AN ALGEBRA OF PIECES OF SPACE —
HERMANN GRASSMANN TO GIAN CARLO ROTA

HENRY CRAPO

ABSTRACT. We sketch the outlines of Gian Carlo Rota's interaction with the ideas that Hermann Grassmann developed in his *Ausdehnungslehre*[13, 15] of 1844 and 1862, as adapted and explained by Giuseppe Peano in 1888. This leads us past what Gian Carlo variously called *Grassmann-Cayley algebra* and *Peano spaces* to the *Whitney algebra* of a matroid, and finally to a resolution of the question "What, really, was Grassmann's *regressive product*?". This final question is the subject of ongoing joint work with Andrea Brini, Francesco Regonati, and William Schmitt.

1. ALMOST TEN YEARS LATER

We are gathered today in order to renew and deepen our recollection of the ways in which our paths intersected that of Gian Carlo Rota. We do this in poignant sadness, but with a bitter-sweet touch: we are pleased to have this opportunity to meet and to discuss his life and work, since we know how Gian Carlo transformed us through his friendship and his love of mathematics.

We will deal only with the most elementary of geometric questions; how to represent pieces of space of various dimensions, in their relation to one another. It's a simple story, but one that extends over a period of some 160 years. We'll start and finish with Hermann Grassmann's project, but the trail will lead us by Giuseppe Peano, Hassler Whitney, to Gian Carlo Rota and his colleagues.

Before I start, let me pause for a moment to recall a late afternoon at the Accademia Nazionale dei Lincei, in 1973, on the eve of another talk I was petrified to give, when Gian Carlo decided to teach me how to *talk*, so I wouldn't make a fool of myself the following day. The procedure was for me to start my talk, with an audience of one, and he would interrupt whenever there was a problem. We were in that otherwise empty conference hall for over two hours, and I never got past my first paragraph. It was terrifying, but it at least got me through the first battle with my fears and apprehensions, disguised as they usually are by timidity, self-effacement, and other forms of apologetic behavior.

2. SYNTHETIC PROJECTIVE GEOMETRY

Grassmann's plan was to develop a purely formal algebra to model natural (synthetic) operations on geometric objects: *flat*, or *linear* pieces of space of all possible dimensions. His approach was to be *synthetic*, so that the symbols in his algebra

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would denote geometric objects themselves, not just numbers (typically, coordinates) that could be derived from those objects by measurement. His was not to be an algebra of numerical quantities, but an algebra of pieces of space.

In the *analytic* approach, so typical in the teaching of Euclidean geometry, we are encouraged to assign "unknown" variables to the coordinates of variable points, to express our hypotheses as equations in those coordinates, and to derive equations that will express our desired conclusions.

The main advantage of a synthetic approach is that the logic of geometric thought and the logic of algebraic manipulations may conceivably remain parallel, and may continue to cast light upon one another. Grassmann expressed this clearly in his introduction to the *Ausdehnungslehre*[13, 14]:

Grassmann (1844): *"Each step from one formula to another appears at once as just the symbolic expression of a parallel act of abstract reasoning. The methods formerly used require the introduction of arbitrary coordinates that have nothing to do with the problem and completely obscure the basic idea, leaving the calculation as a mechanical generation of formulas, inaccessible and thus deadening to the intellect. Here, however, where the idea is no longer strangely obscured but radiates through the formulas in complete clarity, the intellect grasps the progressive development of the idea with each formal development of the mathematics."*

In our contemporary setting, a synthetic approach to geometry yields additional benefits. At the completion of a synthetic calculation, there is no need to *climb back up* from scalars (real numbers, necessarily subject to round-off errors, often rendered totally useless by division by zero) or from drawings, fraught with their own approximations of incidence, to statements of geometric incidence. In the synthetic approach, one even receives precise warnings as to particular positions of degeneracy. The synthetic approach is thus tailor-made for machine computation.

Gian Carlo was a stalwart proponent of the synthetic approach to geometry during the decade of the 1960's, when he studied the combinatorics of ordered sets and lattices, and in particular, matroid theory. But this attitude did not withstand his encounter with invariant theory, beginning with his lectures on the invariant theory of the symmetric group at the A.M.S. summer school at Bowdoin College in 1971.

As Gian Carlo later said, with his admirable fluency of expression in his adopted tongue,

"Synthetic projective geometry in the plane held great sway between 1850 and 1940. It is an instance of a theory whose beauty was largely in the eyes of its beholders. Numerous expositions were written of this theory by English and Italian mathematicians (the definitive one being the one given by the American mathematicians Veblen and Young). These expositions vied with one another in elegance of presentation and in cleverness of proof; the subject became required by universities in several countries. In retrospect, one wonders what all the fuss was about."

"Nowadays, synthetic geometry is largely cultivated by historians, and the average mathematician ignores the main results of this once flourishing branch of mathematics. The claim that has been raised by defenders of synthetic geometry,

that synthetic proofs are more beautiful than analytic proofs, is demonstrably false. Even in the nineteenth century, invariant-theoretic techniques were available that could have provided elegant, coordinate-free analytic proofs of geometric facts without resorting to the gymnastics of synthetic reasoning and without having to stoop to using coordinates."

Once one adopts an invariant-theoretic approach, much attention must be paid to the reduction of expressions to standard form, where one can recognize whether a polynomial in determinants is equal to zero. The process is called *straightening*, and it was mainly to the algorithmic process of straightening in a succession of algebraic contexts that Gian Carlo devoted his creative talents during three decades. We filled pages with calculations such as the following, for the bracket algebra of six points in a projective plane:

$$\begin{array}{ccc} b & c & d \\ a & e & f \end{array} - \begin{array}{ccc} a & c & d \\ b & e & f \end{array} + \begin{array}{ccc} a & b & d \\ c & e & f \end{array} - \begin{array}{ccc} a & b & c \\ d & e & f \end{array} = 0$$

the *straightening* of the two-rowed tableau on the left. This expression we would write in *dotted* form

$$\begin{array}{ccc} \dot{b} & \dot{c} & \dot{d} \\ \dot{a} & e & f \end{array}$$

that would be *expanded* to the above expression by summing, with alternating sign, over all permutations of the dotted letters. The basic principle is that *dotting a dependent set of points yields zero.*

To make a long and fascinating story short, Gian Carlo finally settled upon a most satisfactory compromise, a formal super-algebraic calculus, developed with Joel Stein, Andrea Brini, Marilena Barnabei [2, 3, 4, 5] and a generation of graduate students, that managed to stay reasonably close to its synthetic geometric roots. His brilliant students Rosa Huang and Wendy Chan [7, 8, 4, 18] carried the torch of synthetic reasoning across this new territory. They rendered feasible a unification of super-algebraic and synthetic geometry, but, as we soon realize, the process is far from complete.

First, we should take a closer look at Grassmann's program for synthetic geometry.

3. HERMANN GRASSMANN'S ALGEBRA

Grassmann emphasizes that he is building an *abstract* theory that can then be *applied to real physical space*. He starts not with geometric axioms, but simply with the notion of a skew-symmetric product of letters, which is assumed to be distributive over addition and modifiable by scalar multiplication. Thus if a and b are letters, and A is any product of letters, then $Aab + Aba = 0$. This is the skew-symmetry. It follows that any product with a repeated letter is equal to zero.

