 0.288675\ldots \qquad\qquad \text{\small A020769}$$

Going instead from the plain enclosed area A_k (83) the result is the same

$$\frac{\frac{\sqrt{3}}{4} A_k}{(\sqrt{3})^{2k}} \;=\; \frac{1}{2\sqrt{3}} - \tfrac{\sqrt{3}}{4}\left(\tfrac{2}{3}\right)^k \;\to\; \frac{1}{2\sqrt{3}}$$

Theorem 31. *The shortcut boundary is the Heighway/Harter dragon curve with unfolding angle $\theta = 120°$.*

Proof. In turn sequence $Rturn(i)$ from theorem 22 the -1 turns are eliminated leaving just the dragon turns. The turns before and after the shortcut are both reduced by 60°. In $Rturn(i)$ the turns $+120°$ and 0 become $+60°$ and $-60°$ respectively. Those 60° turns correspond to unfolding the dragon by $\theta = 120°$.
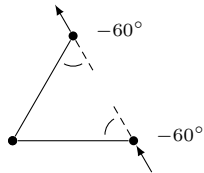
turns before and
after shortcut
reduced by 60°

$\square$

The shortcut area (95) has $ASH_{k+1} = 3\,ASH_k$ for $k \geq 1$ so the area is exactly 3 copies of the previous level, with no join area in between.

k=3
shortcut boundary

join length
$JBSH_3 = 4$
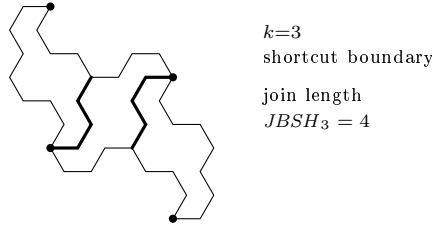
Theorem 32. *The shortcut join boundary length is*

$$JBSH_k = 2^{k-1} \qquad\qquad \textit{for } k \geq 1$$

Proof. For $k \geq 1$ the total shortcut boundary BSH_{k+1} is 3 copies of the previous level boundary less 4 copies of the join boundary (2 in each join).

$$BSH_{k+1} = 3\,BSH_k - 4\,JBSH_k$$
$$JBSH_k = (3.2^{k+1} - 2^{k+2})/4 \; = \; 2^{k-1} \qquad\qquad \square$$

Exact matching of the shortcut sides can also be seen in the dragon curve turn sequence of theorem 31. In a dragon curve with 2^k segments the turns in the second half are reverse order and opposite direction to the first half, so the second half of one boundary matches the first half of the next. (It would then have to be shown that the matching goes no further.)

9 Centroid

The terdragon curve is symmetric in 180° rotation so the centroid of the segments, points or area are all the midpoint of the curve at $b^k/2$. But some measures can be made on just one side of the curve.

Theorem 33. *The centroid of the right boundary triangles of terdragon k is*

$$GRT_k = \frac{7-2\omega_6}{13}\,b^k \; + \; \frac{5-7\omega_6}{39}\left(\frac{\overline{\omega_6}}{2}\right)^k$$

$$= \frac{3-\sqrt{3}i}{6}, \; \frac{9+\sqrt{3}i}{12}, \; \frac{24+14\sqrt{3}i}{24}, \; \frac{33+67\sqrt{3}i}{48}, \; \frac{-99+233\sqrt{3}i}{96}, \; \dots$$

Proof. For $k=0$ the curve is a single line segment with a single triangle. The centroid of the triangle is the mean of its corners.

0 ——→ 1

k=0

$\overline{\omega_6}$

$$GRT_0 = \frac{0 + 1 + \overline{\omega_6}}{3} = \frac{\overline{b}}{3}$$

As in theorem 14, the boundary triangles in a V part are a reversal of the R part, so the centroid is the mean of the two copies in the previous level.
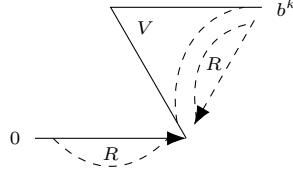
Figure 19

$$GRT_k = \tfrac{1}{2}\big(GRT_{k-1} \; + \; b^k + (\omega_6)^4 \, GRT_{k-1}\big)$$

$$= \tfrac{\overline{\omega_6}}{2} GRT_{k-1} \; + \; \tfrac{1}{2}b^k$$

$$= GRT_0 \left(\tfrac{\overline{\omega_6}}{2}\right)^k \; + \; \tfrac{1}{2}b \sum_{j=0}^{k-1} \left(\tfrac{\overline{\omega_6}}{2}\right)^j b^{k-1-j}$$

$$= \tfrac{\bar{b}}{3} \left(\tfrac{\overline{\omega_6}}{2}\right)^k \; + \; \tfrac{1}{2}b \frac{\left(\tfrac{\overline{\omega_6}}{2}\right)^k - b^k}{\left(\tfrac{\overline{\omega_6}}{2}\right) - b} \qquad \qquad \square$$

Per theorem 31, the line segments of the shortcut boundary are the Heighway/Harter dragon curve unfolding by 120°. The same reversing calculation as above is made for its centroid, but with initial line centroid $GRSH_0 = \tfrac{1}{2}$. Equating the sum parts of the two gives

$$GRSH_k \; - \; GRSH_0 . \left(\tfrac{\overline{\omega_6}}{2}\right)^k \; = \; GRT_k \; - \; GRT_0 . \left(\tfrac{\overline{\omega_6}}{2}\right)^k$$

$$GRSH_k = \frac{7-2\omega_6}{13} b^k \; + \; \frac{-1+4\omega_6}{26} \left(\frac{\overline{\omega_6}}{2}\right)^k \qquad \text{terdragon 120° centroid}$$

$$= \tfrac{2}{4}, \; \tfrac{7+\sqrt{3}i}{8}, \; \tfrac{17+9\sqrt{3}i}{16}, \; \tfrac{22+44\sqrt{3}i}{32}, \; \tfrac{-67+155\sqrt{3}i}{64}, \; \cdots$$

Theorem 34. *The centroid of the right boundary segments of terdragon k is*

$$GR_k = \begin{cases} \tfrac{1}{2} & \text{if } k = 0 \\ GRT_k + \tfrac{\omega_6}{3}\left(\tfrac{b}{2}\right)^k & \text{if } k \geq 1 \end{cases}$$

$$= \tfrac{2}{4}, \; \tfrac{9+3\sqrt{3}i}{12}, \; \tfrac{21+17\sqrt{3}i}{24}, \; \tfrac{24+70\sqrt{3}i}{48}, \; \tfrac{-117+233\sqrt{3}i}{96}, \; \cdots$$

And across a V part (other sides of an R),

$$GV_k = \begin{cases} \tfrac{1}{2} - \tfrac{1}{4}\sqrt{3}i & \text{if } k = 0 \\ GRT_k - \tfrac{\omega_6}{3}\left(\tfrac{b}{2}\right)^k & \text{if } k \geq 1 \end{cases}$$

$$= \tfrac{4-2\sqrt{3}i}{8}, \; \tfrac{9-\sqrt{3}i}{12}, \; \tfrac{27+11\sqrt{3}i}{24}, \; \tfrac{42+64\sqrt{3}i}{48}, \; \tfrac{-81+233\sqrt{3}i}{96}, \; \cdots$$

Proof. The centroid of the R right and V part boundaries are

These parts expand, similar to the R,V expansion of figure 11,
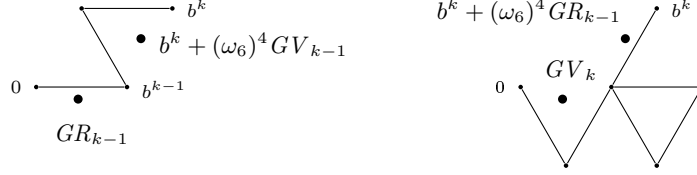
For $k \geq 1$ there are the same number of segments $R_k = V_k$ in each part so the centroids are the mean of the previous level.

$$GR_k = \tfrac{1}{2} GR_{k-1} \; + \; \tfrac{1}{2}\left(b^k + (\omega_6)^4 GV_{k-1}\right) \qquad k \geq 2 \qquad (96)$$

$$GV_k = \tfrac{1}{2} GV_{k-1} \; + \; \tfrac{1}{2}\left(b^k + (\omega_6)^4 GR_{k-1}\right) \qquad\qquad\quad (97)$$

Taking (96) for GV and substituting into (97) gives

$$GR_k = GR_{k-1} - \tfrac{\overline{b}}{4} GR_{k-2} + \tfrac{1}{4} b^k \qquad k \geq 3$$

The characteristic polynomial of the GR terms alone is

$$x^2 - x + \tfrac{\overline{b}}{4} = \left(x - \tfrac{\overline{\omega_6}}{2}\right)\left(x - \tfrac{b}{2}\right)$$

so GR_k is powers of $\tfrac{\overline{\omega_6}}{2}$, $\tfrac{b}{2}$ and the further b. From the initial values the coefficients of b and $\tfrac{\overline{\omega_6}}{2}$ are the same as for GRT_k. The coefficient of the $\tfrac{b}{2}$ power is $\tfrac{\omega_6}{3}$. Substituting into (96) gives GV_k in the same form but coefficient $-\tfrac{\omega_6}{3}$. $\qquad\square$

For the terdragon fractal, all four right boundary centroid forms above can be scaled by b^k for a unit length curve. The limit as $k\to\infty$ is the coefficient of the b^k term and so is the same in each case. Notice this is not the middle horizontally but a little towards the start at $\tfrac{6}{13}$

centroid of right boundary

$$\to GRf = \frac{7 - 2\omega_6}{13} \; = \; \frac{6}{13} - \frac{\sqrt{3}}{13}\,i \qquad (98)$$

$$= 0.461538\ldots - 0.133234\ldots i$$
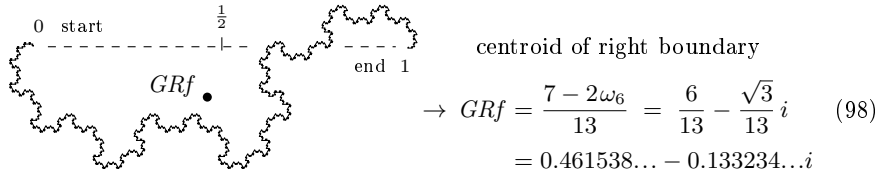
The equivalent of figure 19 in the fractal is two suitably rotated halves whose mean is the centroid of the whole.

$$\frac{GRf/b \; + 1 - GRf/\overline{b}}{2} = GRf$$

$$GRf = \frac{1}{2 - 1/b + 1/\overline{b}}$$

9.1 Centroid of Join

$1 + (3 + \frac{1}{3})\frac{\sqrt{3}}{2}$

$GJ_2 = \frac{3}{4} + \frac{17}{12}\sqrt{3}$

$\frac{1}{2} + (2 + \frac{1}{3})\frac{\sqrt{3}}{2}$

$k=2$ join,
centroid of 2 triangles

0
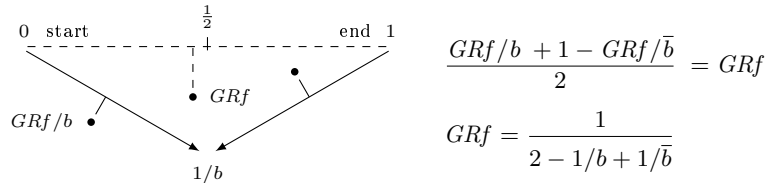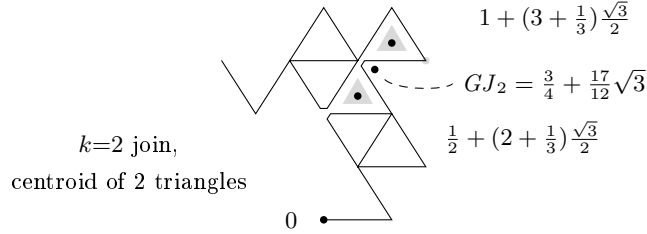
Theorem 35. *For $k \geq 1$ there are enclosed triangles in the join between two level k terdragon curves. The centroid of those triangles is*

$$GJ_k = b^{k+1} - 2\omega_6\, GRT_k \qquad\qquad\qquad k \geq 1$$

$$= \frac{9+3\omega_6}{13}b^k \;+\; \frac{-14+4\omega_6}{39}\left(\frac{\overline{\omega_6}}{2}\right)^k \qquad\qquad (99)$$

$$= 1+\tfrac{2}{3}\sqrt{3}i,\; \tfrac{3}{4}+\tfrac{17}{12}\sqrt{3}i,\; -1+\tfrac{29}{12}\sqrt{3}i,\; -\tfrac{83}{16}+\tfrac{149}{48}\sqrt{3}i,\; \ldots$$

Proof. For $k \geq 1$ the right boundary triangles are two joins, per the triangle arrangement in the second proof of area theorem 23. So, with suitable rotations and offsets, the mean of the join centroids is the right triangles centroid GRT_k.

$\omega_6 b^k$

$\frac{\overline{b}}{3}b^k$
middle

R

0 $\qquad\qquad b^k$
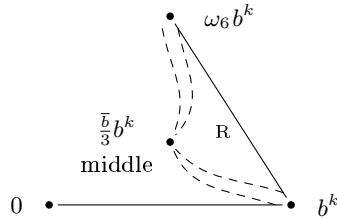
$$\tfrac{1}{2}GJ_k + \tfrac{1}{2}\left(b^k + (\omega_6)^2\, GJ_k\right) \;=\; \omega_6 b^k + (\omega_6)^5\, GRT_k$$

$$GJ_k = \frac{\omega_6 b^k + (\omega_6)^5\, GRT_k - \tfrac{1}{2}b^k}{\tfrac{1}{2} + \tfrac{1}{2}(\omega_6)^2} \qquad\qquad \square$$

Scaled by b^k for a fractal of unit length, the limit is the coefficient of the b^k term in (99).

$$\frac{GJ_k}{b^k} \to GJf = \frac{9 + 3\omega_6}{13} = \frac{21 + 3\sqrt{3}i}{26} = 0.807692\ldots + 0.199852\ldots i \qquad (100)$$

GJf

1
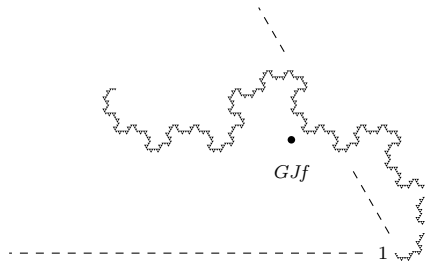
9.2 Centroid of Right Enclosed Area

Theorem 36. *The centroid of the unit triangles enclosed by the right side of the terdragon curve level $k \geq 2$ is*

$$GAR_k = \frac{1}{2}b^k + \frac{1}{156} \cdot \frac{(-3+12\omega_6)2^k b^k - 26\omega_6 b^k + (-10+14\omega_6)\overline{\omega_6}^k}{3^{k-1} - 2^{k-1}}$$

$$= \frac{3+5\sqrt{3}i}{6}, \frac{-12+46\sqrt{3}i}{30}, \frac{-306+248\sqrt{3}i}{114}, \frac{-2769+799\sqrt{3}i}{390}, \ldots \qquad k \geq 2$$

Proof. Each segment is either a right boundary or a side of a right-side enclosed unit triangle. Weighted by the number of segments, the centroid of the enclosed triangles and the boundary segments sum to the centroid of all segments which is the midpoint $\frac{1}{2}b^k$.

$$3^k \cdot \frac{1}{2}b^k = 3AR_k.GAR_k + R_k.GR_k \qquad \square$$

The right side area is three copies of the previous level and one join, so $AR_k = 3AR_{k-1} + J_{k-1}$. The centroids of those give a recurrence for GAR with the join centroid GJ.

$$GAR_k = \frac{1}{AR_k}\begin{pmatrix} AR_{k-1}.GAR_{k-1} \\ + AR_{k-1}.\left(b^{k-1} + \omega_3 GAR_{k-1}\right) \\ + AR_{k-1}.\left(\omega_6 b^{k-1} + GAR_{k-1}\right) \\ + \quad J_{k-1}.\left(b^k + GJ_{k-1}\right) \end{pmatrix}$$

Scaled by b^k to make a fractal of unit length the limit is $\frac{1}{2}$ which is the midpoint of the whole.

$$GARf_k = GAR_k/b^k \to \tfrac{1}{2} \quad \text{as } k \to \infty$$

$GARf_2 = \frac{1}{2} + \frac{1}{18}\sqrt{3}i$

$GARf_3 = \frac{23}{45} + \frac{2}{45}\sqrt{3}i$

limit $\frac{1}{2}$
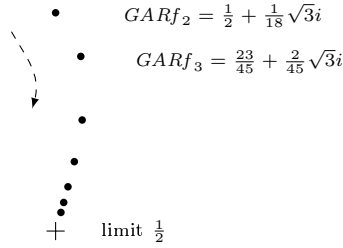
10 Convex Hull

A convex hull is the smallest convex polygon which can be drawn around a given set of points.
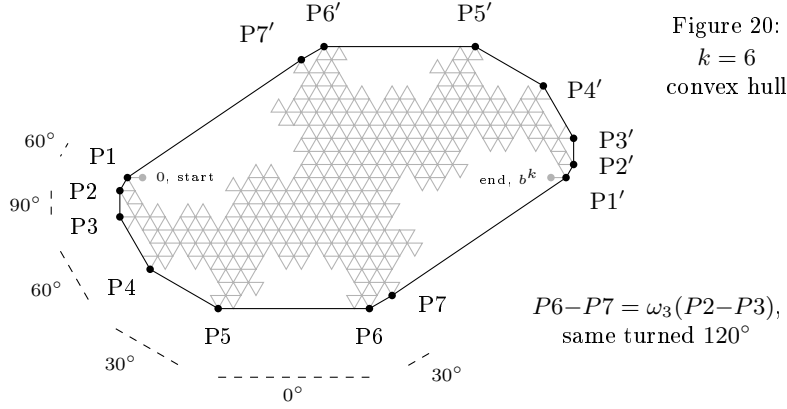
Figure 20:
$k = 6$
convex hull

$P6-P7 = \omega_3(P2-P3)$,
same turned 120°

Theorem 37. *The convex hull around terdragon $k \geq 6$ is a set of 14 vertices located at*

$$P1(k) = -\tfrac{1}{24}\left(b^k \quad + p(k) \quad\right) \tag{101}$$
$$P2(k) = -\tfrac{1}{24}\left(b^{k+1} + p(k{+}1)\right)$$
$$P3(k) = -\tfrac{1}{24}\left(b^{k+2} + p(k{+}2)\right)$$
$$P4(k) = -\tfrac{1}{24}\left(b^{k+3} + p(k{+}3)\right)$$
$$P5(k) = -\tfrac{1}{24}\left(b^{k+4} + p(k{+}4)\right)$$
$$P6(k) = -\tfrac{1}{24}\left(b^{k+5} + p(k{+}5)\right)$$
$$P7(k) = -\tfrac{1}{24}\left((1{+}\tfrac{1}{9}\omega_6)b^{k+5} + p(k{+}6)\right)$$

and their reversals from the end of the curve

$$P1'(k) = b^k - P1(k), \quad P4'(k) = b^k - P4(k), \quad P6'(k) = b^k - P6(k),$$
$$P2'(k) = b^k - P2(k), \quad P5'(k) = b^k - P5(k), \quad P7'(k) = b^k - P7(k)$$
$$P3'(k) = b^k - P3(k),$$

where periodic term

$$p(m) = [\,-9, \qquad 6{+}15\omega_3, \qquad -3{-}3\omega_3, \qquad -3{-}6\omega_3,$$
$$-9\omega_3, \quad (6{+}15\omega_3)\omega_3, \quad (-3{-}3\omega_3)\omega_3, \quad (-3{-}6\omega_3)\omega_3,$$
$$-9\omega_3^2, \quad (6{+}15\omega_3)\omega_3^2, \quad (-3{-}3\omega_3)\omega_3^2, \quad (-3{-}6\omega_3)\omega_3^2\,]$$
$$\text{for } m \equiv 0 \text{ to } 11 \bmod 12$$

Sides P1–P2 through P6–P7 are at successive +30° angles as illustrated in figure 20. Side P6–P7 is the same as P2–P3 but turned +120°. And likewise reversals P1' etc.