He also develops a notion of *dependency*, saying that a letter e is *dependent* upon a set $\{a, b, c, d\}$ if and only if there are scalars $\alpha, \beta, \gamma, \delta$ such that

$$\alpha\, a + \beta\, b + \gamma\, c + \delta\, d = e.$$

Grassmann realizes that such an expression is possible if and only if the point e lies in the *common system*, or projective subspace spanned by the points a, b, c, d. He

uses an axiom of distributivity of product over sums to prove that the product of letters forming a dependent set is equal to zero. With a, b, c, d, e as above:

$$abcde = \alpha\, abcda + \beta\, abcdb + \gamma\, abcdc + \delta\, abcdd = 0$$

the terms on the right being zero as products because each has a repeated letter.

As far as I can see, he establishes no formal axiomatization of the relation of linear dependence, and in particular, no statement of the exchange property. For that, we must wait until 1935, and the matroid theory of Whitney, MacLane and Birkhoff.

The application to geometry is proposed via *an interpretation* of this abstract algebra. The individual letters may be understood as *points*,

The center of gravity of several points can be interpreted as their sum, the displacement between two points as their product, the surface area lying between three points as their product, and the region (a pyramid) between four points as their product.

Grassmann is delighted to find that, in contrast to earlier formalizations of geometry, there need be no a priori maximum to the rank of the overall space being described:

The origins of this science are as immediate as those of arithmetic; and in regard to content, the limitation to three dimensions is absent. Only thus do the laws come to light in their full clarity, and their essential interrelationships are revealed.

Giuseppe Peano, in rewriting Grassmann's theory, chose to assume a few basic principles of *comparison of signed areas and volumes*, and to base all proofs on those principles. For instance, given a line segment ab between distinct points a, b in the plane, and points c, d off that line, then the signed areas abc and abd will be equal (equal in magnitude, and equally oriented CW or CCW) if and only if c and d lie on a line *parallel* to ab. See Figure 1.

The corresponding statement for three-dimensional space is that signed volumes $abcd$ and $abce$, for non-collinear points a, b, c, are equal (equal volume, and with the same chirality, or handedness) if and only if d and e lie on a line *parallel* to the plane abc. See Figure 2. Since Peano restricts his attention to three-dimensional space, this principle is his main axiom, with a notion of parallelism taken to be understood. This means that even the simplest geometric properties are ultimately rephrased as equations among measured volumes. For instance, Peano wishes to show that three points a, b, c are *collinear* if and only if the linear combination $bc - ac + ab$ of products of points is equal to zero. He shows that for every pair p, q of points, if the points a, b, c are in that order on a line, the tetrahedron $acpq$ is the disjoint union of the tetrahedra $abpq$ and $bcpq$, so their volumes add:

$$abpq + bcpq = acpq.$$

A further argument about symmetry shows that $ab + bc = ac$ holds independent of the order of a, b, c on the line. The statement $abpq + bcpq = acpq$, quantified over all choices of p and q, is Peano's *definition* of the equality $ab + bc = ac$. So his proof is complete. Perhaps we can agree that this is putting the cart before the horse.

(Grassmann took expressions of the form $ab + bc = ac$, for three collinear points a, b, c, to be axiomatic in his algebra.)
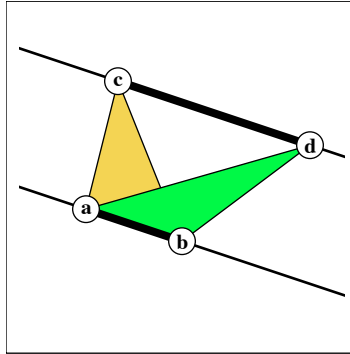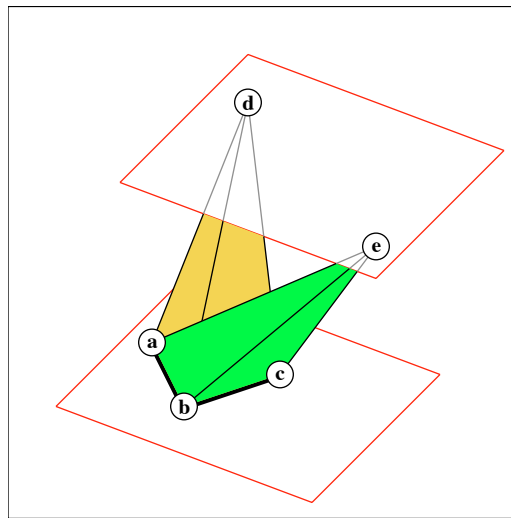

FIGURE 1. $abc = abd$ *iff* line ab parallel to line cd.

FIGURE 2. $abcd = abce$ *iff* line de parallel to plane abc.

I mention this strange feature of Peano's version because it became something of an *idée fixe* for Gian Carlo's work on Peano spaces. It gave rise to the technique of *filling brackets* in order to verify equations involving flats of low rank, a technique that unnecessarily relies on information concerning the rank of the overall space.

4. EXTENSORS AND VECTORS

We are all familiar with the formation of linear combinations of points in space, having studied linear algebra and having learned to replace points by their *position vectors*, which can then be multiplied by scalars and added. The origin serves as reference point, and that is all we need.

Addition of points is not well-defined in real projective geometry, because although points in a space of rank n (projective dimension $n-1$) may be represented as n-tuples of real numbers, the n-tuples are only determined up to an overall non-zero scalar multiple, and addition of these vectors will not produce a well-defined result. The usual approach is to consider *weighted points*, consisting of a position, given by *standard homogeneous coordinates*, of the form $(a_1, \ldots, a_{n-1}, 1)$, and a *weight* μ, to form a *point* $(\mu a_1, \ldots, \mu a_{n-1}, \mu)$ *of weight* μ. This is what worked for Möbius in his *barycentric calculus*. And it is the crucial step used by Peano to clarify the presentation of Grassmann's algebra.

This amounts to fixing a choice of *hyperplane at infinity* with equation $x_n = 0$. The *finite points* are represented as above, with weight 1. A linear combination $\lambda a + \mu c$, for scalars λ and μ positive, is a point b situated between a and c, such that the ratio of the distance $a \to b$ to the distance $b \to c$ is in the inverse proportion μ/λ, and the resulting weighted point has weight equal to $\lambda + \mu$, as illustrated in Figure 3.
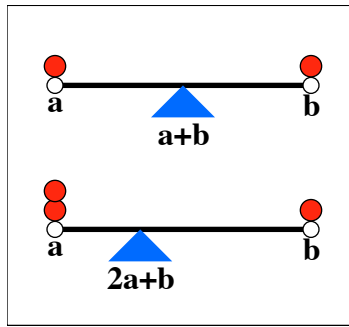

FIGURE 3. Addition of weighted points.

In particular, for points a and b of weight 1, $a + b$ is located the midpoint of the interval ab, and has weight 2, while $2a + b$ is located twice as far from b as from a, and has weight 3.

Both Grassmann and Peano are careful to distinguish between products ab of points, which have come to be called 2-*extensors*, and differences $b - a$ of points, which Grassmann calls *displacements* and Peano calls 1-*vectors*. The distinction between such types of objects is easily explained in terms of modern notation, in homogeneous coordinates.

In Figure 4 we show two equal 2-extensors, $ab = cd$, in red, and their difference vector, $v = b - a = d - c = f - e$ in blue, which represents a projective point, not a line, namely, the point at infinity on the line ab, with weight equal to the length from a to b and sign indicating the orientation form a to b. Check that $ab = av = bv$, so multiplication of a point a on the right by a vector v creates a line segment of length and direction v starting at a.