For $k < 6$ the above points are the hull vertices but with some duplications and some points excluded.

k	vertices	duplication	exclude
0	2	$P3{=}P4{=}P5{=}P6'$ *(P1 on boundary)*	$P2, P7$
1	4	$P2{=}P3{=}P4$ *and* $P5{=}P6{=}P7$	$P1$

2	*6*	*P1=P2=P3 and P4=P5 and P6=P7*
3	*8*	*P1=P2 and P3=P4 (P7 on boundary)*
4	*10*	*P2=P3 and P6=P7*
5	*12*	*P1=P2*

Proof. For $k = 0$ to 5 the convex hulls can be formed explicitly. For $k=0$ the hull is merely a line 0 to 1. $P1 = \frac{1}{3}$ is on that line but not a vertex. For $k=3$ point $P7 = P6 + \omega_3$ is on the boundary but not a vertex.
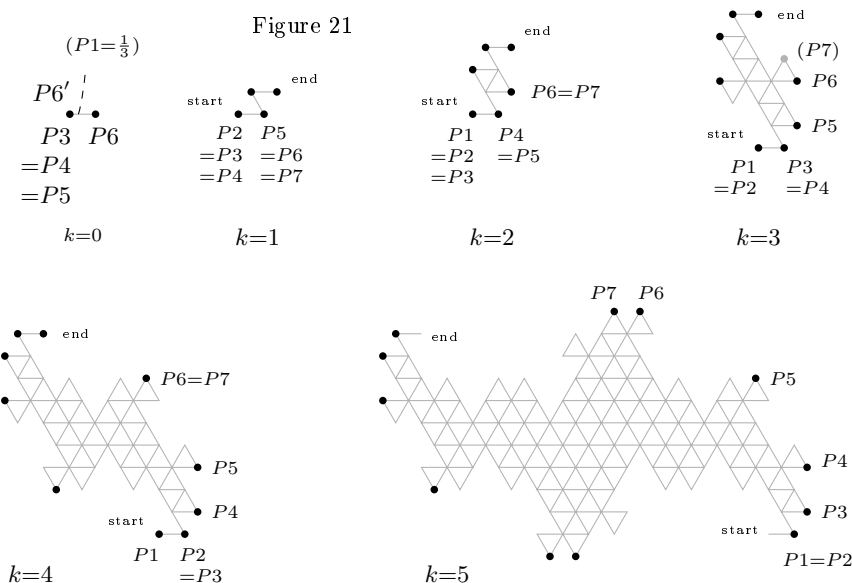
Figure 21

Side P1–P2 is at $60°$ relative to the b^k endpoint since

$$\frac{P1(k) - P2(k)}{b^{k+2}} = \tfrac{1}{72} + \tfrac{1}{24}\,\frac{p(k+1) - p(k)}{b^{k+2}}$$

and the periodic values of $p(m)$ have difference $p(k+1) - p(k)$ which is always aligned to the b^{k+2} direction. These p differences can be illustrated
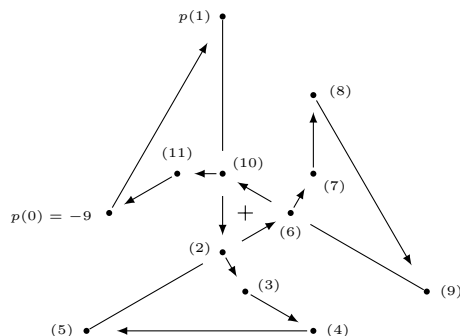
Figure 22: $p(m)$ steps

$p(0)$ to $p(1)$ at the top left is $60°$ since $p(1){-}p(0) = 15\omega_6$, corresponding to b^2. At each point the direction turns $+30°$ the same as $\arg b = 30°$. At $m = 0, 1, 4, 6, 8, 9$ there is an additional reversal $180°$ but still $+30°$.

Similarly the other sides P2–P3 aligned to b^{k+3} etc through P6–P7 aligned to b^{k+7}.

The sides P2–P3 and P6–P7 are the same length but turned $+120°$ since, using $b^4 = 9\omega_3$ and $p(m+4) = \omega_3\,p(m)$,

$$P2(k) - P3(k) \;=\; \omega_3\big(P6(k) - P7(k)\big)$$

For the vertex formulas, proceed by induction. Suppose the formulas are true of $k{-}1$. Terdragon k comprises three $k{-}1$. The convex hull around k is the hull around the hulls of the three sub-parts.

The expansion is shown in the following diagram. 0 is the origin. b^k is the endpoint of level k. The three sub-parts are A,B,C and their vertices are labelled $P1A$, $P1B$, $P1C$ etc.
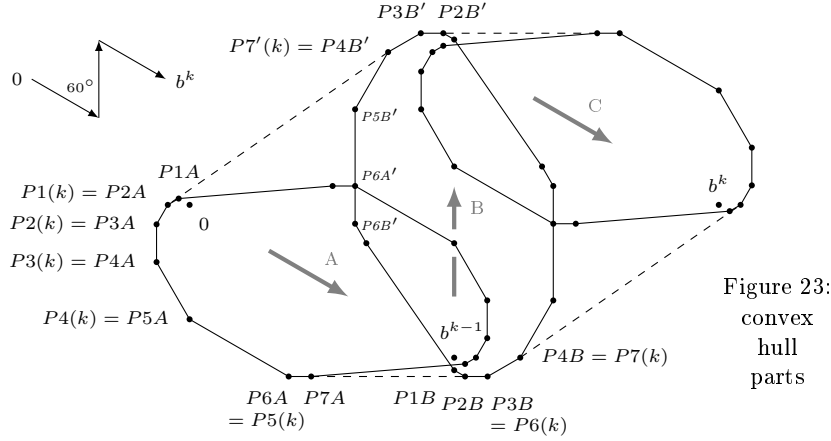
Figure 23: convex hull parts

For the dashed bottom side, both P6A–P7A and P2B–P3B are horizontal (aligned to the b^k endpoint) as per the side angles above and the respective A and B parts turned $-30°$ and $+90°$. They are at the same position vertically since, with $p(k+5) - p(k+4)$ aligned to b^k (the bottom horizontal $p(4)$ to $p(5)$ in figure 22),

$$\operatorname{Im}\frac{P6A - P3B}{b^k} = \operatorname{Im}\left(-\tfrac{3}{8} + \tfrac{1}{24}\frac{p(k+5) - p(k+4)}{b^k}\right) \;=\; 0$$

So the hull is $P5(k)$ at P6A across to $P6(k)$ at P3B.

For the dashed top left P2A–P4B′, the sub-part sides P1A–P2A and P3B′–P4B′ are both $60°$ per the side angles. But P2A–P4B′ is steeper than $60°$ since

$$\operatorname{Im}\frac{P2A - P4B'}{b^{k+2}} = \operatorname{Im}\left(-\tfrac{1}{9} + \tfrac{1}{12}\omega_3 - \tfrac{1}{72}\frac{p(k+10) + p(k+4)}{b^k}\right)$$

$$= \tfrac{1}{24}\sqrt{3}\left(1 - (-\tfrac{1}{3})^{\lceil k/2\rceil}\right) \;>\; 0 \qquad \text{for } k \geq 7$$

So P1A is inside the hull and $P1(k)$ is at P2A. Likewise at the top $P7'(k)$ is at P4B′. The side P1A–P2A is quite short so a little difficult to see in figure 23.

The other new sides are the same rotated $180°$.

So mutual recurrences for the vertices

$$P1(k) = P2(k-1) \qquad P5(k) = P6(k-1)$$
$$P2(k) = P3(k-1) \qquad P6(k) = b^{k-1} + \omega_3 P3(k-1)$$
$$P3(k) = P4(k-1) \qquad P7(k) = b^{k-1} + \omega_3 P4(k-1)$$
$$P4(k) = P5(k-1)$$

The power forms (101) of the theorem satisfy these recurrences starting from an initial $k{=}6$ hull calculated explicitly, which completes the induction. The power forms can be found by writing the recurrences in generating functions and solving simultaneously by some linear algebra or solving directly by expanding. The chain of dependencies is

$$\text{P1} \longrightarrow \text{P2} \longrightarrow \text{P3} \longrightarrow \text{P4} \longleftarrow \text{P7}$$
$$\uparrow \qquad\qquad \downarrow$$
$$\text{P6} \longleftarrow \text{P5}$$
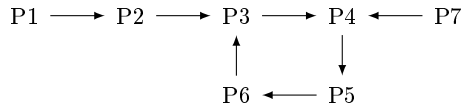
Starting at $P3(k)$ and expanding to reach $P3(k{-}4)$ again,

$$P3(k) = b^{k-4} + \omega_3 P3(k-4)$$

Apply this repeatedly until reaching $k = 6, 7, 8$ or 9. Let this be $q \geq 0$ many times so that $k{-}6 = 4q + r$ with $0 \leq r \leq 3$ so ending at $P1(6{+}r)$.

$$P3(k) = b^{k-4} + \omega_3 b^{k-8} \cdots + \omega_3^{q-1} b^{k-4-4(q-1)} + \omega_3^q P3(6{+}r)$$
$$= \omega_3^q b^{r+6} \frac{(b^4)^q - \omega_3^q}{b^4 - \omega_3} + \omega_3^q P3(6{+}r)$$
$$= -\tfrac{1}{24}\left(b^{k+2} - b^{r+8}\omega_3^q - 24\omega_3^q P3(6{+}r) \right) \qquad (102)$$
$$\text{using } b^{-2}/(b^4{-}\omega_3) = -\tfrac{1}{24}$$

In (102) the right hand terms are periodic in $r = 0, 1, 2, 3$ and $q = 0, 1, 2$. It uses the initial $P3(6)$ through $P3(9)$ which are calculated from the recurrences or by explicitly forming those hulls. The result is the 12 terms of $p(m)$.

$p(m)$ could be numbered starting anywhere mod 12. The choice here is to match the b power in each $P1$ etc. So the expression in (102) is reckoned as $p(k{+}2)$ to match its b^{k+2}.

$$p(k{+}2) = -b^{r+8}\omega_3^q - 24\omega_3^q P3(6{+}r) \qquad\qquad \square$$

In figure 23, the A sub-part vertex $P6A'$ is close to the B sub-part vertical $P5B'$ to $P6B'$. The vertex is on the line for $k \equiv 0, 2, 3 \bmod 4$ but is 1 unit triangle to the left when $k \equiv 1 \bmod 4$.

$$P6A'(k) = P6'(k-1) \qquad P6B'(k) = b^{k-1} + \omega_3 P6'(k-1)$$
$$\text{Re} \frac{P6A'(k) - P6B'(k)}{\omega_{12}{}^k} = \text{Re} \, \tfrac{1}{24} \frac{p(k{+}4) - \omega_3 p(k{+}4)}{\omega_{12}{}^k}$$
$$= \begin{cases} 0 & \text{if } k \equiv 0, 2, 3 \bmod 4 \\ -\tfrac{1}{2}\sqrt{3} & \text{if } k \equiv 1 \quad \bmod 4 \end{cases}$$

P5B′

$0 \longrightarrow b^k$

$k{=}9$
sub-part vertex P6A′
1 unit triangle away
from P5B′–P6B′ vertical

P6A′

P6B′
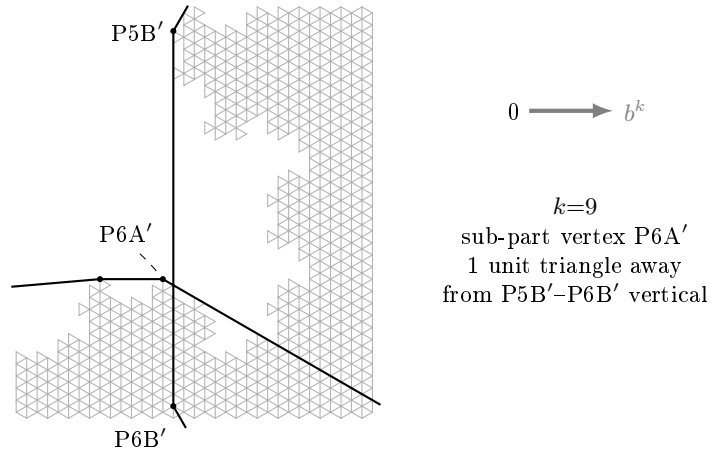
For the curve scaled to a unit length, the limits for the hull vertex locations are the coefficients of the b^k terms in each P1 etc. They and the resulting hull extents are

P7f′ P6f′ P5f′

P4f′
P3f′
P1f = −1/24   end, 1   P2f′
P2f = −b/24   P1f′
P3f = −b²/24   0, start
P4f = −b³/24
                  $(-b^5+b)/24$ = P7f
P5f = −b⁴/24   −b⁵/24 = P6f

$\frac{3}{16}\sqrt{3} = 0.324759\ldots$
(A212952)

$\frac{3}{16}\sqrt{3}$

$\frac{1}{16}$ 1 $\frac{1}{16}$
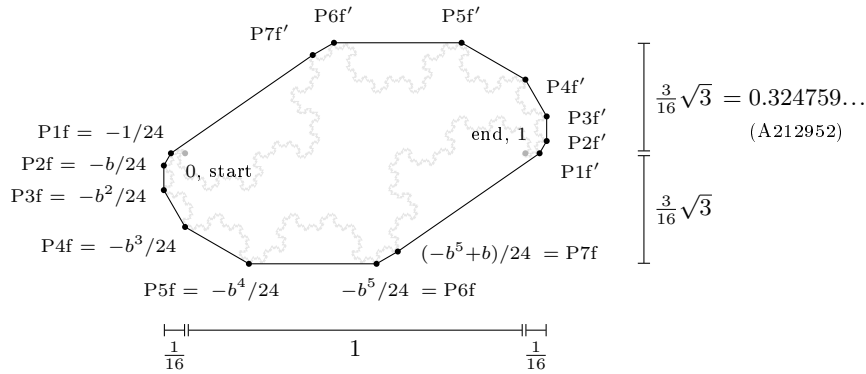
Each hull vertex is a single-visited point since a double or triple has 4 or 6 segments around it so is not a convex vertex. Point numbers n in the curve for each hull vertex follow from the sub-parts similar to the vertex locations.

$$PN1(k) = PN2(k{-}1) \qquad PN5(k) = PN6(k{-}1)$$
$$PN2(k) = PN3(k{-}1) \qquad PN6(k) = 3^{k-1} + PN3(k{-}1)$$
$$PN3(k) = PN4(k{-}1) \qquad PN7(k) = 3^{k-1} + PN4(k{-}1)$$
$$PN4(k) = PN5(k{-}1)$$

P3B and P4B are the middle sub-part (ternary digit 1) so add 3^{k-1} in PN6 and PN7. Initial values at $k{=}6$ determine the low digits and then the 4-cycle P3–P4–P5–P6 is a high repeating pattern 1000. It's convenient to take that pattern as high 1 then repeat 0001 zero or more times, so as to simplify the low digit forms.

$$PN1(k) = \tfrac{1}{720}3^k + \tfrac{1}{80}[-9, 53, -1, -3]$$
$$= \text{ternary } 1\ 0001\ 0001\ldots \text{ empty}, 0, 00 \text{ or } 001 \quad \text{for } k{-}5 \text{ digits}$$

$$= 1, 3, 9, 28, 82, 246, 738, 2215, 6643, \ldots \qquad k \geq 6$$

$$PN2(k) = PN1(k{+}1) \qquad PN4(k) = PN1(k{+}3) \qquad PN6(k) = PN1(k{+}5)$$

$$PN3(k) = PN1(k{+}2) \qquad PN5(k) = PN1(k{+}4)$$

$$PN7(k) = \tfrac{83}{240}3^k + \tfrac{1}{80}[-1, -3, -9, 53]$$

$$= \text{ternary } 1001\ 0001\ 0001\ldots \ \text{ empty}, 0, 00 \text{ or } 001 \quad \text{for } k \text{ digits}$$

$$= 252, 757, 2269, 6807, 20421, 61264, \ldots \qquad k \geq 6$$

In *PN7*, the 3^{k-1} high ternary 1 digit is only 3 places above the rest of $PN4(k{-}1)$ so an initial 100 before the 1000 pattern.

The area of the hull can be calculated by taking triangular sectors from origin to consecutive points $P1, P2$ etc in the usual way. The area of such a sector is $\frac{1}{2}\operatorname{Im}\left(z_1.\overline{z_2}\right)$, for z_1 to z_2 anti-clockwise around. It's convenient to measure the area as number of unit triangle equivalents (the area of each such being $\sqrt{3}/4$). This corresponds to curve area A measured in unit triangles (theorem 23).

$$HA_k = \begin{cases} 0, 2, & \text{if } k = 0, 1 \\ \tfrac{29}{24}3^k - \tfrac{1}{12}[15, 23, 11, 25].3^{\lfloor k/2 \rfloor} - \tfrac{1}{8}[5, 3, 1, 3] & \text{if } k \geq 2 \end{cases} \qquad (103)$$

$$= 0, 2, 8, 26, 86, 276, 856, 2586, \ldots$$

The area of hulls $k = 0, 1$ can be calculated explicitly. $k{=}1$ is a rhombus comprising 2 unit triangles, as seen in figure 21. For $2 \leq k \leq 5$, the duplications and extra vertices on the hull boundary give empty or split sectors but the general formula found from the vertex formulas for $k \geq 6$ still holds.

Scaled by 3^k for curve start to end a unit length, the hull area limit, as a multiple of unit triangles, is the coefficient of the 3^k term in (103).

$$\frac{HA_k}{3^k} \to \frac{29}{24} = 1.208333\ldots \quad \text{unit triangles limit} \qquad \text{1+ A212832}$$

This can be compared to the limit number of triangles enclosed by the curve $A_k/3^k \to \frac{2}{3} = 0.666\ldots$

Then with $\sqrt{3}/4$ unit triangle area, the area of the hull around the fractal is

$$HAf = \frac{29}{24} \cdot \frac{\sqrt{3}}{4} = 0.523223\ldots \quad \text{hull area limit}$$

The hull boundary length is the total side lengths

$$HB_k = \left| P1(k){-}P2(k) \right| + \left| P3(k){-}P2(k) \right| + \cdots$$

$$= \begin{cases} 2, 4 & \text{if } k = 0, 1 \\ \left(\tfrac{13}{12} + \tfrac{5}{12}\sqrt{3}\right)\sqrt{3}^k & \text{if } k \geq 2 \\ + \tfrac{1}{6}\sqrt{37.3^k + [-30, 162, 30, -162].3^{\lfloor k/2 \rfloor} + [9, 63]} \\ + \left[\tfrac{9}{4} - \tfrac{7}{4}\sqrt{3}, \ \tfrac{3}{4} - \tfrac{7}{4}\sqrt{3}, \ \tfrac{3}{4} - \tfrac{5}{4}\sqrt{3}, \ \tfrac{9}{4} - \tfrac{5}{4}\sqrt{3}\right] \end{cases}$$

$$= 2, \ 4, \ 4{+}2\sqrt{3}, \ 10{+}2\sqrt{3}, \ 12{+}2\sqrt{3}{+}2\sqrt{19}, \ 12{+}8\sqrt{3}{+}2\sqrt{73}, \ldots$$

The middle root term $37.3^k + \cdots$ is from sides $P7$–$P1'$ and $P7'$–$P1$ which are not at $30°$ angles.

Scaled down by $\sqrt{3}^k$ for the curve a unit length, the hull boundary length limit is

$$\frac{HB_k}{\sqrt{3}^k} \rightarrow HBf = \frac{13}{12} + \frac{5}{12}\sqrt{3} + \frac{1}{6}\sqrt{37} = 2.818814\ldots$$

hull boundary
side length
limits
(and top sides
repeating these),
total HBf
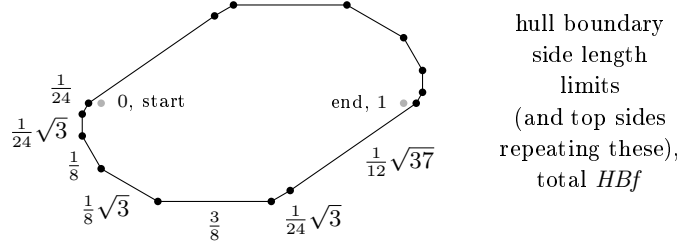
Theorem 38. *The two points of the terdragon curve furthest apart are $P3$ and $P3'$ of the convex hull. For curve k they are at a distance*

$$HD_k = \sqrt[2]{\left(\frac{21}{16}3^k - \frac{1}{8}[3,9,9,15].3^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{16}[1,3,9,19] \right)} \qquad (104)$$

$$= \sqrt[2]{1,3,9,31,103,309,927,2821,\ldots}$$

Proof. The points furthest apart must be vertices of the convex hull. For $k < 9$ the furthest points can be verified explicitly and their distances apart are per the formula.

For $k \geq 9$, points $P1$ through $P7'$ of the convex hull are located at various factors of b^k and offsets $p(m)$ from those powers. The offsets are at most

$$pmax = \max(\tfrac{1}{24}|p(m)|) = \tfrac{1}{8}\sqrt{19}$$

Comparing factors of b^k on the hull vertices, P3–P3' are the furthest apart. Their distance is at least

$$\left| P3(k) - P3'(k) \right| \geq \left| b^k + 2\tfrac{1}{24}b^{k+2} \right| - 2pmax = \tfrac{1}{4}\sqrt{21}.\sqrt{3}^k - 2pmax$$

The second furthest by b^k factors is P2–P2' and their distance, or the distance of any pair with smaller b^k factor, is at most

$$\left| P2(k) - P2'(k) \right| \leq \left| b^k + 2\tfrac{1}{24}b^{k+1} \right| + 2pmax = \tfrac{1}{4}\sqrt{\tfrac{61}{3}}.\sqrt{3}^k + 2pmax$$

For $k \geq 9$ the difference between the two bounds is positive, as seen by decreasing and increasing terms to convenient squares,

$$\left(\tfrac{1}{4}\sqrt{21} - \tfrac{1}{4}\sqrt{\tfrac{61}{3}} \right)\sqrt{3}^9 - 4pmax \qquad\qquad k \geq 9$$

$$> \left(\tfrac{1}{4}\sqrt{\tfrac{458^2}{10000}} - \tfrac{1}{4}\sqrt{\tfrac{451^2}{10000}} \right).140 - 4.\tfrac{1}{8}\sqrt{\tfrac{436^2}{10000}} = \tfrac{27}{100} > 0 \qquad \square$$

Scaled by $\sqrt{3}^k$ for start to end a unit length, the distance is square root of the coefficient of the 3^k term in (104).

curve points
furthest apart,
distance

$$\frac{HD_k}{\sqrt{3^k}} \to HDf = \frac{\sqrt{21}}{4} = 1.145643\ldots$$
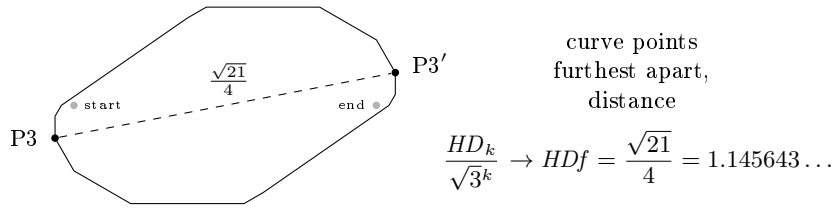
HD is between any two points of the curve. It's also possible to consider only points on some line parallel to curve start to end.

Theorem 39. *Consider a line parallel to curve start to end, and the points of the curve which may be on it. The greatest distance between two points on any such line is uniquely attained between the following locations P1S and P1S′*

$$P1S_k = \begin{cases} 0 & \text{if } k = 0,1 \\ P1(k) + \omega_{12}^{k+1} & \text{if } k \equiv 1 \bmod 4 \text{ and } k \geq 5 \\ P1(k) & \text{otherwise} \end{cases}$$
$$P1S'_k = b^k - P1S_k$$

They are on the curve centreline through curve start and end. Their distance apart is

$$HSD_k = |P1S_k - P1S'_k|$$
$$= \begin{cases} 1, \sqrt{3} & \text{if } k = 0,1 \\ \frac{13}{12}\sqrt{3^k} - [\frac{3}{4}, \frac{3}{4}\sqrt{3}, \frac{1}{4}, \frac{1}{4}\sqrt{3}] & \text{if } k \geq 1 \end{cases} \qquad (105)$$
$$= 1, \sqrt{3}, 3, 3\sqrt{3}, 9, 9\sqrt{3}, 29, 29\sqrt{3}, 87, 87\sqrt{3}, \ldots$$

For $k = 2m$, $HSD_k = $ ternary $10\,02\,02\ldots$ with $m+1$ digits, and for $k = 2m+1$ the same with further factor $\sqrt{3}$.

Proof. Greatest distances can be verified explicitly for $k \leq 6$. For $k \geq 7$, hull vertex P1 is on the start to end line when $k \not\equiv 1 \bmod 4$ since its formula is

$$\text{Im } P1(k)/\omega_{12}^k = [0, -\tfrac{1}{2}, 0, 0]$$

Hull side P1–P2 is at 60° to the line start to end and side P1–P7′ is at less than 60°, so any parallel line points away from P1 are shorter than P1 to P1′.

For $k \equiv 1 \bmod 4$, in the $k-1$ hulls of figure 23, P1 = P2A has adjacent sides 60° and 30° so that anywhere other the overlap arising from $\text{Im } P1(k) = -\tfrac{1}{2}$ is shorter.
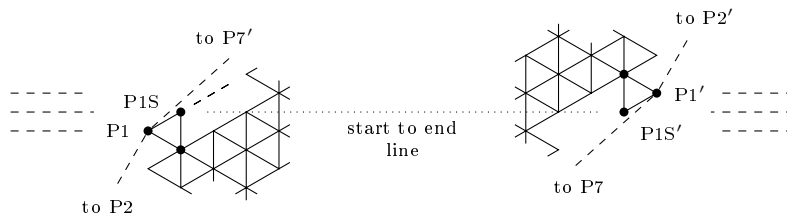
P1 is at 30° down from P1S. Distance P1S to P1S′ could be equalled by P1 to a point below left 30° from P1S′. Or likewise from P1′ to a point above right 30° of P1S. But these points are not in the curve. They are not in $k=9$ and thereafter the P1–P1S segment expands 4 times as follows for new $P1(k+4)$ also without point above right.
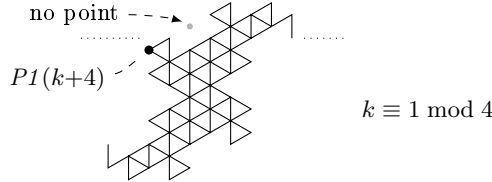
For the curve scaled to a unit length, the limit is the distance P1 to P1′ which is the coefficient of $\sqrt{3}^k$ in (105),

$$\frac{HSD(k)}{b^k} \to \frac{13}{12}$$

A maximum distance between two points on a line perpendicular to start to end is the corresponding points in the middle third of the curve, which is $P1S(k-1)$ and $P1S'(k-1)$ in the $k-1$ middle part of figure 23. These points are not on the whole curve hull boundary. Their width limit is simply $/\sqrt{3}$,

$$\frac{HSD(k-1)}{b^k} \to \frac{13\sqrt{3}}{36} = 0.625462\ldots$$

10.1 Middle Nearest

Theorem 40. *The left boundary point or points nearest to the terdragon middle* $\frac{1}{2}b^k$ *are located at*

$$Lnear_k = \begin{cases} 0 \text{ and } 1 & \text{if } k=0 \\ 1 \text{ and } \frac{1}{2}+\frac{1}{2}\sqrt{3}i & \text{if } k=1 \\ \frac{1}{2}+\frac{1}{2}\sqrt{3}i \text{ and } 1+\sqrt{3}i & \text{if } k=2 \\ \dfrac{19+\sqrt{3}i}{48}\,b^k + \frac{1}{24}pt(k+1) & \text{if } k \geq 3 \\ \quad \text{and when } k=5 \text{ also equal nearest } -\frac{11}{2}+\frac{3}{2}\sqrt{3}i \end{cases}$$

where

$$pt(m) = [\,15, \qquad 6-9\omega_3, \qquad -3-27\omega_3, \qquad -3+18\omega_3, \qquad (106)$$
$$15\omega_3, \quad (6-9\omega_3)\omega_3, \quad (-3-27\omega_3)\omega_3, \quad (-3+18\omega_3)\omega_3,$$
$$15\omega_3^2, \quad (6-9\omega_3)\omega_3^2, \quad (-3-27\omega_3)\omega_3^2, \quad (-3+18\omega_3)\omega_3^2\,]$$

By symmetry the right boundary point nearest the middle is

$$Rnear_k = b^k - Lnear_k$$

Proof. For $k \leq 5$ the points nearest the middle can be calculated explicitly.

For $k \geq 6$, boundary points correspond to corners of triangles on the boundary of surrounding curves. Form the convex hull around segments plus boundary

triangles. This can be calculated the same as the segments hull in theorem 37, since the curve with boundary triangles is an unfold of sub-curves $k-1$ and their boundary triangles. For $k \geq 6$ there are 14 vertices (like the segments hull).

$k = 6$
convex hull
segments and
boundary triangles
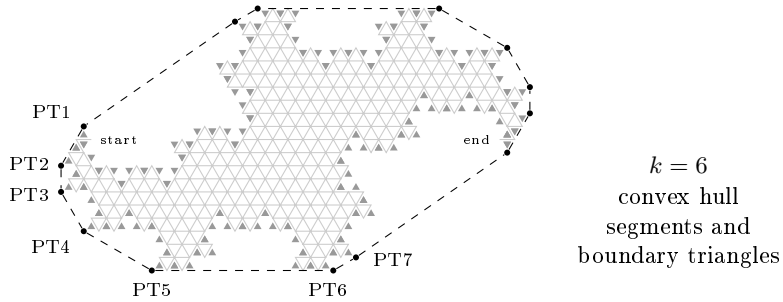
The boundary triangles push the segments hull vertices out by 1 unit triangle on each straight side. The triangles hull vertices are at corners of a triangle, since a curve point would have triangles each side of it and so not be a hull vertex.

Working through the hull recurrences the result is the same location forms as segments P1 etc (101), but different offset terms. Each p in P1 etc becomes pt at (106) in PT1 etc.

$$PT1(k) = -\tfrac{1}{24}\left(b^k + pt(k)\right) \quad \text{etc}$$

Consider then curve k comprising $k-2$ sub-curves and surrounding $k-2$ sub-curves. The triangle hulls around those surrounding sub-curves are

Figure 24:
$k-2$ hulls
surrounding
left side
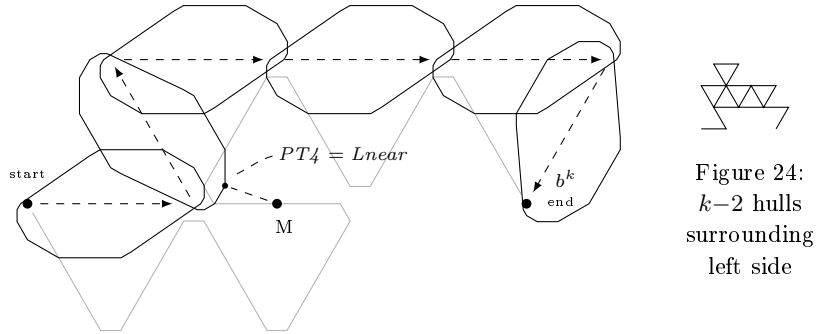
The boundary triangles push into the left boundary L so that minimum extents for the left boundary points are given by maximum extents of the surrounding hulls.

The claimed $Lnear$ is the marked $PT4$ in figure 24, being $PT4$ in that surrounding $k-2$ hull. Its sub-curve starts at $\tfrac{1}{3}b^k$. Its sub-curve endpoint b^{k-2} is directed $-60°$ relative to the b^k end. So $+120°$ direction in figure 24 is total turn $180°$ so negate,

$$Lnear_k = \tfrac{1}{3}b^k - PT4(k-2)$$

Working through the hull formulas it can be verified that this is nearer than the other hulls, and that the slopes of the sides adjacent to $PT4$ are more than $90°$ to a line $M-PT4$ so that nothing else in the surrounding hull is nearer. $\square$

The offsets in pt can be illustrated

$$pt(0) = 15$$
$$pt(1) = 6{-}9\omega_3$$
$$pt(2)$$
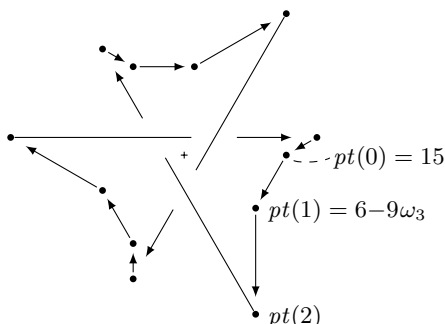
The difference between p and pt is effectively which sides are pushed out by the boundary triangles in the way noted above.

$$\tfrac{1}{24}(pt(k) - p(k)) = [1, \ -\omega_3, -\omega_3, \omega_3, \omega_3, \ -\omega_3^2, -\omega_3^2, \omega_3^2, \omega_3^2, \ -1, -1, 1]$$
$$= \omega_3^{\lfloor (k+3)/4 \rfloor} \cdot (-1)^{\lfloor (k+1)/2 \rfloor}$$

For endpoints scaled to a unit length, the limits for $Lnear$ and $Rnear$ are their b^k coefficients.

$$\frac{Lnear_k}{b^k} \to Lnearf = \frac{19{+}\sqrt{3}i}{48} = \frac{10{+}\omega_3}{24} = 0.3958333... + 0.036084...i \quad (107)$$

$$\frac{Rnear_k}{b^k} \to Rnearf = \frac{29{-}\sqrt{3}i}{48} = \frac{14{-}\omega_3}{24} = 0.6041666... - 0.036084...i$$

A line between $Lnearf$ and $Rnearf$ is the narrowest part through the middle. The length of that line and the angle down from the curve start to end are

$$|Rnearf - Lnear| \ = \tfrac{1}{12}\sqrt{7} \ = 0.220479\ldots$$
$$\arg(Rnearf - Lnear) = -\arctan \tfrac{1}{5}\sqrt{3} = -19.106605° \ldots$$

$Lnearf$
$= \frac{19}{48} + \frac{\sqrt{3}}{48}i$

start

$\frac{1}{12}\sqrt{7}$

end

$Rnearf = \frac{29}{48} - \frac{\sqrt{3}}{48}i$

$-19.1°$

$Lnear, Rnear$ limits, width and angle
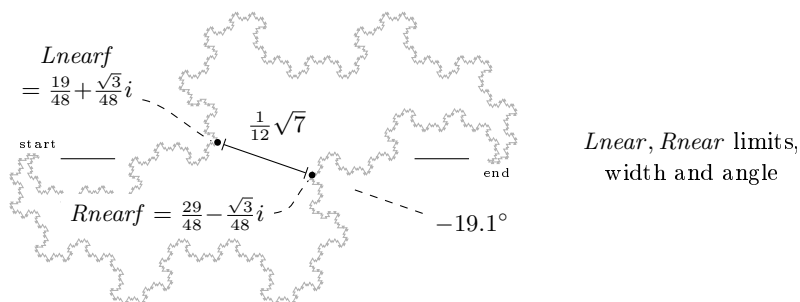
Theorem 41. *For two curves at 60°, the point or points of the left boundary which are nearest to the join are*

$$
Jnear_k = \begin{cases}
1 & \textit{if } k=0 \\
1 \;\; and \;\; \omega_6 & \textit{if } k=1 \\
\sqrt{3}i & \textit{if } k=2 \\
JnearPT2_k \;\; and \;\; JnearPT2_k - \bar{b} & \textit{if } k=3 \\
JnearPT2_k \;\; and \;\; JnearPT2_k - \omega_6 & \textit{if } k=6 \\
JnearPT2_k & \textit{otherwise}
\end{cases}
$$

$$
JnearPT2_k = \omega_6 \left(b^{k-1} + \omega_3 PT2(k{-}1) \right)
$$
$$
= \left(\tfrac{13}{24} + \tfrac{1}{6}\sqrt{3}i \right) b^k + \tfrac{1}{24} pt(k) \tag{108}
$$

Proof. The nearest points for $k \leq 6$ can be calculated explicitly.
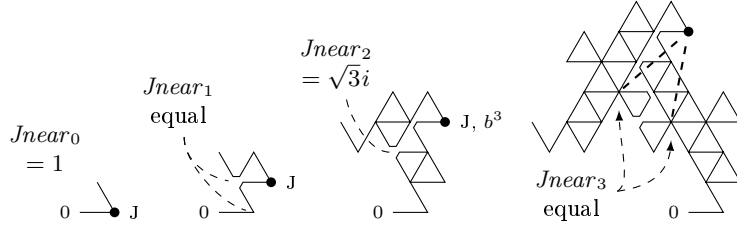
The $k{-}1$ triangle hulls of the absent left k side are
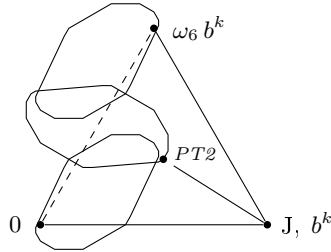
Figure 25: absent $k{-}1$ hulls

Working through the formulas shows the nearest to J is *PT2* of the middle hull. Its adjacent sides are 30° before and 60° after which are past 90° perpendicular to the line from J, so other points are further away. □

The limit for join of curves scaled to unit lengths is the b^k coefficient in (108).

$$
\frac{Jnear_k}{b^k} \to Jnearf = \tfrac{13}{24} + \tfrac{1}{6}\sqrt{3}i \;=\; \tfrac{17}{24} + \tfrac{1}{3}\omega_3 \tag{109}
$$
$$
= 0.541666... + 0.288675...i \qquad \text{\small imag A020769}
$$

$$|Jnearf| = \tfrac{1}{24}\sqrt{217}$$
$$= 0.613788$$

$$|1 - Jnearf| = \frac{13}{24} = 0.541666$$

0   1

$$\arg Jnearf = \arctan \tfrac{4}{13}\sqrt{3}$$
$$= 28.054880...^{\circ}$$

$$\arg\left(1 - Jnearf\right) = \arctan \tfrac{4}{11}\sqrt{3}$$
$$= 32.204227...^{\circ}$$
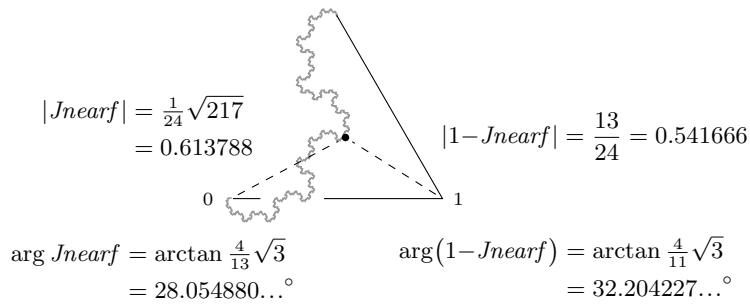
$Jnearf$ is located $+\tfrac{1}{24}$ right of the middle $b/3 = \tfrac{1}{2} + \tfrac{1}{6}\sqrt{3}i$. The middle is a double-visited endpoint of sub-curves. The absent third visit sub-curves are the first two hulls in figure 25. They spiral around the middle, as all curve ends do, giving boundary points which closer to J than the middle is.

$Jnearf - 1$ is the narrowest part through the middle of 6 arm plane filling per section 7 (and by symmetry the same at successive 60°).

ω_3 ω_6

$$Jnearf - 1 = -\tfrac{11}{48} + \tfrac{\sqrt{3}}{6}i$$

-1 1

$$-32.2°$$

$$2\,|1 - Jnear| = \tfrac{13}{12} \text{ across}$$
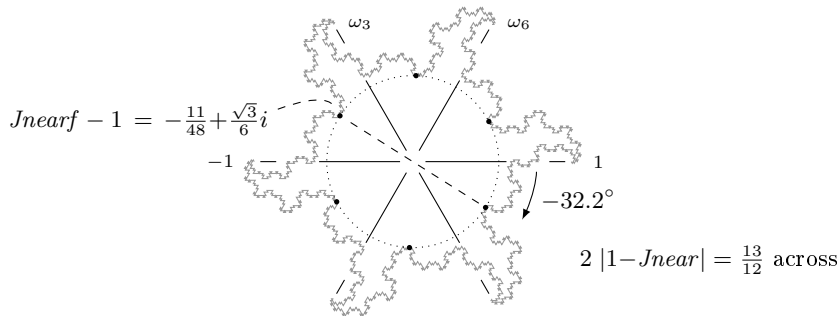
10.2   Minimum Area Rectangle

P7

P4

P1′

end   start

P1

P4′

P7′
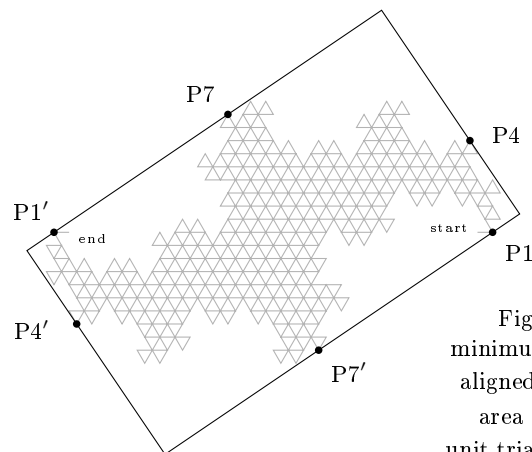
Figure 26: $k{=}6$
minimum area rectangle
aligned to side P7–P1′
area $MR_6 = \frac{205407}{193}$
unit triangles equivalents

Theorem 42. *The minimum-area rectangle around terdragon level k has area MR_k unit triangle equivalents ($\sqrt{3}/4$ each),*

$$MR_k = \begin{cases} 0, 3, 9, 33 & \text{if } k = 0 \text{ to } 3 \\ \dfrac{MrW_k . MrH_k}{MrDen_k} & \text{if } k \geq 4 \end{cases} \tag{110}$$

$$= 0, 3, 9, 33, \tfrac{2187}{19}, \tfrac{25392}{73}, \tfrac{205407}{193}, \ldots$$

where

$$MrW_k = \tfrac{13}{12}3^k + \tfrac{1}{12}[-9, 18, -1, -30].3^{\lfloor k/2 \rfloor} + \tfrac{1}{4}[0, -3, -2, 1]$$
$$= 81, 276, 787, 2302, 7047, 21444, \ldots \quad k \geq 4$$

$$MrH_k = \tfrac{13}{36}3^k + \tfrac{1}{12}[-3, 6, -1, -16].3^{\lfloor k/2 \rfloor} + \tfrac{1}{4}[0, -1, 0, 1]$$
$$= 27, 92, 261, 754, 2349, 7148, \ldots \quad k \geq 4$$

$$MrDen_k = |P7(k) - P1'(k)|^2$$
$$= \tfrac{37}{144}3^k + \tfrac{1}{24}[-5, 27, 5, -27].3^{\lfloor k/2 \rfloor} + \tfrac{1}{16}[1, 7]$$
$$= 19, 73, 193, 532, 1669, 5149, \ldots \quad k \geq 4$$

For $k \geq 4$, the rectangle is aligned to the side $P7$–$P1'$. For $k = 1$ to 3, it is aligned $+30°$ to the curve endpoint. For $k=0$ the curve is a line segment and the minimum rectangle is trivially aligned to that segment.

Proof. A minimum area rectangle has at least one side aligned to a side of the convex hull, so it suffices to consider rectangles on the hull sides.

For $k=0$, the curve is a unit line segment with area $MR_0 = 0$.

For $k=1$, the two rectangle alignments both have area $MR_1 = 3$ triangles.

$$\begin{array}{l} k = 1 \\ MR_1 = 3 \end{array}$$ area $\tfrac{3}{2} \times \tfrac{1}{2}\sqrt{3}$ area $\tfrac{1}{2}\sqrt{3} \times \tfrac{3}{2} = 3\tfrac{\sqrt{3}}{4}$

For $k=2$ and $k=3$, the possible alignments and areas are as follows. In each case the first is the minimum and is per the general formula.

$$\begin{array}{l} k = 2 \\ MR_2 = 9 \end{array}$$ area $\tfrac{3}{2} \times \tfrac{3}{2}\sqrt{3}$ area $\sqrt{3} \times 3 = 12\tfrac{\sqrt{3}}{4}$

$$\begin{array}{l} k = 3 \\ MR_3 = 33 \end{array}$$

$\tfrac{11}{2} \times \tfrac{3}{2}\sqrt{3}$ $= 33\tfrac{\sqrt{3}}{4}$ $3 \times 3\sqrt{3}$ $= 36\tfrac{\sqrt{3}}{4}$ $\tfrac{5}{2}\sqrt{3} \times \tfrac{9}{2} = 45\tfrac{\sqrt{3}}{4}$

For $k \geq 4$, the hull vertices $P1(k)$ through $P7(k)$ from the convex hull theorem 37 can be used, allowing for repetitions which occur in for $k=4$ and $k=5$.