To avoid cumbersome notation, we will henceforth follow Peano's example, and give all examples with reference to a real projective 3-space, rank 4.
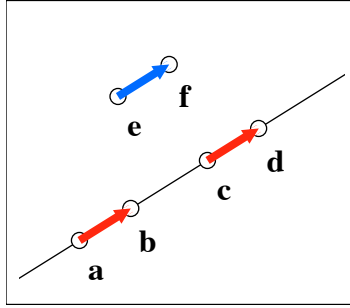
FIGURE 4. $ab = cd$, while $b - a = d - c = f - e$ is a point at infinity.

The homogeneous coordinates of any product of $k \leq 4$ weighted points (called a *k-extensor*), are the $k \times k$ minors of the matrix whose rows are the coordinate vectors of the k points in question. So for any four weighted points a, b, c, d in a space of rank 4,

$$a = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

$$ab = \begin{pmatrix} \mathbf{12} & \mathbf{13} & \mathbf{23} & \mathbf{14} & \mathbf{24} & \mathbf{34} \\ \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_2 & a_2 \\ b_2 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} & \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} & \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} \end{pmatrix}$$

$$abc = \begin{pmatrix} \mathbf{123} & \mathbf{124} & \mathbf{134} & \mathbf{234} \\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} \end{pmatrix}$$

$$abcd = \begin{pmatrix} \mathbf{1234} \\ \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \end{pmatrix}$$

If a, b, c, d are points of weight 1, the 2-extensors ab and cd are equal if and only if the line segments from a to b and from c to d are *collinear*, of equal length, and similarly oriented. More generally, Grassmann showed that two k-extensors $a \ldots b$ and $c \ldots d$ differ only by a non-zero scalar multiple, $a \ldots b = \sigma \, a \ldots d$, if and only if the sets of points obtainable as linear combinations of $a, \ldots, b$ and those from $c, \ldots, d$ form what we would these days call *the same projective subspace*. Such a subspace, considered as a set of projective points, we call a *projective flat*.

Coplanar 2-extensors add the way coplanar forces do in physical systems. Say you are forming the sum $ab + cd$ as in Figure 5. You slide the line segments representing the forces ab and cd along their lines of action until the ends a and c coincide at the point e of incidence of those two lines. The sum is then represented as the diagonal line segment of the parallelogram they generate.
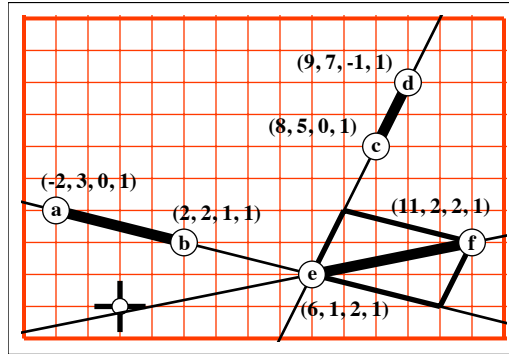
FIGURE 5. $ab + cd = ef$ for these six points of weight 1.

Let's carry out the explicit extensor calculations, so you know I'm not bluffing: let a, b, c, d be four points coplanar in 3-space (rank 4),

	1	2	3	4
a	−2	3	0	1
b	2	2	1	1
c	8	5	0	1
d	9	7	−1	1

That the four points are coplanar is clear from the fact that the four vectors are dependent: $-a + 2b - 3c + 2d = 0$. The 2-extensors ab and cd are, with their sum:

	12	13	23	14	24	34
ab	−10	−2	3	−4	1	−1
cd	11	−8	−5	−1	−2	1
ab + cd	1	−10	−2	−5	−1	0

The point e of intersection of lines ab and cd, together with the point f situated at the end of the diagonal of the parallelogram formed by the translates of the two line segments to e, have homogeneous coordinates

	1	2	3	4
e	6	1	2	1
f	11	2	2	1

and exterior product

	12	13	23	14	24	34
ef	1	−10	−2	−5	−1	0

equal to the sum $ab + cd$.

Cospatial planes add in a similar fashion. For any 3-extensor abc spanning a projective plane Q and for any 2-extensor pq in the plane Q, there are points r in Q such that $abc = pqr$. The required procedure is illustrated in Figure 6. We slide the point c parallel to the line ab until it reaches the line pq, shown in blue, at c'. Then slide the point a parallel to the line bc' until it reaches the line pq at a'. The oriented plane areas abc, abc', and $a'bc'$ are all equal, and the final triangle has an edge on the line pq.

So, given any 3-extensors abc, def, and for any pair p, q of points on the line of intersection of the planes abc and def, there exist points r, s such that $abc = pqr, def = pqs$, so $abc + def$ can be expressed in the form $pqr + pqs = pq(r + s)$, and the problem of adding planes in 3-space is reduced to the problem of adding points on a line. The result is shown in Figure 7, where $t = r + s$.
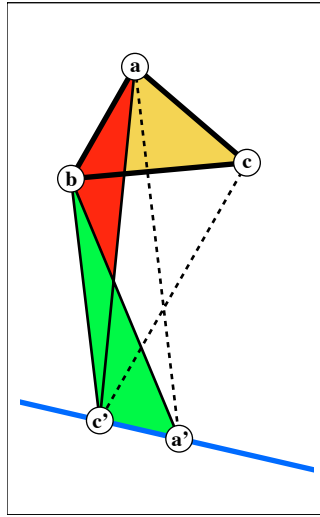
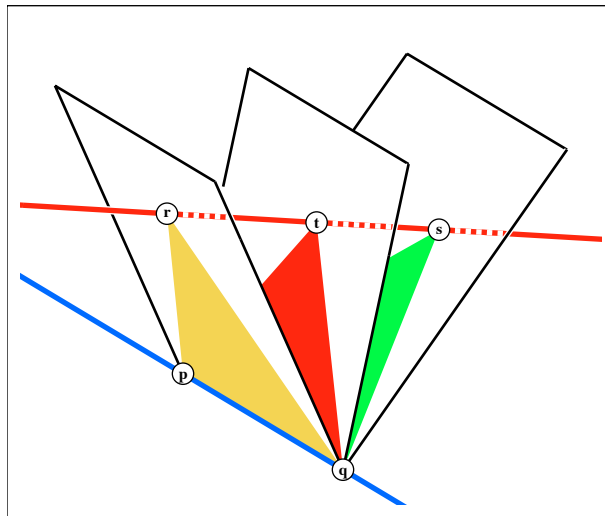

FIGURE 6. The 3-extensors abc and $a'bc'$ are equal.

FIGURE 7. A sum of cospatial 3-extensors, $pqr + pqs = pqt$.

In dimensions ≥ 3, it is necessary to *label* the individual coordinates with the set of columns used to calculate that minor. This practice becomes even more systematic in the subsequent super-algebraic approach of Rota, Stein, Brini and colleagues, where a coordinate abc_{ijk} will be denoted $(abc|ijk)$, with *negative letters*

a, b, c to denote vectors and *negative places* i, j, k to denote coordinate positions, and the minor is calculated by the *Laplace expansion*:

$$
\begin{aligned}
(abc|ijk) &= (a|i)(b|j)(c|k) - (a|i)(b|k)(c|j) + \cdots - (a|k)(b|j)(c|i), \\
&= (a|i)(b|j)(c|k) - (a|i)(c|j)(b|k) + \cdots - (c|i)(b|j)(a|k)
\end{aligned}
$$

as the sum of products of individual *letter-place* elements. We have chosen to list the coordinates of a line segment ab in 3-space in the order $12, 13, 23, 14, 24, 34$ because that makes the negative of the difference vector $b - a$ visible in the last three coordinate places $14, 24, 34$, and the *moments* about the 3rd, 2nd, and 1st coordinate axes, respectively, visible in the first three coordinate places.