There are 7 sides (and 180° reversals). The first 6 are 30° turns which means 90° after the first 3, so total 4 distinct rectangle alignments.

A rectangle aligned $-30°$ to the b^k endpoint, which is the P1–P2 and P4–P5 sides for $k \geq 6$, is
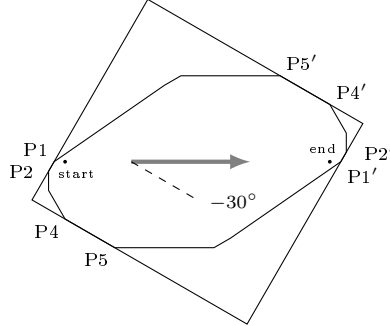
It's convenient to divide by b^{k+2} for the alignment and factor $3^{k+2} = |b^{k+2}|^2$ to scale back up to unit segments. P2–P2$'$ and P5–P5$'$ are suitable rectangle extents for $k \geq 1$. So, measured in unit triangles,

$$MR12(k) = 3^{k+2} . \operatorname{Re} \frac{P5'(k) - P5(k)}{b^{k+2}} . \operatorname{Im} \frac{P2(k) - P2'(k)}{b^{k+2}} \Big/ \frac{\sqrt{3}}{4} \qquad k \geq 1$$

To allow any k, for completeness, $P2$ is used rather than $P1$ since $P1(1)$ is not on the hull boundary, though actually its extents are the same as $P2$ there. Then with a $-30°$ hull explicitly calculated around the $k{=}0$ line segment,

$$MR12(k) = \begin{cases} 1 & \text{if } k{=}0 \\ \frac{91}{48} 3^k - \frac{1}{24}[51, 69, 17, 51].3^{\lfloor k/2 \rfloor} + \frac{1}{16}[9, 3, 1, 3] & \text{if } k \geq 1 \end{cases}$$
$$= 1, 3, 15, 45, 135, 435, \ldots \qquad k \geq 0$$

A rectangle aligned to the b^k curve endpoint, which is sides P2–P3 and P5–P6 sides for $k \geq 5$, is
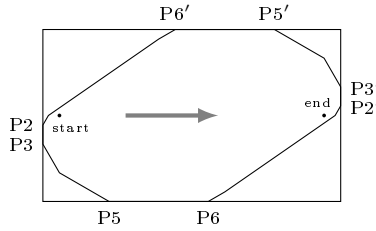
$$MR23(k) = 3^k . \operatorname{Re} \frac{P3(k) - P3'(k)}{b^k} . \operatorname{Im} \frac{P5(k) - P5'(k)}{b^k} \Big/ \frac{\sqrt{3}}{4}$$
$$= \frac{27}{16} 3^k - \frac{1}{8} [15, 9, 9, 27].3^{\lfloor k/2 \rfloor} + \frac{1}{16} [3, 1, 3, 9]$$
$$= 0, 4, 12, 36, 120, 400, \ldots \qquad k \geq 0$$

A rectangle aligned $+30°$ to the b^k endpoint, which is the P3–P4 and P6–P7 sides for $k \geq 4$, is
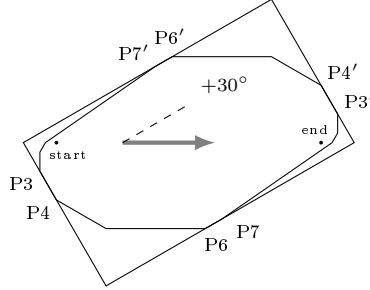
$$MR34(k) = 3^{k+1}.\operatorname{Re}\frac{P3(k)-P3'(k)}{b^{k+1}}.\operatorname{Im}\frac{P6(k)-P6'(k)}{b^{k+1}} \Big/ \tfrac{\sqrt{3}}{4}$$

$$= \tfrac{25}{16}\,3^k - \tfrac{1}{8}\,[5,15,15,25].3^{\lfloor k/2 \rfloor} + \tfrac{1}{16}\,[1,3,9,3]$$

$$= 1,3,9,33,121,363,\ldots \qquad k \geq 0$$

MR34 is the alignment of the minimum area rectangles around $k = 1$ to 3 illustrated above, but not beyond those.

The final alignment is to the P7–P1$'$ side, which is the claimed minimum. That side turns away from P6–P7 and since P3–P4 is at $90°$ to P6–P7, the P4 vertex is the rectangle width, as shown in the sample figure 26.

$$MR71(k) = \left|P7(k)-P1'(k)\right|^2.\operatorname{Re}\frac{P4(k)-P4'(k)}{P7(k)-P1'(k)}.\operatorname{Im}\frac{P7(k)-P7'(k)}{P7(k)-P1'(k)} \Big/ \tfrac{\sqrt{3}}{4}$$

$$= \frac{\operatorname{Re}(P4-P4')(\overline{P7-P1'}).\operatorname{Im}(P7-P7')(\overline{P7-P1'})}{|P7-P1'|^2} \Big/ \tfrac{\sqrt{3}}{4}$$

MrW and *MrH* at (110) are the real and imaginary parts here, and the unit triangle divisor $\sqrt{3}/4$ split between them in a convenient way so as to make integers.

$$MrW_k = \operatorname{Re}(P4-P4')\overline{(P7-P1')} \Big/ \tfrac{1}{2}$$

$$MrH_k = \operatorname{Im}(P7-P7')\overline{(P7-P1')} \Big/ \tfrac{\sqrt{3}}{2}$$

To compare to *MR12*, *MR23*, and *MR34*, divide down to

$$MR71(k) = \tfrac{169}{111}\,3^k - a(k)\,3^{\lfloor k/2 \rfloor} + \frac{b(k)\,3^k + c(k)\,3^{\lfloor k/2 \rfloor} + d(k)}{MrDen_k}$$

$$a(k) = \left[\tfrac{1196}{1369}, \tfrac{3354}{1369}, \tfrac{6994}{4107}, \tfrac{3380}{1369}\right]$$

$$b(k) = \left[-\tfrac{491}{5476}, -\tfrac{855}{21904}, \tfrac{471}{5476}, -\tfrac{23605}{197136}\right]$$

$$c(k) = \left[\tfrac{299}{5476}, \tfrac{3525}{10952}, \tfrac{811}{5476}, \tfrac{4003}{32856}\right]$$

$$d(k) = \left[0, \tfrac{3}{16}, 0, \tfrac{1}{16}\right]$$

Factor $\tfrac{169}{111}$ on 3^k here is smaller than the corresponding $\tfrac{91}{48}$, $\tfrac{27}{16}$ and $\tfrac{25}{16}$ of the other alignments. For $k \geq 4$, the difference exceeds the half-power and constant terms and so *MR71* is the minimum. □

11   Moment of Inertia

The mass moment of inertia $I = \sum mr^2$ of a rigid body rotating around a given axis is the ratio of torque to angular acceleration, similar to the way ordinary mass is the ratio of force to linear acceleration.
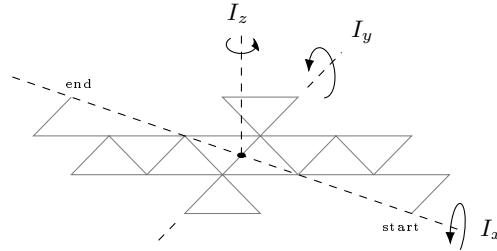
Figure 27: moment of inertia

Rotating about the z axis keeps the curve within the plane. This case is the simplest.

Theorem 43. *The terdragon curve with mass uniformly distributed along its length, at any expansion level and any unfolding angle θ, has the same moment of inertia I_z about its centre as a straight line from start to end.*
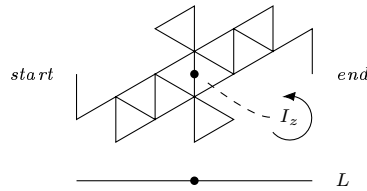
$$I_z = \tfrac{1}{12}mL^2$$

I_z *moment of inertia*
terdragon = line segment

Proof. For $k=0$, the curve is a straight line so the statement is true.

Suppose the statement is true of level k. Let each of its segments have mass μ and length s. The moment of inertia of such a segment about its centre is $I = \frac{1}{12}\mu s^2$. In the next expansion, the segment unfolds by angle θ as follows
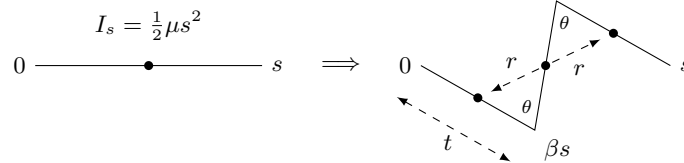
There are now 3 segments each length t and mass $\mu/3$. The centre of mass is still located at the midpoint. The moment of inertia I' of the expanded shape about this centre of mass is also unchanged since

$$\beta = 1/(2 + e^{i(\pi-\theta)}) \qquad\qquad \text{reduction}$$

$$t = s\,|\beta| \qquad\qquad \text{new segment length}$$

$$r = s\left|\frac{1}{2} - \frac{1}{2}\beta\right| \qquad\qquad \text{to midpoints}$$

$$I' = 3\,\frac{1}{12}\frac{\mu}{3}\,t^2 + 2\,\frac{\mu}{3}\,r^2 \qquad\qquad \text{parallel axis theorem} \qquad (111)$$

$$= \frac{1}{12}\mu s^2\left(|\beta|^2 + 2\,|1{-}\beta|^2\right)$$

$$= \frac{1}{12}\,\mu s^2 \; \frac{1 + 2\left(\left(1 + \cos(\pi{-}\theta)\right)^2 + \sin^2(\pi{-}\theta)\right)}{\left(2 + \cos(\pi{-}\theta)\right)^2 + \sin^2(\pi{-}\theta)} \tag{112}$$

$$= \frac{1}{12}\,\mu s^2 \quad = I \hspace{4cm} \square$$

The usual terdragon is $\theta = 60°$. It has $t = s/\sqrt{3}$ and the triangle formed by r is equilateral so $r = t/2$. Applying this to (111) easily gives $I' = I$. For other angles, r and t vary inversely and the sin and cos terms of (112) cancel out so $I' = I$ always.

The following diagram shows the geometry of the expansion. $\overline{AD}$ is length s. $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$ are the three new line segments each length t. B is distance $t/2$ from the middle M.
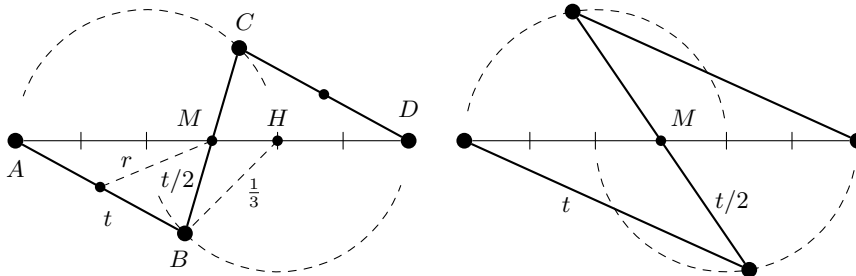
H is at $\frac{2}{3}$ along $\overline{AD}$. The distance $\overline{HB}$ is

$$\overline{HB} = \left|\tfrac{2}{3}s - bs\right| \; = \frac{1}{3}s\sqrt{\frac{(1 + 2\cos)^2 + (2\sin)^2}{(2 + \cos)^2 + \sin^2}} \; = \frac{1}{3}s$$

so B is on a circle of radius $\frac{1}{3}$ centred at H. Likewise by symmetry C on the corresponding circle above.

The midpoint of $\overline{AB}$, which is where r measures to, also follows a circle as in the following diagram. This is simply because the $\overline{AB}$ midpoint follows the circle of B but shrunk by $\frac{1}{2}$ in both x and y directions. So where B arcs from $\frac{1}{3}$ to 1 the $\overline{AB}$ midpoint arcs from $\frac{1}{6}$ to $\frac{1}{2}$.
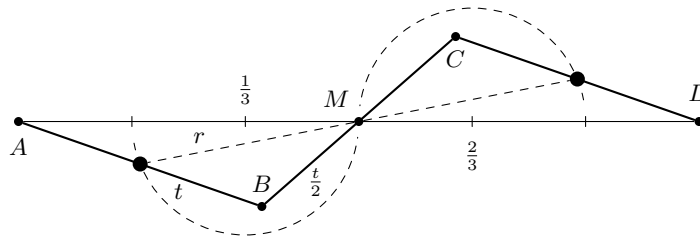
The first circle is centred at $\frac{1}{3}$ with radius $\frac{1}{6}$. It and the corresponding upper arc meet at M since both $\overline{AB}$ and $\overline{CD}$ midpoints are in the middle when fully overlapping $\overline{AB} = \overline{CD} = \overline{AD}$ for no unfold $\theta = 0$.

The points also make circles when the line segments $\overline{AB}$ etc are fixed lengths. This is obvious for C since it pivots from B. D is a fixed offset to the right so is a shift of the C circle.
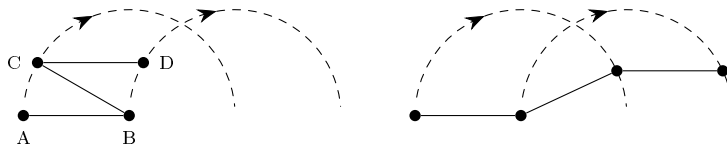
Theorem 44. *Consider each segment of the terdragon to have a unit mass uniformly distributed along its length. The centre of mass is the centre of the curve. With the x axis aligned to the endpoints, the moment of inertia tensor about the centre is*

$$\begin{pmatrix} I_x & -I_{xy} & 0 \\ -I_{xy} & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \qquad \begin{array}{ll} I_x = \sum y^2 & I_{xy} = \sum xy \\ I_y = \sum x^2 & I_z = \sum x^2 + y^2 \end{array}$$

where

$$I_x(k) = \tfrac{1}{84}\left(2.9^k - [2,-3].(-3)^{\lfloor k/2 \rfloor}\right)$$
$$= 0, \tfrac{1}{4}, 2, \tfrac{69}{4}, 156, \tfrac{5625}{4}, 12654, \tfrac{455517}{4}, \ldots$$
$$I_y(k) = \tfrac{1}{84}\left(5.9^k + [2,-3].(-3)^{\lfloor k/2 \rfloor}\right)$$
$$= \tfrac{1}{12}, \tfrac{1}{2}, \tfrac{19}{4}, \tfrac{87}{2}, \tfrac{1563}{4}, \tfrac{7029}{2}, \tfrac{126531}{4}, \tfrac{569403}{2}, \ldots$$
$$I_{xy}(k) = \tfrac{\sqrt{3}}{168}\left(2.9^k - [2,4].(-3)^{\lfloor k/2 \rfloor}\right) \tag{113}$$
$$= \sqrt{3}.\left\{0, \tfrac{1}{12}, 1, \tfrac{35}{4}, 78, \tfrac{2811}{4}, 6327, \tfrac{227763}{4}, \ldots\right\}$$
$$I_z(k) = I_x(k) + I_y(k) = \tfrac{1}{12}9^k \qquad per\ straight\ line$$
$$= \tfrac{1}{12}, \tfrac{3}{4}, \tfrac{27}{4}, \tfrac{243}{4}, \tfrac{2187}{4}, \tfrac{19683}{4}, \ldots \qquad k{\geq}1 \ \ \tfrac{1}{4}\text{A013708}$$

I_x and I_y are the moments of inertia rotating about the x or y axes as in figure 27. They can be combined with I_{xy} in the usual way for inertia about an axis at angle α in the plane

$$I(k, \alpha) = I_x(k).\cos^2\alpha - 2I_{xy}(k).\cos\alpha\sin\alpha + I_y(k).\sin^2\alpha$$
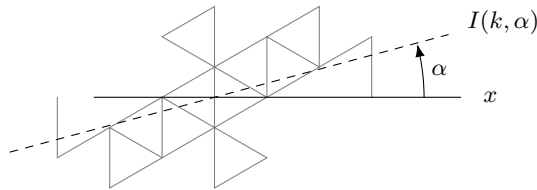
Proof. For $k{=}0$ the curve is a single line segment and that line has inertia $I_x(0) = 0$, $I_{xy}(0) = 0$ and $I_y(0) = \tfrac{1}{12}$ which is per the formulas.

For $k \geq 1$ the inertia is calculated by rotations and the parallel axis theorem from the 3 copies of level $k{-}1$.
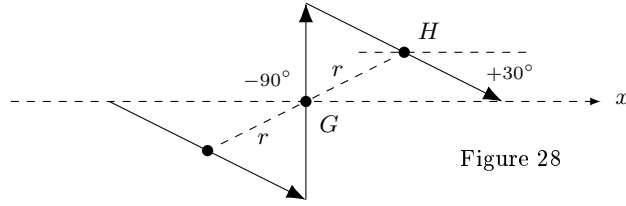
Figure 28

The first and last copies have the x axis at $+30°$ relative to those copies. The axes are turned by a matrix of rotation in the usual way

$$R = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{rotate axes by } +30°$$

Distance r is half the $k{-}1$ curve extent $r = \frac{1}{2}(\sqrt{3})^{k-1}$ and it is at $-30°$ to the axes for shifting the centre of mass of the first and last sub-curves. The middle sub-curve is axes at $-90°$. So total

$$\begin{aligned} I(k) = {}& 2\,R^{-1}.I(k{-}1).R & \text{first and last } +30° \\ & + R^3.I(k{-}1).R^{-3} & \text{middle } -90° \\ & + 2.3^{k-1}.\left(\tfrac{1}{2}\sqrt{3}^{k-1}\right)^2.R\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)R^{-1} & mr^2 \text{ first and last } -30° \end{aligned}$$

Multiplying through is mutual recurrences

$$I_x(k) = \quad \tfrac{3}{2}I_x(k{-}1) \; - \sqrt{3}\,I_{xy}(k{-}1) \; + \tfrac{3}{2}I_y(k{-}1) \; + \tfrac{1}{8}9^{k-1} \qquad (114)$$

$$I_y(k) = \quad \tfrac{3}{2}I_x(k{-}1) \; + \sqrt{3}\,I_{xy}(k{-}1) \; + \tfrac{3}{2}I_y(k{-}1) \; + \tfrac{3}{8}9^{k-1} \qquad (115)$$

$$I_{xy}(k) = \tfrac{1}{2}\sqrt{3}\,I_x(k{-}1) \qquad\qquad - \tfrac{1}{2}\sqrt{3}\,I_y(k{-}1) \; + \tfrac{1}{8}\sqrt{3}.9^{k-1}$$

I_{xy} has difference $I_x - I_y$ and subtracting $(114){-}(115)$ is that $I_x - I_y$ in terms of I_{xy} again so a recurrence for I_{xy} which can be expanded and summed down to either $I_{xy}(0)$ or $I_{xy}(1)$ according as k even or odd.

$$\begin{aligned} I_{xy}(k) &= -3I_{xy}(k{-}2) + \sqrt{3}.9^{k-2} \\ &= \sqrt{3}\,\frac{9^k - 9^{(k \bmod 2)}(-3)^{\lfloor k/2 \rfloor}}{81 - (-3)} \; + \; I_{xy}(k \bmod 2).(-3)^{\lfloor k/2 \rfloor} \end{aligned}$$

where $k \bmod 2$ means 0 or 1 as k even or odd

With initial $I_{xy}(0) = 0$ and $I_{xy}(1) = \frac{1}{12}\sqrt{3}$ from the mutual recurrences (or explicit calculation) this gives (113).

I_z is equivalent to a straight line as from theorem 43. The line here is extent $(\sqrt{3})^k$ and mass 3^k so $I_z = \frac{1}{12}9^k$. $I_z = I_x + I_y$ for any plane figure. Substituting I_{xy} and $I_y = \frac{1}{12}9^k - I_x$ into (114) gives I_x, and from which I_y. $\qquad\square$

Variations can be made with a different mass distribution on each line segment. For example a unit mass at the midpoint of each segment would make the initial $I_y(0)$ zero and change $I_{xy}(1)$ and the factor on $(-3)^{\lfloor k/2 \rfloor}$ in I_{xy}. Subtracting the individual line segments inertia $\frac{1}{12}3^k$ from I_z introduces a 3^k term into I_x and I_y.

An inertia matrix is real and symmetric so can be diagonalized with a suitable matrix of rotation turning to the eigenvectors which are its principal axes. The physical significance of this is that rotation about those axes is perfectly balanced with no torque exerted on the mounting points.