A set of k 1-extensors $a, b, \ldots d$ has a non-zero product if and only if the k points are independent, and thus span a projective subspace of rank k (dimension $k - 1$). This integer k is called the *step* of the extensor.

We have seen that differences of points a, b of weight 1 are vectors, which means simply that they are 1-extensors e *at infinity*, with coordinate $e_4 = 0$.

In Grassmann's theory, there exist k-vectors of all steps k for which k is less than the rank of the entire space. In terms of standard homogeneous coordinates in a space of rank 4, they are those extensors for which all coordinates are zero whose labels involve the place 4. k-vectors are also definable as "boundaries" of $(k + 1)$-extensors, or as products of 1-vectors, as we shall see.

Each k-extensor has an associated $(k - 1)$-vector, which I shall refer to as its *boundary*,

$$
\begin{aligned}
\partial\, ab &= b - a \\
\partial\, abc &= bc - ac + ab \\
\partial\, abcd &= bcd - acd + abd - abc
\end{aligned}
$$

If the points a, b, c, d are of weight 1, then the 4th coordinate of $\partial\, ab$, the $14, 24, 34$ coordinates of $\partial\, abc$, and the $124, 134, 234$ coordinates of $\partial\, abcd$ are all equal to zero. Such extensors are, as elements of our affine version of projective space, pieces of space in the hyperplane at infinity. As algebraic objects they are vectors, so $\partial\, ab = \partial\, cd$, or $b - a = d - c$, for points a, b, c, d of weight 1, if and only if the line segments from a to b and from c to d are parallel, of equal length, and similarly oriented. That is, they are equal as difference vectors of position.

The subspace of k-vectors consists exactly of those k-tensors obtained by taking boundaries of $k+1$-extensors. The k-vectors are also expressible as exterior products of k 1-vectors, since

$$
\begin{aligned}
\partial\, ab &= (b - a) \\
\partial\, abc &= (b - a)(c - a), \\
\partial\, abcd &= (b - a)(c - a)(d - a), \\
&\quad \ldots.
\end{aligned}
$$

Remarkably, $p\,\partial\, abc = abc$ for any point p in the plane spanned by a, b, c. In particular, $a\,\partial\, abc = b\,\partial\, abc = c\,\partial\, abc$.

So $\partial\, abc = bc - ac + ab$ is a 2-vector, and it can only have non-zero coordinates with labels $12, 13, 23$. Geometrically, it can be considered to be a directed line segment in the line at infinity in the plane abc. It is also equal to a *couple* in the plane of the points a, b, c, that is, the sum of two 2-extensors that are parallel to

one another, of equal length and opposite orientation. Couples of forces occur in statics, and cause rotation when applied to a rigid body.

In Figure 8 we show two couples that are equal, though expressed as sums of quite different 2-extensors. The coordinate expressions are as follows.

	1	**2**	**3**		**1**	**2**	**3**
a	1	2	1	**e**	3	−2	1
b	−3	1	1	**f**	2	0	1
c	−1	4	1	**g**	8	−2	1
d	3	5	1	**h**	9	−4	1

$$ab + cd = \begin{pmatrix} \mathbf{12} & \mathbf{13} & \mathbf{23} \\ 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} \mathbf{12} & \mathbf{13} & \mathbf{23} \\ -17 & -4 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{12} & \mathbf{13} & \mathbf{23} \\ -10 & 0 & 0 \end{pmatrix}$$
$$ef + gh = \begin{pmatrix} \mathbf{12} & \mathbf{13} & \mathbf{23} \\ 4 & 1 & -2 \end{pmatrix} + \begin{pmatrix} \mathbf{12} & \mathbf{13} & \mathbf{23} \\ -14 & -1 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{12} & \mathbf{13} & \mathbf{23} \\ -10 & 0 & 0 \end{pmatrix}$$

$$ab + cd = (7, 4, 1) + (-17, -4, -1) = (-10, 0, 0)$$
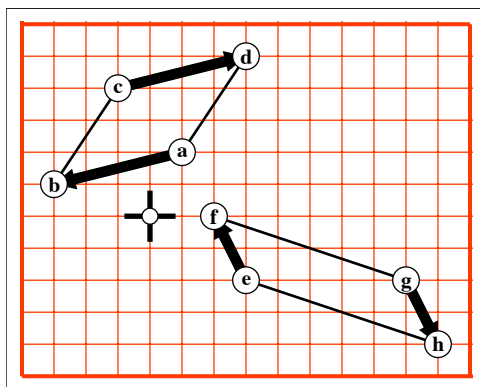$$ef + gh = (4, 1, -2) + (-14, -1, 2) = (-10, 0, 0)$$

FIGURE 8. The couples $ab + cd$ and $ef + gh$ are equal.

5. Reduced Forms

Grassmann did much more than create a new algebra of *pieces of space*. First he showed that every sum of 1-extensors (weighted points), with sum of weights not equal to zero, is equal to a single weighted point (with weight equal to the sum of the weights of the individual points). If the sum of the weights is zero, the resulting 1-extensor is a 1-vector (which may itself be zero). He then went on to show that every sum of coplanar 2-extensors is equal to a single 2-extensor, or to the sum of two 2-extensors on parallel lines, with equal length and opposite orientation, forming a couple, or is simply zero.

The situation for linear combinations of 2-extensors in 3-space is a bit more complicated. The sum of 2-extensors that are not coplanar is not expressible as a product of points, and so is not itself an extensor. It is simply an antisymmetric

tensor of step 2. Grassmann showed that such a linear combination of non-coplanar extensors can be reduced to the sum of a 2-extensor and a 2-vector, or couple.

Not bad at all, for the mid 19th century.

At the beginning of the next millenium, in 1900, Sir Robert Ball, in his *Theory of Screws* will use a bit of Euclidean geometry to show that a sum of 2-extensors in 3-space can be expressed as the sum of a force along a line plus a moment in a plane perpendicular to that line. He called these general antisymmetric tensors *screws*. Such a combination of forces, also called a *wrench*, when applied to a rigid body produces a *screw motion*.

Much of the study of screws, with applications to statics, is to be found at the end of Chapter 2 in Grassmann. He discusses coordinate notation, change of basis, and even shows that an anti-symmetric tensor S of step 2 (a screw) is an extensor if and only if the exterior product SS is equal to zero. This is the first and most basic invariant property of anti-symmetric tensors.

Any linear combination of 3-extensors in rank 4 (3-space) is equal to a single 3-extensor. This is because we are getting close to the rank of the entire space. The simplicity of calculations with linear combinations of points is carried over by duality to calculations with linear combinations of copoints, here, with planes.

The extreme case of this duality becomes visible in that k-extensors in k-space add and multiply just like scalars. For this reason they are called *pseudo-scalars*.

6. Grassmann-Cayley Algebra, Peano Spaces

Gian Carlo chose to convert pseudo-scalars to ordinary scalars by taking a determinant, or *bracket* of the product[12]. These brackets provide the scalar coefficients of any invariant linear combination. Thus, for any three points a, b, c on a projective line, there is a linear relation, unique up to an overall scalar multiple

$$[ab]c - [ac]b + [bc]a = 0$$

and for any four points a, b, c, d on a projective plane

$$[abc]d - [abd]c + [acd]b - [bcd]a = 0$$

So for any four coplanar points a, b, c, d,

$$[abc]d - [abd]c = [bcd]a - [acd]b,$$

and we have two distinct expressions for a projective point that is clearly on both of the lines cd and ab.