In the usual way for a 2×2 matrix, the eigenvectors are in direction d where

$$d^2 = \big(I_x(k) - I_y(k)\big) \, - \, 2I_{xy}(k)\,i$$

$$\alpha = \tfrac{1}{2}\arctan\frac{-2\,I_{xy}(k)}{I_x(k) - I_y(k)} \, + (0\,\text{or}\,\tfrac{\pi}{2})$$

$$= \tfrac{1}{2}\arctan\big(\tfrac{2}{\sqrt{3}} - \epsilon_k\big) \, + (0\,\text{or}\,\tfrac{\pi}{2})$$

$$\epsilon_k = \begin{cases} \dfrac{14\sqrt{3}}{9.(-27)^{k/2} + 12} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$$

$\epsilon_k \to 0$ as $k \to \infty$ so the limit for the principal axes is the $\tfrac{1}{2}\arctan\frac{2}{\sqrt{3}}$. The hypotenuse $|2+\sqrt{3}i| = \sqrt{7}$ and a double-angle can square that up to hypotenuse 7 in an arccos,

$$\alpha_{min} \to \tfrac{1}{4}\arccos\tfrac{-1}{7} \; = 24.553302...° \qquad \text{arccos second quadrant} \quad (116)$$
$$\alpha_{max} = \alpha_{min} + \tfrac{\pi}{2}$$
$$\to \pi - \tfrac{1}{4}\arccos\tfrac{-1}{7} \; = 114.553302...° \quad \text{arccos third quadrant}$$
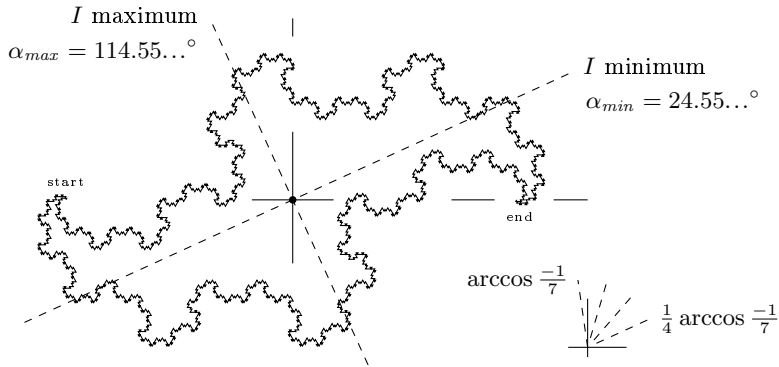
Roughly speaking, the minimum inertia is where the curve is closest to the axis and the maximum is where the curve is furthest from the axis, as measured by mr^2.

For the curve scaled to unit length, unit mass, and rotated $-\alpha_{min}$ so x axis minimum, the inertia limit is

$$\begin{pmatrix} \frac{1}{24}-\frac{1}{168}\sqrt{21} & 0 & 0 \\ 0 & \frac{1}{24}+\frac{1}{168}\sqrt{21} & 0 \\ 0 & 0 & \frac{1}{12} \end{pmatrix}$$

The inertia of the convex hull can be compared to that of the curve it surrounds. The inertia of the hull is calculated from its polygon. Its limit scaled to a unit length and with mass equal to its area is

$$HI_x = \tfrac{7261}{884736}\sqrt{3} \quad = 0.0142148\ldots \qquad \text{hull inertia}$$
$$HI_y = \tfrac{58999}{2654208}\sqrt{3} \quad = 0.0385008\ldots$$
$$HI_{xy} = \tfrac{449}{55296} \qquad\quad = 0.0081199\ldots$$
$$H\alpha_{min} = \tfrac{1}{2}\arctan\left(\tfrac{449}{1163}\sqrt{3}\right) = 16.885199°\ldots$$
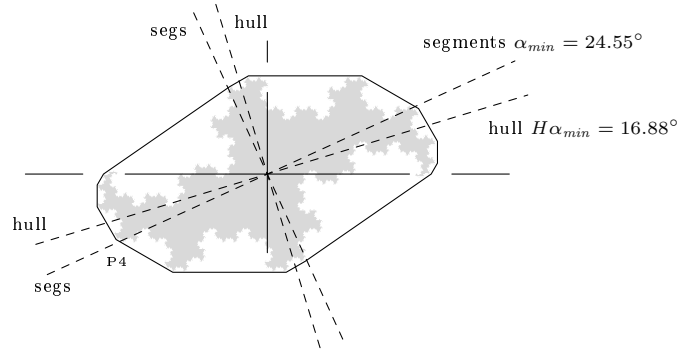
The segments axis α_{min} is close to hull vertex P4 but does not pass through it since P4 is at a slightly smaller slope,

$$\frac{P4(k)}{b^k} \to P4f = -\tfrac{1}{8}\sqrt{3}\,i$$
$$\arg\left(\tfrac{1}{2} - P4f\right) = \arctan\tfrac{1}{4}\sqrt{3} \quad = 23.413224°\ldots$$
$$\qquad\qquad = \tfrac{1}{2}\arctan\tfrac{8}{13}\sqrt{3} \quad < \alpha_{min} = \tfrac{1}{2}\arctan\tfrac{8}{12}\sqrt{3}$$

12 Terdragon Graph

The terdragon as a graph has an Euler path from start to end (traverse all edges exactly once) simply by its construction.

There is no Hamiltonian path start to end (visit all vertices exactly once) for $k \geq 3$ since the vertices in hanging triangles cannot be visited without repeating the vertex they attach to. There is no such path in $k=2$ either.

Theorem 45. *The path length between the endpoints of the terdragon curve as a graph is*

$$EndLength_k = \begin{cases} 3 & \text{if } k=1 \\ \tfrac{1}{8}\Big([11,19]\,3^{\left\lfloor\frac{k}{2}\right\rfloor} + 2k + [-3,-5,3,1]\Big) & \text{if } k\neq 1 \end{cases} \tag{117}$$
$$= 1, 3, 5, 8, 13, 22, 39, 66, 113, 194, \ldots$$

k=8 graph path
start to end
$EndLength_8 = 113$
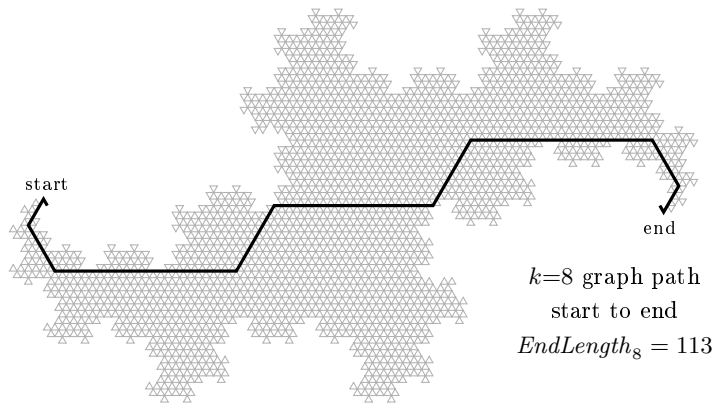
Proof. Firstly take k even and let $h = k/2$. Curve k comprises 9 sub-curves,
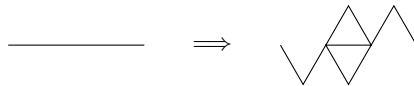
The shortest path start to end would be a straight line which is 3^h segments. But it's necessary to detour away from that midline up and down to go around the V shaped indent at start and end.
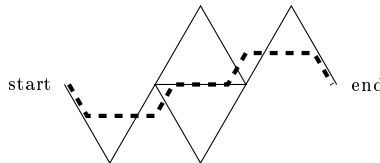
Making such a detour on a triangular grid adds a distance equal to the detour extent,

detour by 2,
adds 2 to distance
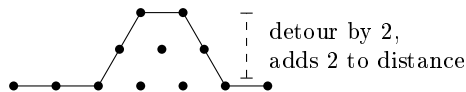
A straight line has a V indent sub-curve as shown above. Such a V comprises 18 sub-curves

~ 3 straight segments
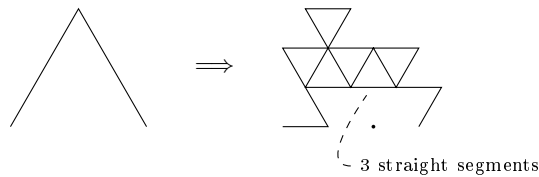
The three straight lines then are V indent sub-curves again. The first and last might be partly enclosed by the angled curves adjacent to them, but the middle is not. All are located at 1 sub-curve length into the V, which is 3^{h-1}. So the sub-curves alternating straight or V down to $h=0$ give

$$IndentV_0 = 1, \quad IndentS_0 = 0$$
$$IndentV_h = 3^{h-1} + IndentS_{h-1}$$

$$IndentS_h = IndentV_{h-1}$$
$$= \tfrac{1}{8}\left(3^h + [-1, 5]\right)$$
$$= 0, 1, 1, 4, 10, 31, 91, 274, 820, \ldots \qquad \text{\small A006342}$$

ternary 1010... ending 101 or 1011 for $h-1$ digits, $\quad h \geq 2$

The detour around the indent reaches the centre line of the end sub-curves. They then have further perpendicular indents. This can be illustrated in the following $k=6$ curve. The dots are the ends of the final sub-curve. The path shown detours around $IndentS_3$ and reaches the centre line of that end sub-curve. The arrow shown cannot go straight but must take a further detour out.
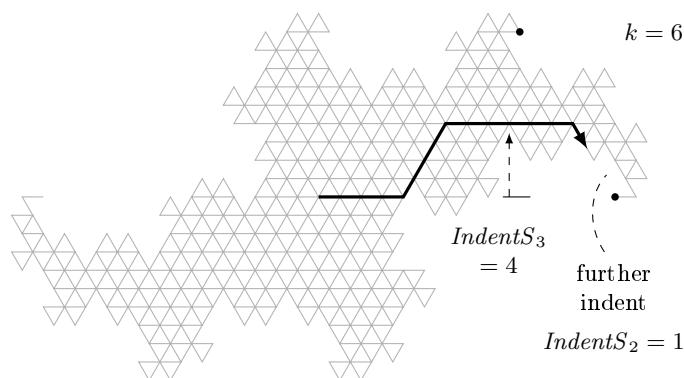
There is always a straight path across the tops of the indent since level k comprising 81 sub-curves of $k-4$ is
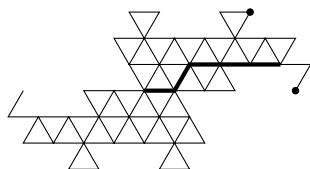
The top horizontal lines indent at most $IndentS_{h-2}$ downwards and the path shown indents at most $IndentS_{h-2}$ up. But

$$2\, IndentS_{h-2} < 3^{h-2}$$

so the top does not interfere with the path. Likewise on the diagonal up from the middle.

So for k even the distance start to end is its length 3^h plus detours at both ends which are sum of $IndentS$ spiralling around. This is k even of (117).

$$EndLengthEven_h = 3^h + 2\sum_{j=0}^{h} IndentS_j$$

For k odd let $h = \lfloor k/2 \rfloor$. The shortest path start to end would be straight across stepping along the sides of rhombus shaped pairs of triangles. This is distance 2.3^h. The following diagram shows a k curve expanded 3 times to 27 sub-curves.
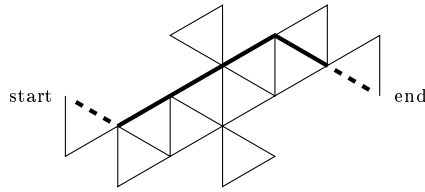
The dashed section is an indent across a V the same as for even k. A path start to end must detour around these at each end. *EndLengthEven* includes one 3^k, so adding another gives 2.3^h and two detours. This is k odd of the theorem (117).

$$EndLengthOdd_h = 3^h + EndLengthEven_h \qquad\qquad h \geq 1$$

This odd case effectively cuts an even path in half and inserts an extra 3^h segments which is the 3-long line in the middle of the diagram above. That middle part goes along parallel straight sides so per above the indent on its two sides do not interfere and there is a straight path of segments. □

12.1 Turn Tree

When the terdragon revisits a location z, the second and third visits are the same turn as the first. This is so for any non-crossing closed curve or curve continuing infinitely and not encircling its start. An opposite turn would enclose either the end or the start,

opposite turns would
enclose curve end

opposite turns would
enclose curve start
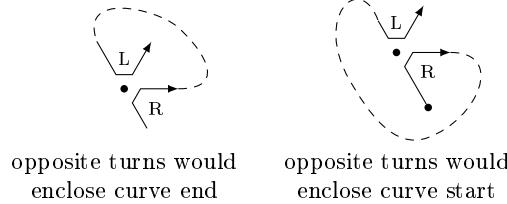
When three terdragons are arranged in a triangle, the locations with right turns and the segments between them form a tree.

Figure 29:
triangle of k=4 terdragons,
right turn points
and segments between
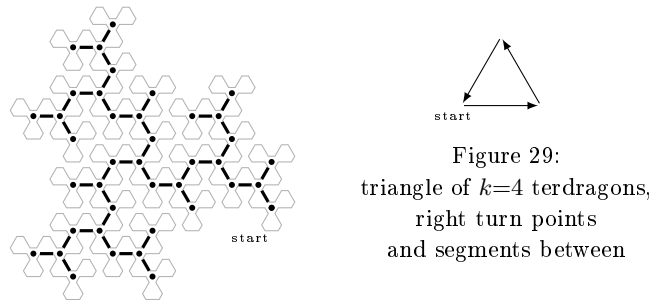
Each unit triangle has a right turn at the corner where it connects to the rest of the curve. Each unit triangle expands per figure 14 to a new right turn in the middle. The curve segments in that expansion go from the connection corners to that new right turn. An existing edge across a side becomes two segments going through the new point.

So a bottom-up expansion rule is to increase all existing vertices to degree-3 by adding new leaf vertices, and insert a new vertex in the middle of each old edge. Or equivalently a kind of star-replacement where each vertex is replaced with a claw (4-star) and each existing edge becomes a vertex in common between the new claws.

claw
replacement,
new vertex
in common
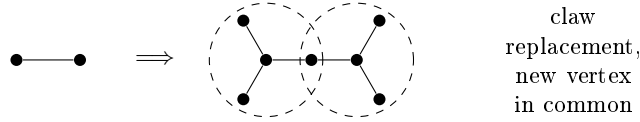
Three triangles of terdragons interlock per theorem 2 plane filling, The following diagram has each turn tree vertex drawn as a hexagon.

Figure 30:
3 triangles of k=5,
right turn points
(as hexagons)
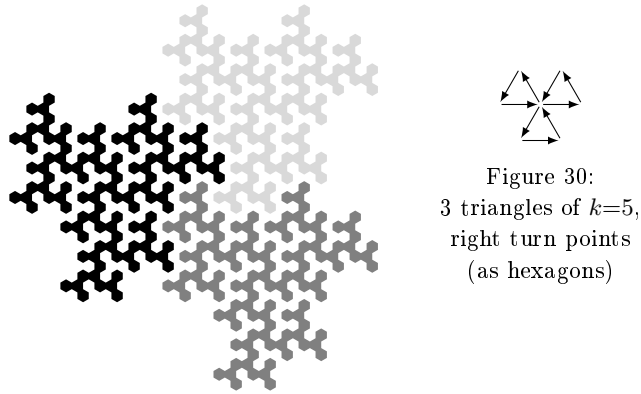
The tree copy shown in black is the terdragon triangle with first segment East per figure 29. The spiralling of the terdragons directs it around to the right.

Taking only arms of the triangles at the origin continued infinitely gives the trees continuing infinitely. If curve arms are considered all going outward the 3 interlocking trees are right turns in the even arms $0, 2, 4$ and left turns in the odd arms $1, 3, 5$.

The gaps between the hexagons in figure 30 are left turn points from the terdragon triangles. They are the same tree structures as the right turns, as can be seen by rotating the pattern 60° to swap the odd and even arms and so swap which of left or right turn is taken in the arms.

The number of vertices in the tree follows from the claw replacement. Each vertex becomes 4, but in each edge there is 1 in common so

$$TTV_k = 4\,TTV_{k-1} + \bigl(TTV_{k-1} - 1\bigr) \qquad \text{starting } TTV_1 = 1$$
$$= \tfrac{1}{2}\bigl(3^k - 1\bigr) \qquad\qquad \text{\small A003462}$$

Theorem 46. *Vertices of the terdragon triangle turn tree are degrees 1,2,3 after the initial degree-0 in k=1. The number of each degree in tree k are*

$$TTDegCount(k, 0) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{\small A000007}$$

$$TTDegCount(k,1) = \begin{cases} 0 & \text{if } k = 0, 1 \\ \frac{1}{2}\left(3^{k-1} + 3\right) & \text{if } k \geq 2 \end{cases}$$

$$= 0, 0, 3, 6, 15, 42, 123, 366, \ldots \qquad \text{A067771}$$

$$TTDegCount(k,2) = \begin{cases} 0 & \text{if } k = 0, 1 \\ \frac{1}{2}\left(3^{k-1} - 3\right) & \text{if } k \geq 2 \end{cases}$$

$$= 0, 0, 0, 3, 12, 39, 120, 363, \ldots \qquad \text{A029858}$$

$$TTDegCount(k,3) = \begin{cases} 0 & \text{if } k{=}0 \\ \frac{1}{2}\left(3^{k-1} - 1\right) & \text{if } k \geq 1 \end{cases}$$

$$= 0, 0, 1, 4, 13, 40, 121, 364, \ldots$$

Proof. Claw replacement gives degree-3 vertices as the preceding total vertices

$$TTDegCount(k,3) = TTV_{k-1}$$

Degree-2 vertices are likewise in each existing edge. There are $TTV - 1$ edges once the tree is not empty.

$$TTDegCount(k,2) = TTV_{k-1} - 1 \qquad k \geq 2$$

The claw replacement leaves only degree 1,2,3 vertices so the remainder of TTV in level k are degree-1. Or alternatively the claw replacement gives 1 degree-1 for each previous degree-2, and 2 for each previous degree-1 and 3 for each previous degree-0.

$$TTDegCount(k,1) = 3\,TTDegCount(k{-}1,0) + 2\,TTDegCount(k{-}1,1)$$
$$+ \, TTDegCount(k{-}1,2) \qquad\qquad \square$$

Theorem 47. *The diameter of terdragon triangle turn tree k is*

$$TTdiameter_k = \begin{cases} none & \text{if } k{=}0 \\ 2^k - 2 & \text{if } k \geq 1 \end{cases}$$

$$= 0, 2, 6, 14, 30, 62, 126, \ldots \qquad k \geq 1 \qquad \text{A000918}$$

The total number of paths attaining the diameter is

$$TTdiameterCount_k = \begin{cases} 0, 1 & \text{if } k{=}0, 1 \\ 3 \cdot 4^{k-2} & \text{if } k \geq 2 \end{cases}$$

$$= 1, 1, 3, 12, 48, 192, 768, \ldots \qquad \text{A002001}$$

The number of diameter endpoints, and total number of vertices on some diameter are

$$TTdiameterEnds_k = \begin{cases} 0, 1 & \text{if } k = 0, 1 \\ 3 \cdot 2^{k-2} & \text{if } k \geq 2 \end{cases}$$

$$= 0, 1, 3, 6, 12, 24, 48, \ldots \qquad \text{A003945}$$

$$TTdiameterVertices_k = \begin{cases} 0 & \text{if } k{=}0 \\ \frac{3}{4}(k{-}1)2^k + 1 & \text{if } k \geq 1 \end{cases}$$
$$= 0, 1, 4, 13, 37, 97, 241, \ldots \qquad \text{\scriptsize A048474}$$

Proof. For any path in level $k{-}1$, the bottom-up replacement inserts 1 further edge into it for level k, so $2\times$ the length. A path between any of those new vertices is shorter. If the path in $k{-}1$ ends at a degree-1 vertex then the replacement there has new leaf vertices attached.

A diameter must be between degree-1 vertices (otherwise could be extended). So the longest is between new leaf vertices on what was a longest path in level $k{-}1$. Starting then from diameter 0 for the single vertex of $k{=}1$,

$$TTdiameter_k = 2\,TTdiameter_{k-1} + 2 \qquad \text{starting } TTdiameter_1 = 0 \quad (118)$$

There are 2 new leaves at the end of the new path in k. They give endpoints, once the diameter is not 0,

$$TTdiameterEnds_k = 2\,TTdiameterEnds_{k-1} \quad \text{starting } TTdiameterEnds_2 {=} 3$$

and combinations of the 2 new at each end is 4 new paths for each existing one

$$TTdiameterCount_k = 4\,TTdiameterCount_{k-1}$$
$$\text{starting } TTdiameterCount_2 = 3$$

For total vertices of diameters, on bottom-up replacement each existing diameter vertex has 1 new vertex towards the middle of the tree, except at the middle vertex itself. The new $TTdiameterEnds_k$ outer vertices are immediately adjacent to existing diameter vertices. So

$$TTdiameterVertices_k = 2\,TTdiameterVertices_{k-1} - 1 + TTdiameterEnds_k$$
$$\text{starting } TTdiameterVertices_1 {=} 1 \qquad \qquad \square$$

In $k = 1, 2$ all the degree-1 vertices are diameter endpoints, but in $k \geq 3$ some degree-1 are not diameter endpoints. The degree-1 vertices grow as 5^k whereas the diameter endpoints grow only as 3^k.

$$TTdiameterEnds_k = TTDegCount(k, 1) \qquad k = 2, 3$$
$$TTdiameterEnds_k < TTDegCount(k, 1) \qquad k \geq 4$$

A top-down definition of the tree is to take the expansion of figure 14 as a level k triangle comprising 3 level $k{-}1$ triangles with a new vertex in between which is where what were left turns at connection corners are a right turn going to the next copy.
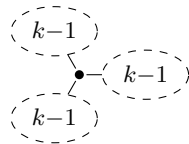
Figure 31: turns tree k as 3 copies of $k{-}1$ and new vertex in between

The connections to the $k{-}1$ are at vertices there attaining the diameter, so that the total is per (118). The three trees filling the plane can be considered like this too if the origin point is included.

The tree is half the Sierpinski triangle as a tree. That triangle has various definitions, among them is to take integer points x, y where $x\,\mathrm{BITAND}\,y = 0$. Tree edges are between points a unit distance apart.
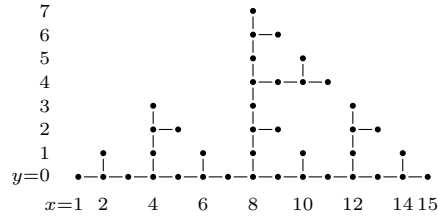
Figure 32:
Sierpinski triangle half $k{=}4$,
eighth plane $0 \le y < x$
to depth $x{+}y \le 2^k{-}1 = 15$

This Sierpinski triangle has the same definition as figure 31. In figure 32 the middle vertex is at $x{=}8, y{=}0$ and the 3 sub-trees attached to it are the same.

The 3 copies in figure 31 or usual properties of the Sierpinski triangle give number of vertices $v_k = 3v_k + 1$ starting $v_0 = 0$ so $v_k = (3^k{-}1)/2$.

Theorem 48. *Take the terdragon triple turn tree vertex nearest the curve start as the root. The width (number of vertices there) at a given depth d, starting $d{=}0$ as the root, is*

$$TTwidth_\infty(d) = 2^{CountOneBits(d+1)-1} \tag{119}$$

$$= 1, 1, 2, 1, 2, 2, 4, 1, 2, 2, 4, 2, 4, 4, 8, \ldots \qquad \text{A048896}$$

Proof. In the top-down figure 31, the new trees attach at the diameter of the first, so the first does not overlap the others.

The distance to those others is $TTdiameter_{k-1} + 2 = 2^{k-1}$ for $k \ge 2$, so that a depth e into them has

$$TTwidth(2^{k-1} + e) = 2\,TTwidth(e) \qquad \text{for } 0 \le e \le TTdiameter_{k-1}$$

so factor 2 for a 1-bit in d. The vertex in between is at $d = 2^{k-1}{-}1$ and its width is 1. That is conveniently handled by taking $d{+}1$ giving $CountOneBits(2^{k-1})$ $= 1$. Together with the initial values gives (119). □

$n \ne 0 \bmod 3$ are right turns when n are $n \equiv 2 \bmod 3$. They are at locations $z \equiv \omega_6 \bmod b$, from the lowest base figure expansion. This is a repeating pattern,
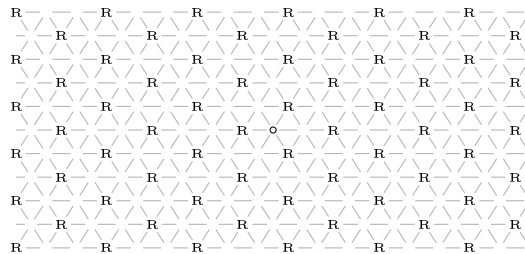
These locations are like $k{=}1$ trees formed from the surrounding 9 segments. Right turns with one low 0 digit on n is the same pattern with a factor of b. Those further points connect to make $k{=}2$ claws, and so on, generating the trees from a simple repeating pattern.

A related "area graph" can be formed by a vertex for each unit triangle inside the terdragon triangle and edges between those which are consecutive in the curve. Or equivalently, if corners of the curve are chamfered off to leave little gaps then edges are between unit triangles touching through those gaps.

triangle of $k{=}4$ terdragons,
vertex for each inside unit triangle,
edge between consecutive in the curve
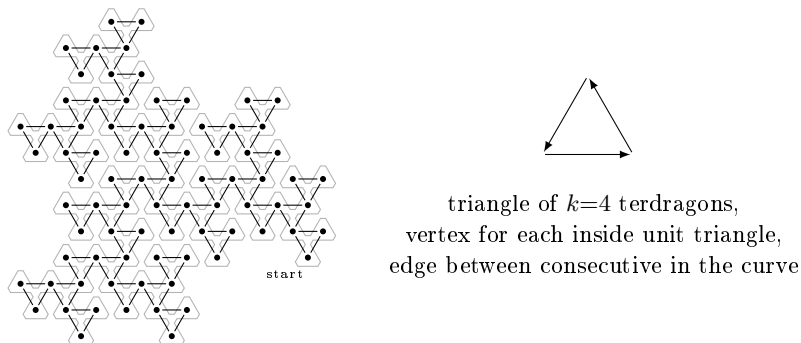
Inside unit triangles occur in connected 3s as from the figure 14 expansion again. Each original side expands to have 2 new triangles consecutive, so the 3 new unit triangles are consecutive in pairs and so a 3-cycle in the graph.

These 3-cycles are connected like the turns tree. Each turn tree vertex is a 3-cycle and turn tree edges are where those 3-cycles share a vertex. This is a "contact triangles" form of the tree.

Or equivalently, increase all existing vertices to degree-3 by adding new leaf vertices (like the second bottom-up form above). Then the area graph is the line graph of this padded tree.

13 Fractional Locations

The location of a point $0 \leq f \leq 1$ along the terdragon fractal is a limit

$$fpoint(f) = \lim_{k \to \infty} \frac{point(\lfloor f.3^k \rfloor)}{b^k} \qquad \text{fractional point}$$

$n = \lfloor f.3^k \rfloor$ is the first k digits below the ternary point of f written in ternary. The location is powers b^j at each digit per (43), with rotation below each 1-digit. The sub-curve there has extent decreasing as $1/\sqrt{3}^k$ so the limit converges to some location z.

When f is rational, its digits are an initial fixed part then repeating periodic part (of length at most denominator-1). The b powers are then likewise periodic and give a location as some $x{+}\omega_3 y$ with rational x, y.

If the periodic part of f has number of 1-digits not a multiple of 3 then there is a net rotation in the periodic part. That can be accounted for in the calculation, or repeating the part 3 times gives a multiple of 3 and so purely periodic b^j powers.

13.1 Fractional Boundary

Theorem 49. *The only points on both left and right boundary of the terdragon fractal are curve start and end $f = 0, 1$.*

Proof. $k{=}3$ sub-curves and convex hulls around them are as follows

$k{=}3$ sub-curves
right and left hulls
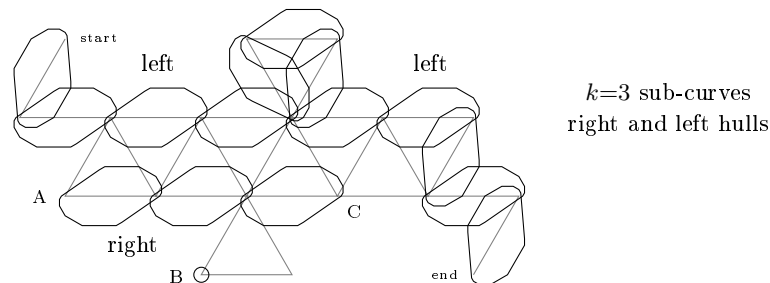
The curve is non-crossing so all left boundary locations are within the convex hulls around the left boundary segment sub-curves.

The right boundary is within corresponding convex hulls around right boundary segments. The hulls drawn A through C, and the hanging triangle on the right side, are disjoint from the left boundary hulls. So the spiralling and curling within those parts of the right boundary never reaches the left boundary.

The sub-curves expand to $k{=}4$ as

$k{=}4$
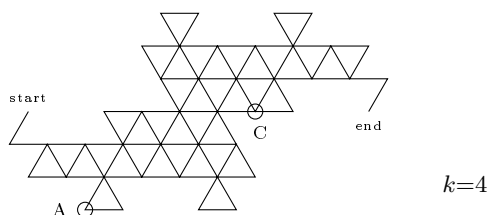
Expanded right boundary parts from start to A are the same as start to B of $k{=}3$. That leaves only sub-curves through to the corresponding new smaller A as possible both boundary. Repeating this excludes points an arbitrarily small distance away from the start, leaving only the start as both left and right boundary.

Expanded right boundary parts end to C are the same as end to B of $k{=}3$. Likewise this leaves only sub-curves through to the corresponding new smaller C as possible both boundary and so anything except the end as not both left and right boundary. □

Theorem 50. *The terdragon fractal has no cut points, ie. is a topological disc.*

Proof. If a cut point separates start and end then it is on both left and right boundary, but from theorem 49 there are no such points.

Suppose a cut point separates a lobe from the boundary. If this point is somewhere within a sub-curve then it separates start and end of that sub-curve, but again no such point exists.

Otherwise the point is always at the start or end of some sub-curve. The only cut points in the finite iterations are the hanging triangle attachments, but they are triple-visited so by the plane filling they are not on the boundary and so not cut points of the fractal. □

Theorem 51. *Fractional f on the boundary of the terdragon fractal are characterized by the ternary digits of f as*

$$fRpred(f) = 1 \ \ \text{if no ternary digit pair } 11, 12, 20$$

$$\text{except } 11 \text{ allowed if all } 0s \text{ below, and } 20 \text{ allowed if all } 2s \text{ below} \qquad (120)$$

$$fLpred(f) = fRpred(1{-}f)$$

$$= 1 \ \ \text{if no ternary digit pair } 02, 10, 11,$$

$$\text{except } 02 \text{ allowed if all } 0s \text{ below, and } 11 \text{ allowed if all } 2s \text{ below}$$

$$fBpred(f) = fRpred(f) \ \text{or} \ fLpred(f)$$

The digit pairs disallowed are the same as the finite *Rpred* and *Lpred*, but with exceptions for certain exact $f = n/3^k$. The 11 at (120) is n ending 11. The 20 is n ending 20222... which is $= 21000...$ in the usual way. These exceptions introduce an extra state each into *Rpred* high to low figure 12.
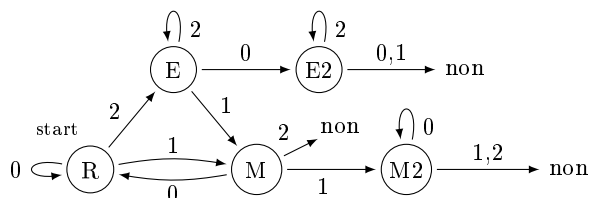
Figure 33:
$fRpred(f)$ by
ternary digits
high to low

Proof. An *Rpred* non-boundary segment has 2 enclosing segments on its right side. Since those sub-curves have no cut points, they enclose all of that side except start and end.
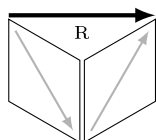
right side enclosed by 2 sub-curves
when *Rpred* non-boundary

Segment start is on the right boundary when it is single or double visited and turn left (since the curve does not overlap). Single visited turn left is accepted by *Rpred* already, since there is no first segment beside it. Double-visited left turn arises from a 2-side triangle in manner of figure 14. From *Rpred* state M this is
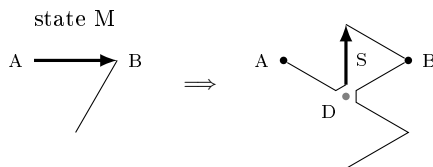
Segment A–B expands to have S fully enclosed. This is ternary digit 1 to reach state M then digit 1 for part S. For *fRpred* the start of S is on the boundary, which is f with all 0 digits below.

A double-visited left turn from *Rpred* state E is
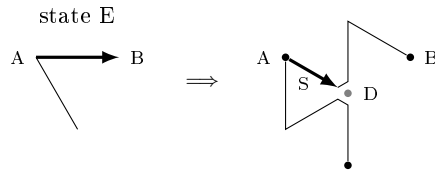
state E

Segment A–B expands again to S fully enclosed for *Rpred*, but its end is on the boundary for *fRpred*. This is digit 2 to reach state E then digit 0 for part S. The end of S is all 2s per .222... = 1.

A triple-visited start or end is not on the boundary, since the 6 sub-curves enclose that point per the curve plane filling.

For *fLpred* similarly with *Lpred* and double-visited turn right on the left boundary. □

Second Proof of Theorem 51. A sub-curve m has its convex hull touched or overlapped by the hulls of the following surrounding sub-curves,
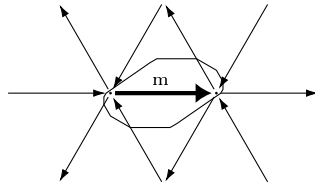
Figure 34:
surrounding segments
whose hulls touch
or overlap
hull of m

If m has all segments of figure 34 surrounding it then it is non-boundary since, by construction, it is does not touch or overlap the hull of any absent outside. Conversely, if m has one or more of the segments of figure 34 absent, then that is some part of the hull of m which is outside the curve and therefore some of m possibly on the boundary.

Hulls beyond figure 34, so not touching m, can be illustrated
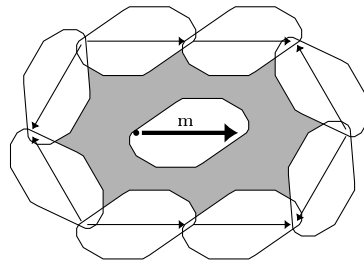
Figure 35:
non-touching
hulls

When m is surrounded by all segments of figure 34, the grey area here is a minimum amount of filled region surrounding m. The closest approach of an absent outside is $\frac{1}{8}/\sqrt{3}^k$ between the horizontal sides.

The hulls in figure 35 are those necessary to delimit the surrounding grey gap region shown. Actually that region may be bigger, since the curve turns $\pm 120°$ at the ends of the segments in figure 34 and so some more in figure 35, but knowing that is not necessary.

A given sub-curve m has some of figure 34 surrounding segments. The initial single segment $k{=}0$ has none. On expansion there are new segments around the three new sub-curves. The segments of figure 34 suffice to determine the corresponding set of segments around each new segment. A finite set of segment configurations arise and give a state machine traversed by ternary digits of f.

A fully surrounded configuration expands to fully surrounded for any next digit $0, 1, 2$. So if the digits of f ever reach fully surrounded then it remains so always. If f never reaches fully surrounded then that is an absent sub-curve at distance $\leq 1/\sqrt{3}^k$ so m an arbitrarily small distance from the outside, and hence a boundary point.

$$fBpred(f) = \begin{cases} 0 & \text{if ever reach fully surrounded} \\ 1 & \text{if never fully surrounded} \end{cases}$$

To distinguish right and left boundary, segments of the curve always turn left or right and so divide the plane into alternating left or right side triangles (eg. as previously for area in figure 13). The actual sub-curves are curling spiralling shapes, but they divide into logical triangles.

possible boundary triangles,
left and right sides
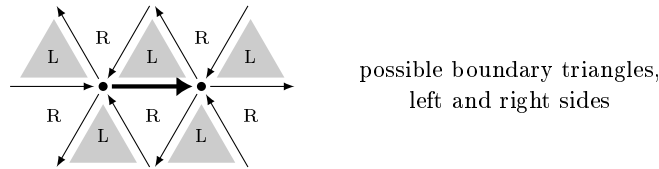
Triangles are shown with just the touching hull segments of figure 34. If a triangle has at least 1 of its sides segments shown, but not all of them, then this is some of its R or L as boundary for m.

A configuration with no R expands to no R again for next digit $0, 1, 2$. Similarly L.

$$fRpred(f) = \begin{cases} 0 & \text{if ever reach no } 1, 2 \text{ side R triangles} \\ 1 & \text{if always a } 1, 2 \text{ side R triangle} \end{cases}$$

$$fLpred(f) = \begin{cases} 0 & \text{if ever reach no } 1, 2 \text{ side L triangles} \\ 1 & \text{if always a } 1, 2 \text{ side L triangle} \end{cases}$$

Total 31 configurations arise. There are 11 with R fully enclosed and 11 with L fully enclosed. 1 configuration is both L and R fully enclosed, being the full set of segments.

Some usual state machine comparison shows the result is the same as *fRpred* in figure 33. Likewise *fLpred*. □

This second proof does not use theorem 49 for no points on both left and right boundary. That theorem can follow mechanically from the state machine by getting the intersection of *fRpred* and *fLpred*. State machine manipulations show the only arbitrarily long strings matched by both are $f = .000... = 0$ and $f = .222... = 1$.

For computer calculation or similar, it might be decided to take only the low 0s representation of exact $f = n/3^k$. In that case state E2 in figure 33 is not needed and can go straight to non-boundary. Similarly instead if only low 2s representation is taken then M2 is not needed.

A given f might be known or proved to be not an exact $/3^k$ so that neither E2 nor M2 is needed, leaving just ternary without $11, 12, 20$ the same as *Rpred*.

The number of f which are *fNonRpred* is uncountably infinite, since once reaching "non", further ternary digits of f can be an arbitrary real.

The number of f which are boundary *fRpred* is uncountably infinite too. That can be seen in the *fRpred* state machine where there are various different ways digits of f can loop among R,M,E so as to always stay away from "non". For example 10 is R→M returning to R, and 210 is R→E→M returning to R. The bits of an arbitrary real can be coded to ternary digits as 0-bit $\to 10$, 1-bit $\to 210$ so there are at least as many *fRpred* as reals. The same holds for *fLpred*, and then union for *fBpred*.

Theorem 52. *The number of visits to the location of a given f in the terdragon fractal is*

$$fVisits(f) = \begin{cases} Visits_k(n) & \text{if } f = n/3^k \text{ for integer } n, k; \text{ and otherwise} \\ 2 & \text{if fNonBpred but sub-digits eventually fBpred} \\ 1 & \text{otherwise} \end{cases}$$

Proof. An exact fraction $f = n/3^k$ is a vertex of curve k and the visits there are the same as $Visits_k$ from (88). By plane filling, those visits enclose the point so no other sub-curves touch it.

The claimed cases whole curve *fBpred* boundary or not, and sub-curve eventually or never *fBpred*, are

	whole curve	
	fBpred	*fNonBpred*
sub-curve eventually *fBpred*	1	2
sub-curve never *fBpred*	no such	1

An f which is on the boundary of some sub-curve, meaning its digits at some digit position and below are *fBpred*, might have an adjacent sub-curve like
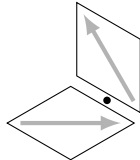
Figure 36:
f on sub-curve boundary
and adjacent other sub-curve

If it has this further sub-curve then by plane filling and no cut points the two enclose the location so visits there are only visits arising from the two.

If no such further sub-curve then the visits are only those arising from the f sub-curve itself. An f which is on a sub-curve boundary like this has only 1 visit because any other would be, for suitable yet smaller sub-curves, an adjacent enclosing further sub-curve like figure 36 and so not on the boundary. So in the table the first row cases are sub-curve boundary 1 or 2 visits according as whole curve boundary or not.

An f which is *fNonBpred* non-boundary, and its digits at all positions below are also *fNonBpred*, is never on the boundary of any sub-curve and so always a non-zero distance away from any other sub-curve and so just 1 visit. □

fNonBpred of a non-3^k means somewhere a ternary digit pair $02, 10, 11$ so *fNonLpred* and also somewhere $11, 12, 20$ so *fNonRpred*. Pair 11 is common to these so a 11 anywhere is *fNonBpred*.

The *fVisits* $= 2$ case is therefore at least one each $02, 10, 11$ and $11, 12, 20$, so as to be non-boundary, but only finitely many of one of them so eventually on a sub-curve boundary.

The *fVisits* $= 1$ case is the converse. Either none at all of $02, 10, 11$ or $11, 12, 20$ so whole curve *fBpred*, or infinitely many of both of them so always *fNonBpred* in all sub-curves.

The latter case, infinitely many of both, can be either rational or irrational. Suitable pairs in an infinite repeating pattern is rational, or non-repeating is irrational. The simplest rational is $f = .111... = \frac{1}{2}$ which is infinite 11 pairs. This is the middle of the curve, then middle of the middle sub-curve, and so on.

It can be noted *fVisits* is not decided by initial digits of f. After some digits, a suitable exact $/3^k$ below can be *Visits* $= 3$. Or all 1s below is middle of the sub-curve *fVisits* $= 1$. Or a sub-curve boundary by suitable pairs is *fVisits* $= 2$.

Theorem 53. *For fVisits$(f)$$=$$2$ by eventually sub-curve right boundary, its other visit fOther(f) is digit runs flipped*

$$
\begin{array}{r|c|c|c|c}
 & \text{high} & \multicolumn{2}{l}{\text{\textit{fRpred} disallowed pair}} & \text{low} \\
\end{array}
$$

	high	*fRpred* disallowed pair			low
f	...	122...22	100...00	022...22	...
fOther(f)	...	200...00	022...22	100...00	...
		$+1$	-1	$+1$	

(121)

Runs begin at and including the lowest fRpred disallowed pair 11, 12 or 20. A 11 is initial run single digit 1, then next run 100....

The runs are alternating 0222 and 1000, except the highest which are 1222 and 2000. Each run is ≥ 1 digit. fOther flips between their two respective forms.

For fVisits$(f)$$=$$2$ by eventually sub-curve left boundary, the same but digit pattern reversed $0 \leftrightarrow 2$, starting from the lowest fLpred disallowed pair

	high	*fLpred* disallowed pair			low
f	...	100...00	122...22	200...00	...
fOther(f)	...	022...22	200...00	122...22	...
		-1	$+1$	-1	

(122)

All runs are maximal in the sense that they take as many of their repeating digits as possible, consistent with the next run. So 100...00 in (121) takes all 0s except one for the following 022...22.

The effect for the right side is runs begin at 1, and at 0 with non-0 below it. Or for the left side at 1, and at 2 with non-2 below it.

Proof. For the right side, the sub-curves on its right are calculated high to low as per *other* table (51). There are 2 segments on the right, but in the next expansion only one of them is used. n digits 0 or 1 use only s, and digit 2 uses only e. So a digit of *fOther* is determined by two digits of f.

R	f pair	00	01	02	10	11	12	20	21	22
	output	2	1	1	0	$f2$	$f2$	$f1$	0	0

(123)

When table (51) has an "n" it is a copy of f for the output. This is shown as output f in (123) here. It occurs for the *Rpred* disallowed pairs $11, 12, 20$ so that *fOther* is unchanged above such a pair.

At the lowest disallowed pair, following the pairs there onwards in f and the output digits in (123) gives the run forms (121).

For the left side similarly, with the pairs being

L	f pair	00	01	02	10	11	12	20	21	22	
	output	2	2	$f1$	$f0$	$f0$	2	0	1	1	$\square$

The left side cases are $0\leftrightarrow2$ digit reversals of the right and its outputs. This is since the curve is the same in $180°$ reverse, so that $1-f$ measures back from the end and then $1 - fOther(1-f)$ measures again from the start. $1-f$ is an $0\leftrightarrow2$ reversal.