This is the key point in the development of the Grassmann-Cayley algebra [12, 17, 22], as introduced by Gian Carlo with his coauthors Peter Doubilet and Joel Stein. If A is an r-extensor and B is a s-extensor in a space of overall rank n, then the *meet* of A and B is defined by the two equivalent formulae

$$A \wedge B = \sum_{(A)_{r-k,k}} [A_{(1)}B]A_{(2)} = \sum_{(B)_{k,s-k}} [AB_{(2)}]B_{(1)}$$

This is the *Sweedler notation* (see below, Section 7) from Hopf algebra, where, for instance, in the projective 3-space, rank 4, a line ab and a plane cde as in Figure 9

will have meet

$$ab \wedge cde \;\; = \;\; [acde]b - [bcde]a \;\; = \;\; [deab]c - [ceab]d + [cdab]e.$$

This meet is equal to zero unless ab and cde together span the whole space, so the line meets the plane in a single point. You can check that the second equality checks with the generic relation
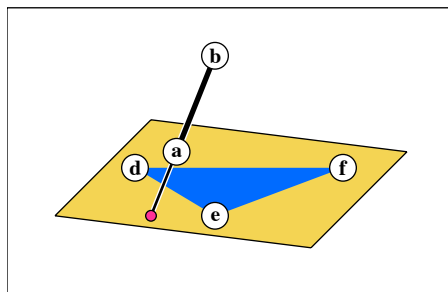
$$[abcd]e - [abce]d + [abde]c - [acde]b + [bcde]a \;\; = \;\; 0.$$

FIGURE 9. The meet of ab with cde.

The serious part of Grassmann-Cayley algebra starts when you try to use these simple relations to detect properties of geometric configuration in *special position*, or to classify possible interrelations between subspaces, that is, when you start work in *invariant theory*.

Gian Carlo and his colleagues made considerable progress in expressing the better-known invariants in terms of this new double algebra, and in reproving a certain number of theorems of projective geometry. Matters got a bit complicated, however, when they got to what they called the *alternating laws*, which "alternated" the operations of join (exterior product) and meet in a single equation. This required maintaining a strict accounting of the ranks of joins and meets, in order to avoid unexpected zeros along the way. They employed Peano's old technique of *filling brackets* whenever the situation got delicate. It took some real gymnastics to construct propositions of general validity when mixing joins and meets.

7. Whitney algebra

The idea of the Whitney algebra of a matroid starts with the simple observation that the relation

$$a \otimes bc - b \otimes ac + c \otimes ab = 0$$

among tensor products of extensors holds if and only if the three points a, b, c are collinear, and this *in a projective space of arbitrary rank*.

We need the notion of a *coproduct slice* of a word. For any non-negative integers i, j with sum n, the (i, j)-slice of the coproduct of an n-letter word W, written $\partial_{i,j} W$, is the Sweedler sum

$$\sum_{(W)_{i,j}} W_{(1)} \otimes W_{(2)},$$

that is, it is the sum of tensor products of terms obtained by decomposing the word W into two subwords of indicated lengths, both in the linear order induced from that on W, with a sign equal to that of the permutation obtained by passing from W to its reordering as the concatenation of the two subwords. For instance,

$$\partial_{2,2}\, abcd = ab \otimes cd - ac \otimes bd + ad \otimes bc + bc \otimes ad - bd \otimes ac + cd \otimes ab.$$

We define the *Whitney algebra of a matroid* $M(S)$ as follows. Let E be the exterior algebra freely generated by the underlying set S of points of the matroid M, T the direct sum of tensor powers of E, and W the quotient of T by the homogeneous ideal generated by coproduct slices of words formed from *dependent* sets of points.

Note that this is a straight-forward analogue of the principle applied in the bracket algebra of a Peano space, that *dotting a dependent set of points yields zero*.

This construction of a Whitney algebra is reasonable because *these very identities hold in the tensor algebra of any vector space.* Consider, for instance, a set of four coplanar points a, b, c, d in a space of rank n, say for large n. Since a, b, c, d form a dependent set, the coproduct slice $\partial_{2,2}\, abcd$ displayed above will have $ij \otimes kl$ -coordinate equal to the determinant of the matrix whose rows are the coordinate representations of the four vectors, calculated by Laplace expansion with respect to columns i, j versus columns k, l. This determinant is zero because the vectors in question form a dependent set. Compare reference [19], where this investigation is carried to its logical conclusion.

The Whitney algebra of a matroid M has a geometric product reminiscent of the meet operation in the Grassmann-Cayley algebra, but the product of two extensors is not equal to zero when the extensors fail to span the entire space. The definition is as follows, where A and B have ranks r and s, the union $A \cup B$ has rank t, and $k = r + s - t$, which would be the "normal" rank of the intersection of the flats spanned by A and B if they were to form a modular pair. The *geometric product* is defined as either of the Sweedler sums

$$A \diamond B = \sum_{(A)_{r-k,k}} A_{(1)}B \otimes A_{(2)} = \sum_{(B)_{k,s-k}} AB_{(2)} \otimes B_{(1)}$$

This product of extensors is always non-zero. The terms in the first tensor position are individually either equal to zero or they span the flat obtained as the span (closure) E of the union $A \cup B$. For a represented matroid, the Grassmann coordinates of the left-hand terms in the tensor products are equal up to an overall scalar multiple, because $A_{(1)}B$ and $AB_{(2)}$ are non-zero if and only if $A_{(1)} \cup B$ and $A \cup B_{(2)}$, respectively, are spanning sets for the flat E. These extensors with span E now act like scalars for a linear combination of k-extensors representing the meet of A and B, equal to that meet whenever A and B form a modular pair, and equal to the vector space meet in any representation of the matroid.

The development of the Whitney algebra began with an exciting exchange of email with Gian Carlo in the winter of 1995-6. On 18/11/95 he called to say that he agreed that the "tensor product" approach to non-spanning syzygies is correct, so that

$$a \otimes bc - b \otimes ac + c \otimes ab$$

is the zero tensor whenever a, b, c are collinear points in any space, and gives a Hopf-algebra structure on an arbitrary matroid, potentially replacing the "bracket ring"[21], which had the disadvantage of being commutative.

Four days later he wrote: *"I just read your fax, it is exactly what I was thinking. I have gone a little further in the formalization of the Hopf algebra of a matroid, so far everything checks beautifully. The philosophical meaning of all this is that every matroid has a natural coordinatization ring, which is the infinite product of copies of a certain quotient of the free exterior algebra generated by the points of the matroid (loops and links allowed, of course). This infinite product is endowed with a coproduct which is not quite a Hopf algebra, but a new object closely related to it. Roughly, it is what one obtains when one mods out all coproducts of minimal dependent sets, and this, remarkably, gives all the exchange identities. I now believe that everything that can be done with the Grassmann-Cayley algebra can also be done with this structure, especially meets."*

On 29/11/95: *"I will try to write down something tonight and send it to you by latex. I still think this is the best idea we have been working on in years, and all your past work on syzygies will fit in beautifully."*

On 20/12/95: *"I am working on your ideas, trying to recast them in letterplace language. I tried to write down something last night, but I was too tired. Things are getting quite rough around here."*

Then, fortunately for this subject, the weather turned bad. On 9/1/96: *"Thanks for the message. I am snowbound in Cambridge, and won't be leaving for Washington until Friday, at least, so hope to redraft the remarks on Whitney algebra I have been collecting. ..."*

"Here are some philosophical remarks. First, all of linear algebra should be done with the Whitney algebra, so no scalars are ever mentioned. Second, there is a new theorem to be stated and proved preliminarily, which seems to be a vast generalization of the second fundamental theorem of invariant theory (Why, Oh why, did I not see this before?!)"

Here, Gian Carlo suggests a comparison between the Whitney algebra of a vector space V, when viewed as a matroid, and the exterior algebra of V.

" I think this is the first step towards proving the big theorem. It is already difficult, and I would appreciate your help. The point is to prove classical determinant identities, such as Jacobi's identity, using only Whitney algebra methods (with an eye toward their quantum generalizations!) Only by going through the Whitney algebra proofs will we see how to carry out a quantum generalization of all this stuff."

"It is of the utmost importance that you familiarize yourself with the letterplace representation of the Whitney algebra, through the Feynman operators, and I will write this stuff first and send it to you."

On 11/1/96, still snowbound in Cambridge, Gian Carlo composed a long text proposing two projects:

(1) the description of a module derived from a Whitney algebra $W(M)$,

(2) a faithful representation of a Whitney algebra as a quotient of a supersymmetric letter-place algebra.

This supersymmetric algebra representation is as follows. He uses the supersymmetric letter-place algebra $\mathrm{Super}[S^-|P^+]$ in negative letters and positive places. . A tensor product $W_1 \otimes W_2 \otimes \ldots \otimes W_k$ is sent to the product

$$(W_1|p_1^{|W_1|})(W_2|p_2^{|W_2|}) \ldots (W_k|p_k^{|W_k|})$$

where the words $p_i^{(|w_i|)}$ are divided powers of positive letters, representing the different possible positions in the tensor product. The letter-place pairs are thus anticommutative. The linear extension of this map on $W(M)$ he termed the *Feynman entangling operator*.

Bill Schmitt joined the project in the autumn of '96. The three us met together only once. Bill managed to solve the basic problem, showing that the Whitney algebra is precisely a *lax Hopf algebra*, the quotient of the tensor algebra of a free exterior algebra by an ideal that is not a co-ideal. The main body of this work Bill and I finished [11] in 2000, too late to share the news with Gian Carlo. It was not until eight years later that Andrea Brini, Francesco Regonati, Bill Schmitt and I finally established that Gian Carlo had been completely correct about the super-symmetric representation of the Whitney algebra.

For a quick taste of the sort of calculations one does in the Whitney algebra of a matroid, and of the relevant geometric signification, consider a simple matroid L on five points: two coplanar lines bc and de, meeting at a point a. In Figure 10 we exhibit, for the matroid L, the geometric reasoning behind the equation $ab \otimes cde = ac \otimes bde$.
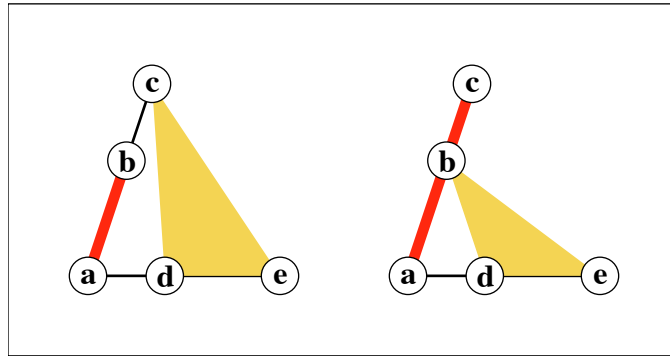

FIGURE 10. $ab \otimes cde = ac \otimes bde$.

First of all,

$$ab \otimes c - ac \otimes b + bc \otimes a = 0$$

because abc is dependent. Multiplying by de in the second tensor position,

$$ab \otimes cde - ac \otimes bde + bc \otimes ade = 0,$$

but the third term is zero because ade is dependent in L, so we have the required equality $ab \otimes cde = ac \otimes bde$.

This equation of tensor products expresses the simple fact that the ratio r of the lengths of the oriented line segments ac and ab is equal to the ratio of the oriented areas of the triangular regions cde and bde, so, in the passage from the product on the left to that on the right, there is merely a shift of a scalar factor r from the second term of the tensor product to the first. *As tensor products they are equal.*

The same fact is verified algebraically as follows. Since b is collinear with a and c, and d is collinear with a and e, we may write $b = (1-\alpha)a + \alpha\, c$ and $d = \beta\, a + (1-\beta)e$ for some choice of non-zero scalars α, β. Then

$$ab \otimes cde = (a \vee ((1 - \alpha)a + \alpha\, c)) \otimes (c \vee (\beta\, a + (1 - \beta)e) \vee e)$$
$$ac \otimes bde = (a \vee c) \otimes (((1 - \alpha)a + \alpha\, c) \vee (\beta\, a + (1 - \beta)e) \vee e)$$

both of which simplify to $-\alpha\beta\, (ac \otimes ace)$

Note that this equation is independent of the dimension of the overall space within which the triangular region ace is to be found.

8. GEOMETRIC PRODUCT

For words $u, v \in W^1$, with $|u| = r, |v| = s$, let $k = r + s - \rho(uv)$. The *geometric product* of u and v in W, written $u \diamond v$, is given by the expression

$$u \diamond v = \sum_{(u)_{r-k,k}} u_{(1)} v \circ u_{((2)}$$

For words A, B and integers r, s, k as above,

$$\sum_{(A)_{r-k,k}} A_{(1)} B \circ A_{((2)} = \sum_{(B)_{k,s-k}} A_{(1)} B \circ A_{((2)}.$$

So the geometric product is commutative:

$$A \diamond B = (-1)^{(r-k)(s-k)} B \diamond A$$

In Figure 11, we see how the intersection point (at b) of the line ef with the plane acd can be computed as a linear combination of points e and f, or alternately as a linear combination of points a, c and d, using the two formulations of the geometric product of ef and acd.
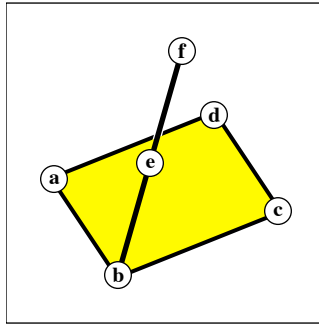

FIGURE 11. The geometric product of a line with a plane.

The calculation is as follows:
$$
\begin{aligned}
acd \diamond ef & = & acef \circ d - adef \circ c + cdef \circ a \\
& = & acdf \circ e - acde \circ f
\end{aligned}
$$

Figure 12 shows how the line of intersection (incident with points a, c, f) of planes abc and def can be computed as a linear combination of lines bc and ac, or can be obtained as a single term by the alternate form of the geometric product.
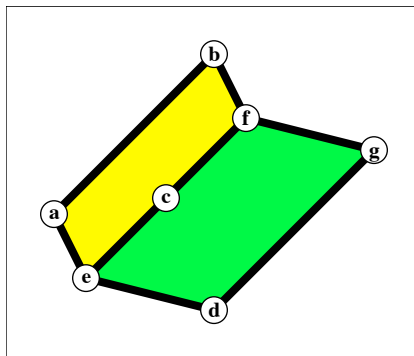

FIGURE 12. The geometric product of two planes.

The calculation is
$$
\begin{aligned}
abc \diamond def & = & adef \circ bc - bdef \circ ac \\
& = & abcd \circ ef
\end{aligned}
$$

Compare section 56-57, pages 88-89 in Peano[20].