Differences $\big|f - fOther(f)\big|$ which occur follow from the runs (121),(122). The ±1 shown under each run is the increment to add to go from f to *fOther*, being difference of the two ternary numbers in the runs. The runs alternate so the signs alternate.

For exact $f = n/3^k$, in theorem 10 differences are also alternating signs. For $fVisits(f)\!=\!2$ by eventually sub-curve boundary, there are infinite such terms. For exact f there are finite terms. All differences of this form occur by choosing suitable run lengths in f.

$$\big|f - fOther(f)\big| = \frac{1}{3^{k_0}} - \frac{1}{3^{k_1}} + \frac{1}{3^{k_2}} - \frac{1}{3^{k_3}} + \cdots$$
$$\text{where } 0 > k_0 > k_1 > k_2 > \cdots$$
$$= \text{fraction ternary } .0 \text{ then digits } 0 \text{ or } 2, \text{ or } 1 \text{ if all 0s below}$$

The ternary form is digit 2s for each pair $1/3^{k_0} - 1/3^{k_1}$ etc, and possible 1 like *other* differences from (57).

The runs have no choices within, so for the $fVisits(f)\!=\!2$ case a difference determines f below its first $1/3^{k_0}$ term. When this is $k_0\!=\!1$ first digit, each difference occurs for just one $f, fOther$ pair.

14   Locations Summary

Various limit locations in the terdragon curve can be illustrated together,

GRf = right boundary centroid (98)
GJf = join area centroid (100)
$Lnearf, Rnearf$ = boundary nearest middle (107)
$Jnearf$ = join nearest (109)
$P1, \ldots$ = convex hull vertices, section 10
$\alpha_{min}, \alpha_{max}$ = inertia principal axes (116)

Figure labels: α_{max}, P6′, $fpoint(\tfrac{2}{3})$, P5′, P7′, $Jnearf$, α_{min}, P4′, GJf, P3′, P1, start, $Lnearf$, 0, P2′, P2, 0, $\tfrac{1}{2}$, end, 1, P1′, P3, $Rnearf$, P4, GRf, $fpoint(\tfrac{1}{3})$, P7, P5, P6
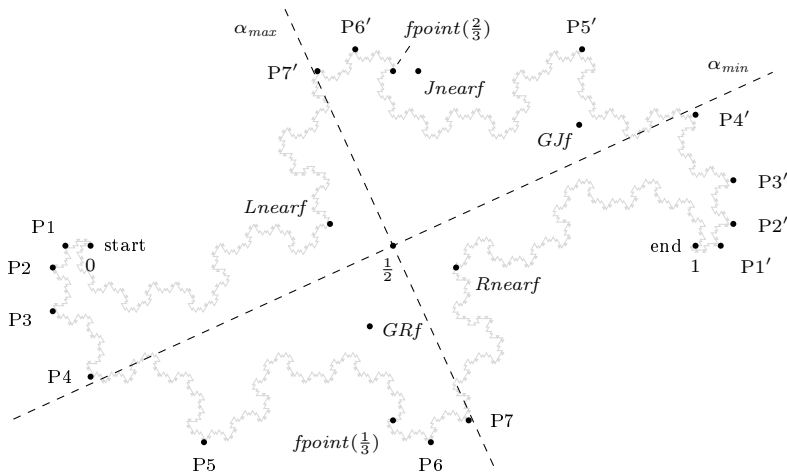
15   Alternate Terdragon

Davis and Knuth[3] also consider an alternate terdragon where the unfolding is to the opposite side L↔R in alternate expansion levels.

Figure 37

$k=0$ $k=1$ $k=2$ $k=3$ $k=4$
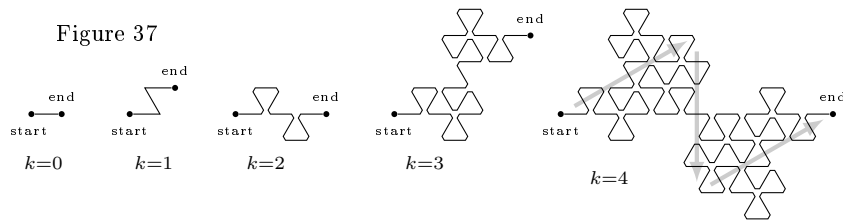
The expansion is
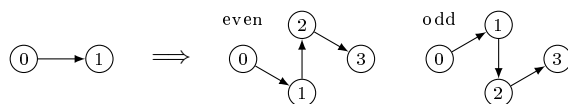
Figure 38:
alternate terdragon
expansions

The form here is taken so level k is a prefix of $k+1$. For unfolding, this means first unfold "even" is the same as the terdragon, then second "odd" is opposite, and so on.

For segment replacement, the last replacement is to be even, so begin with whichever odd or even to give that last. Another equivalent is to conjugate (mirror vertically) the existing curve and each time replace by the terdragon "even" base.

Taking two expansions is a flip and flip back again so becomes a plain 9 segment replacement,
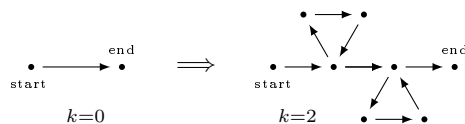
Figure 39:
alternate terdragon
two expansions

$k=0$ $k=2$

The turn sequence goes as *LowestNonZero* but the sense is flipped when that digit is at an odd position (least significant digit as position 0).

$$AltTurn(n) = \begin{cases} +1 & \text{if } LowestNonZero(n) + CountLowZeros(n) \equiv 1 \bmod 2 \\ -1 & \text{if } LowestNonZero(n) + CountLowZeros(n) \equiv 0 \bmod 2 \end{cases}$$

$$= -(-1)^{LowestNonZero(n)+CountLowZeros(n)}$$

$$= + - + - ++ - ++ - + - ++ - + - + - ++ - + - + - ++ \ldots$$

$CountLowZeros(n) = 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, \ldots \quad n \geq 1$ A007949

Or next turn, for $n \geq 0$,

$$AltTurn(n+1) = \begin{cases} +1 & \text{if } LowestNonTwo(n) + CountLowTwos(n) \equiv 0 \bmod 2 \\ -1 & \text{if } LowestNonTwo(n) + CountLowTwos(n) \equiv 1 \bmod 2 \end{cases}$$

$$= (-1)^{LowestNonTwo(n)+CountLowTwos(n)}$$

$CountLowTwos(n) = CountLowZeros(n+1)$

A turn recurrence differs from the terdragon (2) in negating the $3n$ case,

$$AltTurn(3n) = -AltTurn(n), \quad AltTurn(3n+1) = 1, \quad AltTurn(3n+2) = -1 \tag{124}$$

The two expansions of figure 39 gives turns from base-9 digits of n (as for example in program code by Arndt[1], who calls the curve R9 there),

$$AltTurn(n) = \begin{cases} +1 & \text{if } Base9LowestNon0(n) = 1, 4, 6, 7 \\ -1 & \text{if } Base9LowestNon0(n) = 2, 3, 5, 8 \end{cases}$$

$Base9LowestNon0(n) = 1, 2, 3, 4, 5, 6, 7, 8, 1, 1, 2, 3, 4, \ldots \quad n \geq 1$ A277547

Predicates for left and right turns are

$$AltTurnLpred(n) = \begin{cases} 1 & \text{if } AltTurn(n) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= 1,0,0,1,0,1,1,0,1,1,0,0,1,0,1,1,0, \ldots \quad n \geq 1$$

$= 1$ at $n = 1, 4, 6, 7, 9, 10, 13, 15, 16, 19, 22, 24, \ldots$ A189715

$$AltTurnRpred(n) = \begin{cases} 1 & \text{if } AltTurn(n) = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$= 0,1,1,0,1,0,0,1,0,0,1,1,0,1,0,0,1, \ldots \quad n \geq 1$$ A156595

$= 1$ at $n = 2, 3, 5, 8, 11, 12, 14, 17, 18, 20, 21, 23, \ldots$ A189716

Generating functions for these in the style of (5) have powers in L or R form in alternate terms,

$$gAltTurnLpred(x) = \sum_{k=0}^{\infty} \frac{x^{[1,2]_k \cdot 3^k}}{1 - x^{3^{k+1}}} \qquad gAltTurnRpred(x) = \sum_{k=0}^{\infty} \frac{x^{[2,1]_k \cdot 3^k}}{1 - x^{3^{k+1}}}$$

Combining them as $AltTurn = AltTurnLpred - AltTurnRpred$ has both $1, 2$ powers in opposite signs and alternating with k. Then like (6), cancel a common factor with the denominator.

$$gAltTurn(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(x^{3^k} - x^{2.3^k}\right)}{1 - x^{3^{k+1}}} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3^k}}{1 + x^{3^k} + x^{2.3^k}}$$

Segment direction is ± 1 for each ternary digit 1 sub-part, with sign according as its digit position which is its level. Or base-9 parts $1, 3, 5, 7$ which are the directions shown in figure 39.

$$AltDir(n) = \sum_{j=0}^{n-1} AltTurn(j)$$

$$= \sum_{n \text{ ternary}} \begin{cases} +1 & \text{if } digit\!=\!1 \text{ and even position} \\ -1 & \text{if } digit\!=\!1 \text{ and odd position} \\ 0 & \text{otherwise} \end{cases}$$

$$= \big(\text{count}\, 1, 7\big) - \big(\text{count}\, 3, 5\big) \quad \text{of } n \text{ base-9 digits}$$

$$= 0, 1, 0, -1, 0, -1, 0, 1, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, \dots$$

The number of left and right turns from 1 to n inclusive are

$$AltTurnsL(n) = \sum_{j=1}^{n} AltTurnLpred(n)$$

$$= 1, 1, 1, 2, 2, 3, 4, 4, 5, 6, 6, 6, 7, 7, 8, 9, 9, 9, \dots$$

$$AltTurnsR(n) = \sum_{j=1}^{n} AltTurnRpred(n)$$

$$= 0, 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 6, 6, 7, 7, 7, 8, 9, \dots \qquad \text{\small A189717}$$

All turns are left or right so total lefts plus rights is simply n. The difference lefts minus rights is *AltDir*.

$$AltTurnsL(n) + AltTurnsR(n) = n \tag{125}$$

$$AltTurnsL(n) - AltTurnsR(n) = AltDir(n) \tag{126}$$

Sum and difference of (125),(126) are

$$AltTurnsL(n) = \tfrac{1}{2}\big(n + AltDir(n)\big)$$

$$AltTurnsR(n) = \tfrac{1}{2}\big(n - AltDir(n)\big)$$

Clark Kimberling in OEIS A189717 gives a recurrence

$$AltTurnsR(n) = \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor - AltTurnsR\big(\left\lfloor \tfrac{n}{3} \right\rfloor\big)$$

The last two parts are *AltTurnsL* in terms of *AltTurnsR* from (125) so

$$AltTurnsR(n) = \left\lfloor \frac{n+1}{3} \right\rfloor + AltTurnsL\big(\left\lfloor \tfrac{n}{3} \right\rfloor\big)$$

This can be seen in the *AltTurn* expansions, similar to the terdragon *TurnsR* at (36) in its turn expansions. Here $\lfloor (n+1)/3 \rfloor$ is the same new L,R as there, but here the "existing" turns are the opposite *AltTurnsL* count because the unfolding

direction alternates.

The alternating unfolding is new curve end at factor b or $\bar{b}$ of the preceding,

$$
\begin{aligned}
AltEnd_k &= [b, \bar{b}] \, . \, AltEnd_{k-1} \qquad \text{starting } AltEnd_0 = 1 \\
&= b^{\lceil k/2 \rceil} . \bar{b}^{\lfloor k/2 \rfloor} \qquad\quad \text{curve end} \\
&= [1, b] . 3^{\lfloor k/2 \rfloor} \\
&= 1, \; 1+\omega_6, \; 3, \; 3+3\omega_6, \; 9, \; 9+9\omega_6, \; \ldots
\end{aligned}
$$

$$
\text{Re} = \tfrac{1}{2} \, \text{A038754}, \quad \text{Im} = \tfrac{1}{2}\sqrt{3} \, . \, \text{A254006}
$$

As noted above, expansion can be treated as conjugate to flip turns then multiply b. The low digit is then the terdragon *digit* positions. So the equivalent of (44) is

$$
AltPoint(3n+a) = b . \overline{AltPoint(n)} + digit(a) . \omega_3^{AltDir(3n)}
$$

This form uses $AltDir(3n)$, which is the direction of the new first segment, to rotate $digit(a)$ suitably. From (124) this is equivalent to $-AltDir(n)$. That negation is a conjugate the same as $\overline{AltPoint(n)}$.

The alternate terdragon boundary is a different shape than the terdragon, but its length is the same. That follows since the "odd" expansion in figure 38 gives R,V parts like figure 11 but opposite order, R $\to$ V,R and V $\to$ V,R
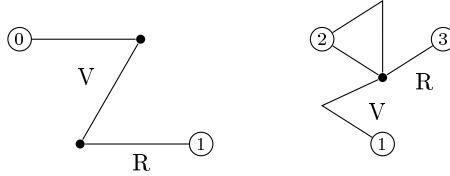
The enclosed area is likewise a different shape but the same number of unit triangles since each V expands to enclose a new unit triangle and each existing enclosed triangle expands to 3 new.

The number of single, double and triple visited points are likewise the same as terdragon S, D, T since they arise from the middles of 1,2,3 side triangles.

In figure 37 it can be seen a single bridge segment goes between two blobs of triangles. By symmetry this is the middle segment $n = \frac{1}{2}(3^k - 1)$.

$k{=}2$ is the first with a bridge segment. On expansion to $k{=}3$ there is the same form, with some additional triangles before and after, so on repeated expansion there is a single such segment in each level.

The bridge in $k{+}1$ is not the same segment number as in k. The k bridge is enclosed by the continuing curve, and the new bridge is the middle of the middle k sub-curve.
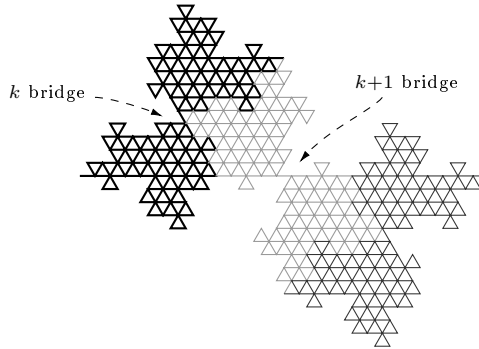
Theorem 54. *Number segments of the alternate terdragon curve starting from* $n{=}0$. *The right boundary segments are given by*

$$AltRpred_k(n) = \begin{cases} 1 & \text{if } FlipOdd_k(n) \text{ has no digit pair } 10, 21, 22 \\ 0 & \text{if any such pair} \end{cases}$$

$FlipOdd_k(n) = n$ in k many ternary digits, flip $0 \leftrightarrow 2$ at odd positions

Proof. Take right boundary sub-curves in parts R,M,E similar to theorem 17.
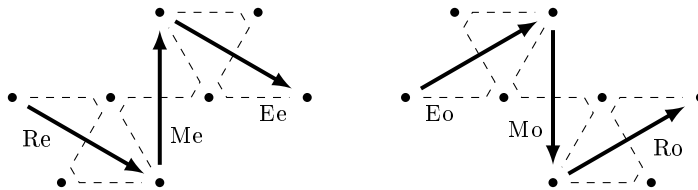
Re,Me,Ee are even k sub-curves. They comprise odd $k-1$ sub-curves in the conjugate base pattern. Ro,Mo,Eo are odd k sub-curves. They comprise even $k-1$ sub-curves in the plain base pattern. Ro has no adjacent sub-curves, like Re has none, and then Mo and Eo are back from there. Consequently for example Re goes to sub-curves in order Eo,Mo,Ro for digit $0, 1, 2$ respectively.

The expansions give the following state machine of new sub-curve type or "non" when enclosed and so not right boundary.
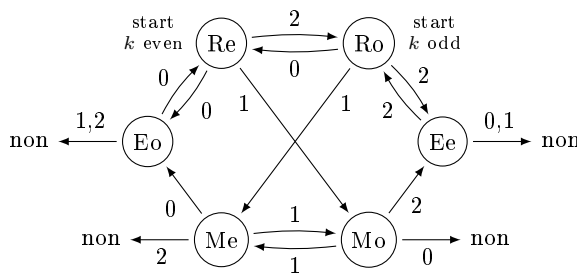
Figure 40:
$AltRpred_k(n)$
state machine,
ternary
high to low

Each successive digit goes alternately to an e or o state. Transitions out of e or o are the same but digit flipped $0 \leftrightarrow 2$. e is an even k sub-curve so its next digit is an odd position. Reckoning those odd positions flipped for unified R,M,E states means they are always reached by digit $0, 1, 2$ respectively. Transitions to "non" are then the disallowed pairs $10, 21, 22$. □

Some state machine manipulations can make a single starting state, rather than Re,Ro for k even,odd. This becomes 8 states by what is effectively simultaneous traversal with accepting or not for when no more digits (or when both reach non).

Some state machine manipulations or just following the disallowed pairs gives the following low to high form. This is a single starting state. o states are an odd number of digits below. o0 is 0 immediately below. o12 is either 1 or 2 immediately below. e states are an even number of digits below, with e01 having either 0 or 1 immediately below and e2 having 2 immediately below.
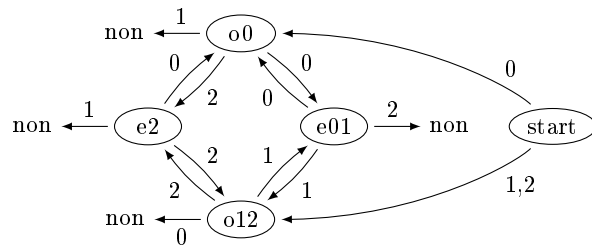
Figure 41:
$AltRpred(n)$
state machine,
ternary
low to high

A yet further approach is to take base-9 digits. Going high to low is R,M,E parts expanding per base-9 figure 39. The start state is R for even k, or an extra O for odd k and the high digit goes to R,M,E.
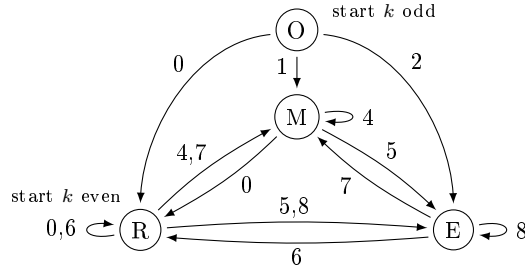
$AltRpred_k(n)$
state machine,
base-9
high to low

The alternate terdragon is symmetric in 180° rotation, so left boundary segment numbers are the right boundary counted from the end, which means $0 \leftrightarrow 2$ digit reversal. Disallowed pairs are the same by flipping at even instead of odd positions,

$$AltLpred_k(n) = AltRpred_k(3^k - 1 - n)$$
$$= \begin{cases} 1 & \text{if } FlipEven_k(n) \text{ has no digit pair } 10, 21, 22 \\ 0 & \text{if any such pair} \end{cases}$$

$FlipEven_k(n) = n$ in k many ternary digits, flip $0 \leftrightarrow 2$ at even positions

For the alternate terdragon continued infinitely, an infinite number of high 0 digits can be considered. After one high 0, the *AltRpred* result is unchanged. This can be seen in low to high figure 41 where a 0 goes to "non" from o12 and otherwise 0s go to and bounce between o0 and e01.

$$AltRpred_\infty(n) = AltRpred_k(n) \quad \text{for } k \text{ with } 3^k > 3n$$
$$= 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, \ldots$$

$$=1 \text{ at } n = 0, 4, 5, 6, 7, 8, 36, 40, 41, \ldots$$
$$AltLpred_\infty(n) = AltLpred_k(n) \quad \text{for } k \text{ with } 3^k > 3n$$
$$= 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$$
$$=1 \text{ at } n = 0, 1, 2, 12, 13, 17, 18, 19, 20, \ldots$$

Those n which are *AltRpred* can be formed from an index m written in mixed radix ternary low and rest binary, similar to Rn theorem 18. Working low to high through figure 41, at each state after the start there are two digits continuing. For example at o0 either $0, 2$. The binary digits choose those two.

The R,M,E states of figure 40 give a count of how many sides the triangle on the right of a segment has, like *Rsides* from (78). This is 1 or 2 for a boundary segment or 3 for a non-boundary.

$$AltRsides_k(n) = \begin{cases} 1 & \text{if } AltRpred_k(n) \text{ state Re or Ro} \\ 2 & \text{if } AltRpred_k(n) \text{ state Me, Mo, Ee or Eo} \\ 3 & \text{if } AltRpred_k(n) \text{ state "non"} \end{cases}$$
$$= 3 - AltRpred_k(n).[2, 1, 1]_n \tag{127}$$
$$AltRsides_\infty(n) = AltRsides_k(n) \quad \text{for } k \text{ with } 3^k > 3n$$
$$= 1, 3, 3, 3, 2, 2, 1, 2, 2, 3, 3, 3, \ldots$$

For (127), the least significant digit of n goes from an o to an e, and those transitions in figure 40 are 0 to Re, or $1, 2$ to Me,Ee so $n \bmod 3$ determines the reduction from 3 sides.

Since the number of 1 or 2 side triangles on the alternate terdragon boundary are the same as the terdragon boundary, $AltRsides_k$ is a permutation of terdragon *Rsides* and so for example the total is the same as from (80).