9. REGRESSIVE PRODUCT

When I first lectured on Gian Carlo's work on the Grassmann-Cayley algebra to a seminar at McMaster University run by two eminent algebraists, Evelyn Nelson and Bernard Banaschewski, they insisted that the meet operation was *just* the dual of Grassmann's exterior product, and referred me to Bourbaki. I was never comfortable with this view, since I felt that a veritable geometric product would not be restricted to the two *extremities* of the lattice of subspaces, yielding non-trivial results only for independent joins and co-independent meets.

Lloyd Kannenberg performed an outstanding service to mathematics when he published his English translations [14, 16, 20] of these seldom-consulted works by Grassmann and Peano, the two *Ausdehnungslehre*, published under the titles *A New Branch of Mathematics* (1844) and *Extension Theory* (1862) and the *Geometric Calculus* of Peano, which, as Kannenberg says, "was published in a small print run in 1888, and has never been reissued in its entirety." Lloyd tells me that his translations of these classics were undertaken with Gian Carlo's active encouragement.

The *Ausdehnungslehre* of 1844 has an entire chapter on what Grassmann calls the regressive product. At the outset (¶125) Grassmann explains that he wants

a multiplication that will produce a non-zero value for the product of magnitudes A, B that are dependent upon one another. *"In order to discover this new definition we must investigate the different degrees of dependence, since according to this new definition the product of two dependent magnitudes can also have a nontrivial value."* (We will put all direct quotations from the English translation in *italics*. We also write ∘ for the regressive product, this being somewhat more visible than Grassmann's period "." notation.)

To measure the different degrees of dependence, Grassmann argues that the set of points dependent upon both A and B forms a *common system*, what we now call a projective subspace, the intersection of the spaces spanned by A and by B. *"To each degree of dependence corresponds a type of multiplication: we include all these types of multiplication under the name regressive multiplication."* The *order* of the multiplication is the value chosen for the rank of the common system.

In ¶126 Grassmann studies the modular law for ranks:

$$\rho(A) + \rho(B) = \rho(C) + \rho(D)$$

where $\rho()$ is the rank function, C is the common system (the lattice-theoretic meet) and D is the nearest covering system (the lattice-theoretic join). In ¶129 he explains the *meaning* of a geometric product. *" In order to bring the actual value of a real regressive product under a simple concept we must seek, for a given product whose value is sought, all forms into which it may be cast, without changing its value, as a consequence of the formal multiplication principles determined by the definition. Whatever all these forms have in common will then represent the product under a simple concept."* So the meaning of the regressive product is synonymous with the equivalence relation *"have the same geometric product"*.

He sees that the simplest form of a product is one in *subordinate form*, that is when it is a *flag* of extensors. He thus takes the *value*, or meaning, of the product to be the *"combined observation"* of the flag of flats *"together with the (scalar) quantity to be distributed on the factors."* A scalar multiple can be transferred from term to term in a product without changing the value of the product, that is, he is introducing a tensor product.

As a formal principle he permits the dropping of an additive term in a factor if that term has a higher degree of dependence on the other factors of the product. For instance, in the figure of three collinear points a, b, c together with a line de not coplanar with a, b, c, $ab \circ (ce + de) = ab \circ de$. We will subsequently show that this product is also equal to $1 \circ abde$.

¶130 gives the key definition. *"If A and B are the two factors of a regressive product and the magnitude C represents the common system of the two factors, then if B is set equal to CD, AD represents the nearest covering system and thus the relative system as well if the product is not zero."* That is, we represent one of the factors, B, as a product CD of an extensor C spanning the intersection of the subspaces spanned by A and B, times an extensor D that is complementary to C in the subspace spanned by B. We then transfer the factor D to the multiplicative term involving A. The result is a flag of extensors, which Grassmann decides to write in decreasing order. He concludes that this flag expression is unique (as tensor product). In ¶131: *"The value of a regressive product consists in the common and*

nearest covering system of the two factors, if the order of the factors is given, apart from a quantity to be distributed multiplicatively on the two systems."

Also in ¶131, he states that the regressive product of two extensors is *equal* to its associated flag representation. With A, B and $B = CD$ as above, he writes

$$A \circ B = A \circ CD = AD \circ C$$

Perhaps even more clearly (¶132), he states that *"we require that two regressive products of nonzero values $A \circ CD$ and $A' \circ C'D'$ are equal so long as generally the product of the outermost factors with the middle one is equal in both expressions, or if they stand in reciprocal proportion, whether the values of the orders of the corresponding factors agree or not. In particular, with this definition we can bring that regressive product into subordinate form."* That is, any regressive product of extensors $A \circ B$ is equal to a flag product, say $E \circ C$, where $E = AD$ in the previous calculation.

He also figures out the sign law for commutativity of the regressive product. If $A \circ B = E \circ C$, and the right hand side is a flag, where the ranks of the extensors A, B, C, E are a, b, c, e, respectively, then the sign for the exchange of order of the product is the parity of the product $(a-c)(b-c)$, the product of the *supplementary numbers* (See Figure 13). It's fascinating that he managed to get this right!
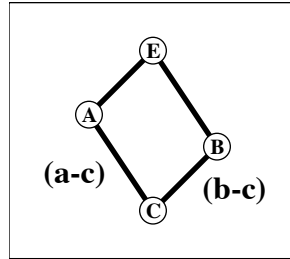

FIGURE 13. Values of rank in a modular lattice.

So we have (skew) commutativity for the regressive product. *How about associativity?* Grassmann calls associativity *the law of combination.* This is the extraordinary part of the story.

He has a well defined product (in fact, one for every degree of dependence), so, in principle, any multiple product is well defined (and even the correct degrees for each successive product of two factors), but he soon recognizes, to his dismay, that the regressive product is **not** generally associative.

The best way to see the non-associativity is by reference to the free modular lattice on three generators, as first found by Dedekind, in Figure 14. (The elements a, b, c are indicated with lower-case letters, but they could be extensors of any rank. I'll also write flags in increasing order.)

In any lattice, for any pair of elements in the order $x < z$, and for any element y,

$$x \vee (y \wedge z) \leq (x \vee y) \wedge z$$

A lattice is *modular* if and only if, under these same conditions, equality holds:

$$x \vee (y \wedge z) = (x \vee y) \wedge z$$

The lattice of subspaces of a vector space is modular (and complemented), so any calculus of linear pieces of space will use the logic of modular lattices.

On the left of Figure 15, we indicate in blue the passage from $b \circ c$ to $b \wedge c \circ b \vee c$. On the right of Figure 15, we show in red the passage from $b \wedge c \circ b \vee c$ to

$$a \circ ((b \wedge c) \circ (b \vee c)) = (a \wedge b \wedge c) \circ ((a \vee (b \wedge c)) \wedge (b \vee c)) \circ (a \vee b \vee c)$$

A value of the triple product will always land at one of the three central elements of the inner green sublattice, but the exact position will depend on which factor entered last into the combined product. *The result carries a trace of the order in which the factors were combined!*

Try this with the simple case of three collinear points a, b, c. Then

$$a \circ (b \circ c) = a \circ (1 \circ bc) = 1 \circ a \circ bc,$$

but

$$(a \circ b) \circ c = (1 \circ ab) \circ c = 1 \circ c \circ ab.$$

Parentheses are not necessary in the final flags, because flag products are associative. All products in which a enters last will be the same, up to sign: $a \circ (b \circ c) = (b \circ c) \circ a = -(c \circ b) \circ a$.
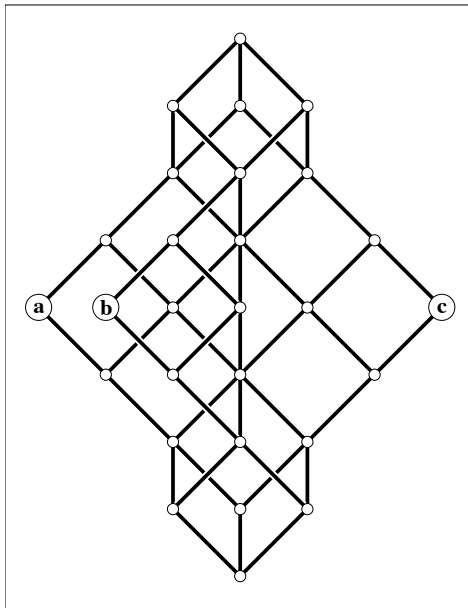

FIGURE 14. The free modular lattice on 3 generators.