As noted above, the alternate terdragon number of single, double and triple visited points are the same as the plain terdragon. The argument of theorem 28 therefore gives the same $Lines_k$. But the line lengths and arrangement is different.

$$d=0 \qquad\qquad d=1 \qquad\qquad d=2$$
$$AltLines_4(0) = 11 \quad AltLines_4(1) = 10 \quad AltLines_4(2) = 10$$
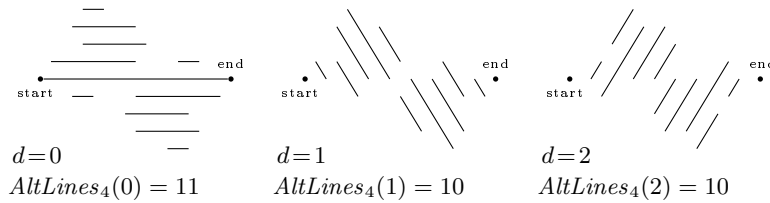
Figure 42: $k=4$, total $Lines_4 = 31$

Theorem 55. *The number of lines in direction $d = 0, 1, 2 \times 120°$ of alternate terdragon k are*

$$AltLines_k(0) = \tfrac{1}{3}\left(2^{k+1} + [1, 2]\right)$$
$$= 1, 2, 3, 6, 11, 22, 43, 86, 171, \ldots \qquad \text{A005578}$$
$$AltLines_k(1) = \tfrac{1}{3}\left(2^{k+1} - [2, 1]\right)$$
$$= 0, 1, 2, 5, 10, 21, 42, 85, 170, \ldots \qquad \text{A000975}$$

$$AltLines_k(2) = \tfrac{1}{3}\left(2^{k+1} - [2,4]\right)$$
$$= 0, 0, 2, 4, 10, 20, 42, 84, 170, \ldots \qquad \text{\small A167030}$$

Proof. Single and double visited points as line endpoints of each direction can be counted the same as theorem 29 $Lines(d)$. The alternate terdragon expands by a conjugate then replace. The conjugate reverses directions d, so $AltRTS$ recurrence in $-d$ and $-(d{+}1)$.
$$AltRTS_k(d) = AltRTS_{k-1}(-d) + AltRTS_{k-1}(-d{-}1)$$

Starting $AltRTS_0(0) = 1$ and $AltRTS_0(1) = AltRTS_0(2) = 0$ is then as follows. Case $d \equiv 2$ is the Jacobsthal numbers.

$$AltRTS_k(d) = \tfrac{1}{3}\left(2^k + \begin{cases} [2,1] & \text{if } d \equiv 0 \\ [-1,-2] & \text{if } d \equiv 1 \\ [-1,1] & \text{if } d \equiv 2 \end{cases}\right) \qquad \text{1+2 side triangles by } d$$

$$AltRTS_k(0) = 1, 1, 2, 3, 6, 11, 22, 43, 86, 171, \ldots \qquad \text{\small A005578}$$
$$AltRTS_k(1) = 0, 0, 1, 2, 5, 10, 21, 42, 85, 170, \ldots \qquad \text{\small A000975}$$
$$AltRTS_k(2) = 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \ldots \qquad \text{\small A001045}$$

Singles and doubles by direction follow from these $AltRTS$ triangles expanding, but with relative directions from the triangles alternating with k,

$$AltSD_k(d) = \sum_{j=0}^{k-1} AltRTS_j(d + (-1)^j) \qquad \text{single, double points by } d$$
$$= 2\, AltRTS(k, d) \ - (2 \text{ if } d{=}0)$$
$$AltSD_k(0\,\text{or}\,1) = 0, 0, 2, 4, 10, 20, 42, 84, 170, 340, \ldots \qquad \text{\small A167030}$$
$$AltSD_k(2) = 0, 2, 2, 6, 10, 22, 42, 86, 170, 342, \ldots \qquad \text{\small A014113}$$

Then lines from single and double endpoints,

$$AltLines_k(d) = \tfrac{1}{2}\left(AltSD_k(d{+}1) + AltSD_k(d{+}2) \ + (2 \text{ if } d{=}0)\right) \qquad \square$$

$AltLines_k(1) = AltLines_k(2)$ when k even, so the same number of line, but in general the set of line lengths are not the same. In $k{=}4$ figure 42 they are the same set of lengths, but for example in $k{=}6$ they are not.

The convex hull around the alternate terdragon is a rectangle with truncated corners.
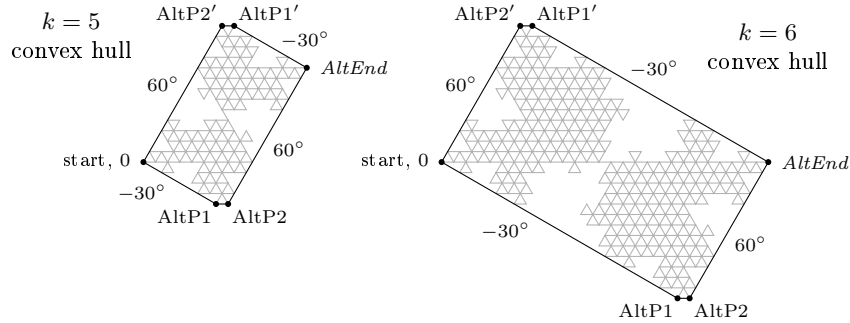
Theorem 56. *The convex hull around alternate terdragon $k \geq 2$ is a 6 sided polygon comprising curve start, end, and further points*

$$AltP1_k = \tfrac{1}{2}\overline{b}\big(3^{\lfloor k/2 \rfloor} - 1\big) \qquad AltP1'_k = AltEnd_k - AltP1_k = AltP2' + 1 \quad (128)$$

$$AltP2_k = AltP1_k + 1 \qquad AltP2'_k = AltEnd_k - AltP2_k = \tfrac{1}{2}\omega_6\big(3^{\lceil k/2 \rceil} - 1\big)$$

For k=0 these are the hull points, but $AltP1_0 = AltP2'_0 = 0$ and $AltP2_0 = AltP1'_0 = 1$ coincide with curve start and end.

For k=1 these are the hull points, but $AltP1_1 = 0$ and $AltP1'_1 = b$ coincide with curve start and end.

Proof. The hull around curve k is formed from the hulls around its three $k-1$ sub-curves. The odd and even cases are

Figure 43:
alternate terdragon
hull sub-curves
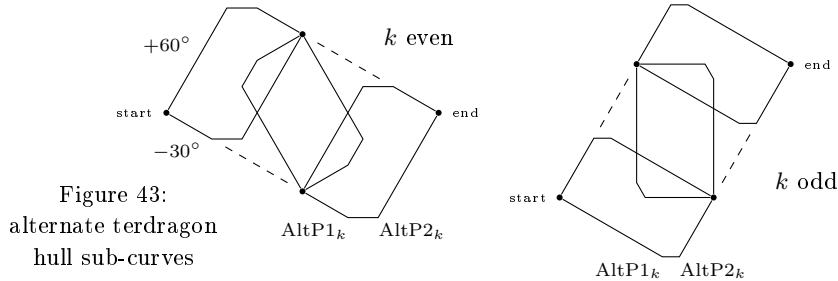
AltP1, AltP2 shift down along the $\overline{b}$ line to form k even, or are unchanged for k odd. Conversely *AltP1′, AltP2′*, giving (128). □

AltP1, AltP2 are horizontal 1 segment apart. They cut off a half unit triangle from what is otherwise a rectangle. The rectangle corners are

$$AltPC_k = \tfrac{1}{2}\overline{b}\,3^{\lfloor k/2 \rfloor} \qquad AltPC'_k = AltEnd_k - AltPC_k = \tfrac{1}{2}\omega_6\,3^{\lceil k/2 \rceil}$$

The hull sub-curves in figure 43 also give the points which are on the hull boundary. The side 0 down to *AltP1* replicates in k even at a point which is $(1 + \overline{\omega_3})\,AltEnd_{k-1} = \overline{b}.3^{k/2-1}$. The first of these is $k=2$ at $\overline{b}$. So locations $z = m.\overline{b}$ where m in ternary is digits $0, 1$ only. Similarly by replication, these are point numbers n with base-9 digits $0, 6$ only.

The side 0 up to $AltP2'$ replicates for k odd. Its first replication is $k{=}1$ to ω_6. So locations $z = m.\omega_6$ where m in ternary has only digits $0, 1$, and point numbers n with base-9 digits $0, 2$ only.

The hull extents are at angles $-30°$ to $+60°$ for total arc $90°$. Adjacent arms of the curve at $60°$ have $30°$ of interlacing with their adjacent arms before and after, and $30°$ which is solely the arm itself.
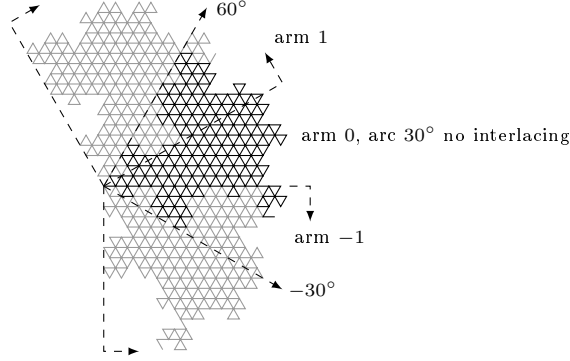
Moment of inertia of the alternate terdragon rotating about the z axis (within the x, y plane), at its centre of mass, is the same as the terdragon I_z. Terdragon theorem 43 applies for any unfold angle, including (by symmetry) what would be negative angles at odd expansions in the alternate terdragon.

Theorem 57. *Consider each segment of the alternate terdragon to have a unit mass uniformly distributed along its length. The centre of mass is the centre of the curve. With the x axis aligned to the endpoints, the moment of inertia tensor about the centre is*

$$\begin{pmatrix} AltI_x & -AltI_{xy} & 0 \\ -AltI_{xy} & AltI_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \qquad \begin{matrix} AltI_x = \sum y^2 & AltI_{xy} = \sum xy \\ AltI_y = \sum x^2 & I_z = \sum x^2{+}y^2 \end{matrix}$$

where I_z is from the plain terdragon and the rest are

$$AltI_x(k) = \tfrac{1}{156}\left(5.9^k - [5,6].3^{\lfloor k/2 \rfloor}\right)$$
$$= 0, \tfrac{1}{4}, \tfrac{5}{2}, \tfrac{93}{4}, 210, \tfrac{7569}{4}, \tfrac{34065}{2}, \ldots$$
$$AltI_y(k) = \tfrac{1}{156}\left(8.9^k + [5,6].3^{\lfloor k/2 \rfloor}\right)$$
$$= \tfrac{1}{12}, \tfrac{1}{2}, \tfrac{17}{4}, \tfrac{75}{2}, \tfrac{1347}{4}, \tfrac{6057}{2}, \tfrac{109017}{4}, \ldots$$
$$AltI_{xy}(k) = -(-1)^k . \tfrac{\sqrt{3}}{156}\left(2.9^k - [2,5].3^{\lfloor k/2 \rfloor}\right)$$
$$= \sqrt{3}.\left\{0, \tfrac{1}{12}, -1, \tfrac{37}{4}, -84, \tfrac{3027}{4}, -6813, \ldots\right\}$$

Proof. Odd k comprises even $k{-}1$ sub-parts the same as figure 28 and mutual recurrences (114). Even k is odd $k{-}1$ sub-parts turning the other way. Working through those is opposite signs on $AltI_{xy}$ for mutual recurrences

$$AltI_x(k) = \tfrac{3}{2}AltI_x(k{-}1) + (-1)^k\sqrt{3}\,AltI_{xy}(k{-}1) + \tfrac{3}{2}AltI_y(k{-}1) + \tfrac{1}{8}9^{k-1}$$

$$AltI_y(k) = \tfrac{3}{2}\,AltI_x(k{-}1) \; - \; (-1)^k\sqrt{3}\,AltI_{xy}(k{-}1) \; + \; \tfrac{3}{2}\,AltI_y(k{-}1) \; + \; \tfrac{3}{8}9^{k-1}$$

$$AltI_{xy}(k) = (-1)^k\left(-\tfrac{1}{2}\sqrt{3}\,AltI_x(k{-}1) \; + \; \tfrac{1}{2}\sqrt{3}\,AltI_y(k{-}1) \; - \; \tfrac{1}{8}\sqrt{3}.9^{k-1}\right) \qquad \Box$$

Principal axes are at

$$Alt\alpha = \tfrac{1}{2}\arctan\frac{-2\,AltI_{xy}(k)}{AltI_x(k) - AltI_y(k)} \; + \; (0\,\text{or}\,\tfrac{\pi}{2})$$

$$= \tfrac{1}{2}\arctan\!\left((-1)^k\tfrac{4}{\sqrt{3}} - \epsilon_k\right) \; + \; (0\,\text{or}\,\tfrac{\pi}{2})$$

$$\text{where } \epsilon_k = \frac{26.[-2,3].3^{\lfloor k/2\rfloor}}{3\big(3.9^k + [10,12].3^{\lfloor k/2\rfloor}\big)}$$

$\epsilon_k \to 0$ as $k \to \infty$ so the limit is term $\tfrac{1}{2}\arctan\tfrac{4}{\sqrt{3}}$. Factor $(-1)^k$ is conjugate negating y when k odd. Taking the fractal as repeated "unpoint", which means the even case, limits are

$$Alt\alpha_{min} \to -\tfrac{1}{2}\arctan\tfrac{4}{\sqrt{3}} \quad = -33.293387...^\circ$$

$$Alt\alpha_{max} = Alt\alpha_{min} + \tfrac{\pi}{2} \quad \to 56.706612...^\circ$$
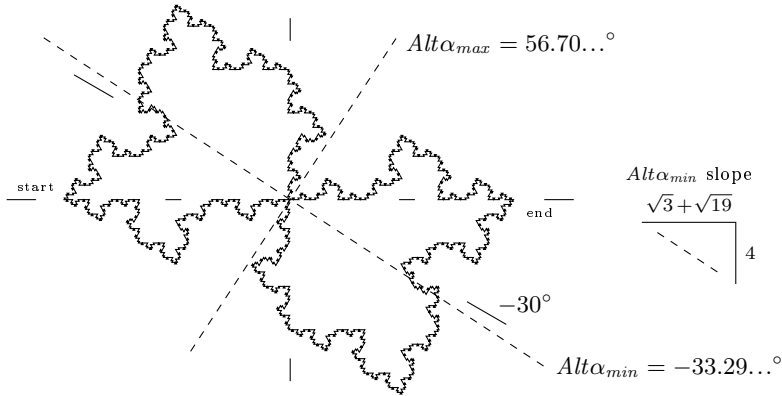
The convex hull around the curve has limit rectangle aligned at $-30°$. $Alt\alpha_{min}$ is a little below that. Roughly speaking, there is a little more curve in the quarters clockwise from it than anti-clockwise.

For the curve scaled to unit length, unit mass, and rotated $-Alt\alpha_{min}$ so that x axis is the minimum, the inertia limit is

$$\begin{pmatrix} \tfrac{1}{24} - \tfrac{1}{312}\sqrt{57} & 0 & 0 \\ 0 & \tfrac{1}{24} + \tfrac{1}{312}\sqrt{57} & 0 \\ 0 & 0 & \tfrac{1}{12} \end{pmatrix}$$

15.1   Alternate Terdragon Graph

The diameter of the alternate terdragon curve as a graph is attained between curve start and end, and for even k between some additional points. In both cases all diameter paths cross the middle bridge.

Figure 44

$AltDiameter_5 = 19$
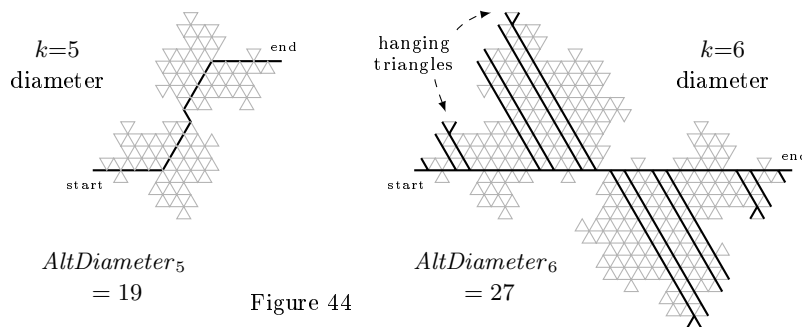
$AltDiameter_6 = 27$

Theorem 58. *The diameter of the alternate terdragon as a graph is*

$$AltDiameter_k = \begin{cases} 3^{\lfloor k/2 \rfloor} & \text{if } k \text{ even} \\ 2.3^{\lfloor k/2 \rfloor} + 1 & \text{if } k \text{ odd} \end{cases}$$

$$= 1, 3, 3, 7, 9, 19, 27, 55, 81, 163, \ldots$$

Diameter endpoints (ie. vertices with eccentricity AltDiameter, peripheral vertices) are the curve start and end, and for k even also points on the upper left and lower right convex hull boundaries and one to the right or left respectively when a hanging triangle there. The number of endpoints is

$$AltDiameterEndpoints_k = \begin{cases} 2 & \text{if } k=0 \text{ or } k \text{ odd} \\ 6 & \text{if } k=2 \\ 5.2^{k/2-1} & \text{if } k \text{ even} \geq 4 \end{cases}$$

$$= 2, 2, 6, 2, 10, 2, 20, 2, 40, \ldots$$

Proof. For k even, the claimed *AltDiameter* is the geometric distance through the grid. For k odd, it is geometric distance $+1$, that extra being since the middle bridge segment crossed is perpendicular to curve start to end.

To see that suitable segments of the curve exist for these distances, for k even curve start to end is a straight line and remains so on replacement by the base-9 figure 39. For points on the diagonal hull boundaries, suppose they exist in some even $k-2$, then the first half of curve k made from those sub-curves is

$k-2$ first half

k first half
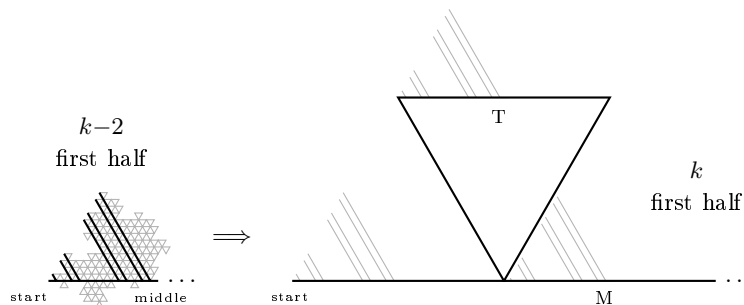
T is a $k-2$ sub-curve. The hull points in it have lines down to the big triangle. That triangle has straight sides and is a full grid inside by plane filling. Then sub-curve M has the same lines as T.

The longest of those lines just precedes the sub-curve midpoint, again by replication. So for k odd, the longest line in the middle sub-curve $k-1$ goes down to the first sub-curve.
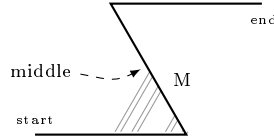
To see no other paths attain or are shorter than the claimed *AltDiameter*, when a segment expands twice, new points are at most $+2$ from an existing,
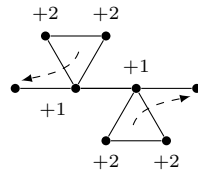
Figure 45:
new vertices distance
to expanded originals

For paths entirely within the first half of the curve, meaning up to and not including the middle bridge segment, it can be verified explicitly all are $< AltDiameter$ through to $k=4$. In $k=3,4$ those paths are lengths at most $3, 6$ respectively. Thereafter on each expansion points are at most $+2$ each end so an upper bound

$$h_k = 3h_{k-2} + 4 \qquad \text{starting } h_3 = 3, \ h_4 = 6$$
$$= [\tfrac{8}{9}, \tfrac{5}{3}].3^{\lfloor k/2 \rfloor} - 2 \qquad \text{for } k \geq 3$$
$$< AltDiameter_k$$

For paths crossing the middle bridge segment, points not a diameter endpoint are distance $\leq AltDiameter - 1$ and on expansion the $+2$ at that end is new $3. -1 + 2 = -1$ shorter than diameter still. So only new points of figure 45 which are $\leq +2$ from what was an existing diameter endpoint need be considered.

For k odd the only existing diameter endpoint is curve start and end. It expands

k odd
curve start
expansion
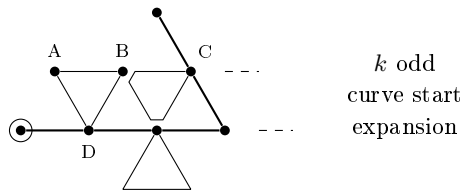
A,B,D are at $+1, +2$ from the start. D is on the expanded existing diameter path so is shorter. A,B can go to C. There is a horizontal line there across to the diameter path, since there is trivially in $k=1$ and thereafter that line passes through enclosed triangles underneath the corresponding line of $k-2$.

For k even,

k even
curve start
expansion

At T, new points A,B are diameter ends per the theorem. C,E are $1,2$ shorter down from A so are not. D is 1 shorter down from T so is not. At S, the same except its I is not a diameter end because it can go to the right to the line down from T so 1 shorter.

At U, all of U,E,F,G,K are ≤ 2 from J and that J is -3 on the diameter path of A, so U,E,F,G,K not diameter ends.

For *AltDiameterEndpoints* in even *k*, after *k*=4 the sets of 5 diagonal points and extra hanging triangle are replicated. The convex hull around the sub-curves and the replication locations mean there is no touching of those points and hanging triangles. Two sets of 5 are shown in figure 44 sample *k*=6.

$$AltDiameterEndpoints_k = 2\,AltDiameterEndpoints_{k-2} \qquad k \text{ even} \geq 6 \qquad \square$$

For both *k* even and odd there are various different paths between diameter endpoints. For *k* even the lines can go across as far as the last line before going down. For *k* odd the line shown in figure 44 can variously go up and then across.

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    Note a misprint page 97 case p=3 formula for $y^{(0)}$ (as used here) shows $-\frac{2}{3}$ cos but should be $+\frac{2}{3}$ cos.

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