However, the regressive product of two flag products is well defined. Grassmann showed this to be true, *an incredible feat, given the tools he had at hand!* In fact, he proved that the extensors of one flag can progressively multiplied into another flag, and that the result is independent not only of the order of these individual
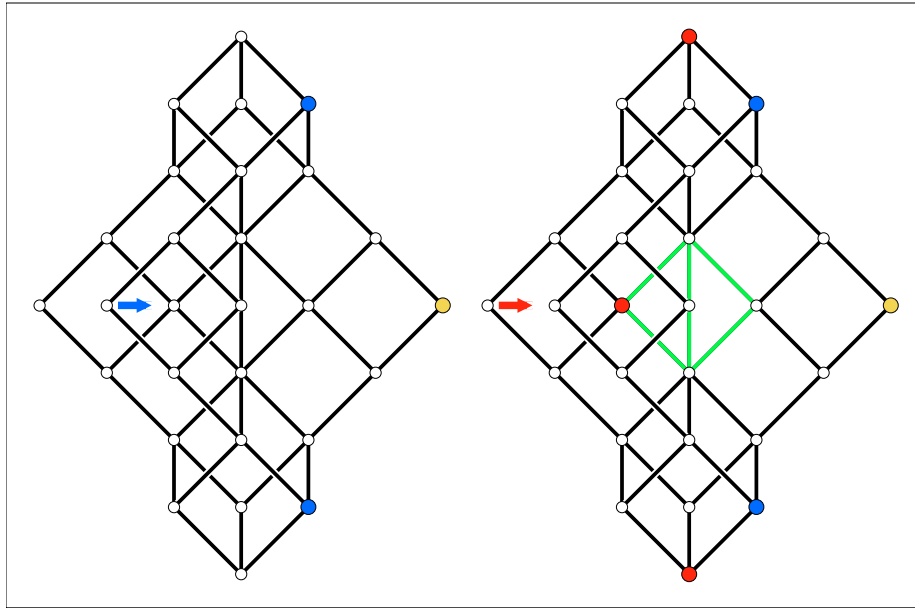
FIGURE 15. The regressive product of a with $b \diamond c$.

multiplications, but also independent of which flag is multiplied into which! In ¶136 Grassmann says *"Instead of multiplying by a product of mutually incident factors one can progressively multiply by the individual factors, and indeed in any order. "*

Garrett Birkhoff, in the first edition of his *Lattice Theory*, proved that the free modular lattice generated by two finite chains is a finite *distributive* lattice. This changes the whole game.

In Figure 16 we show the free modular lattice generated by two 2-chains. In the center of Figure 17 we show the result of multiplying the extensor a into the flag $c \circ d$, then, on the right, the result of multiplying b into that result. We end up on what might well be termed the *backbone* of the distributive lattice. In Figure 18 we show what happens when the flag $c \circ d$ is multiplied into the flag $a \circ b$, but starting with the top element d, instead. The result is the same.

By the end of his chapter on the regressive product, Grassmann seems rather disheartened. He admits clearly in ¶139 that *"the multiplicative law of combination . . . is not generally valid for the mixed product of three factors."*

He includes a footnote to say that *cases can be found in which our law still finds its application* via the results available for the product of an extensor by a flag of extensors, but concludes, with a certain degree of disillusion, that *"these cases are so isolated, and the constraints under which they occur so contrived, that no scientific advantage results from their enumeration."*

Then, having investigated duality and having proven associativity for joins of independent extensors and meets of co-independent extensors, he concludes with the note: *"the theoretical presentation of this part of extension theory now appears*
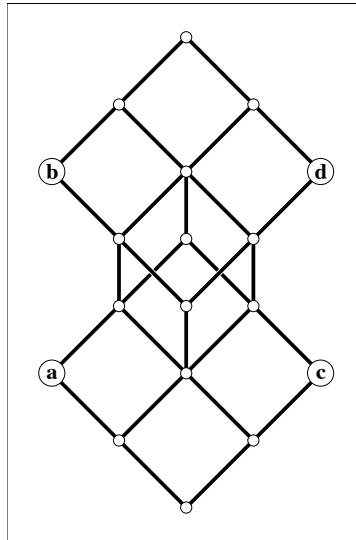
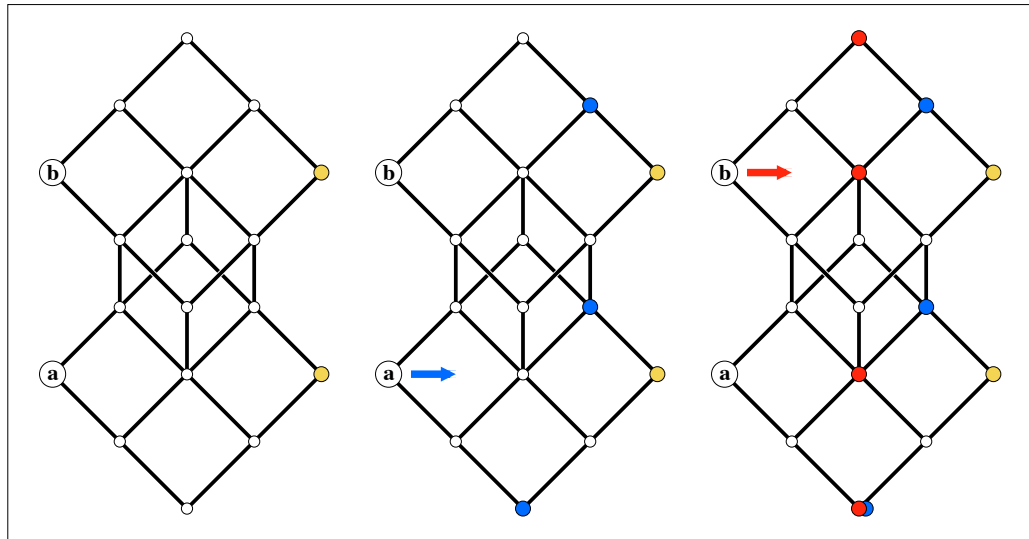FIGURE 16. The free modular lattice generated by two 2-chains.

FIGURE 17. The product $b \to (a \to (c < d))$

as completed, excepting consideration of the types of multiplication for which the law of combination is no longer valid." He adds the footnote: *"How to treat such products, which to be sure have manifold applications, I have sought to indicate at the conclusion of this work."*

Clarification of these questions involving the regressive product of two flags has been joint work with Andrea Brini, Francesco Regonati, and Bill Schmitt, last year in Bologna.
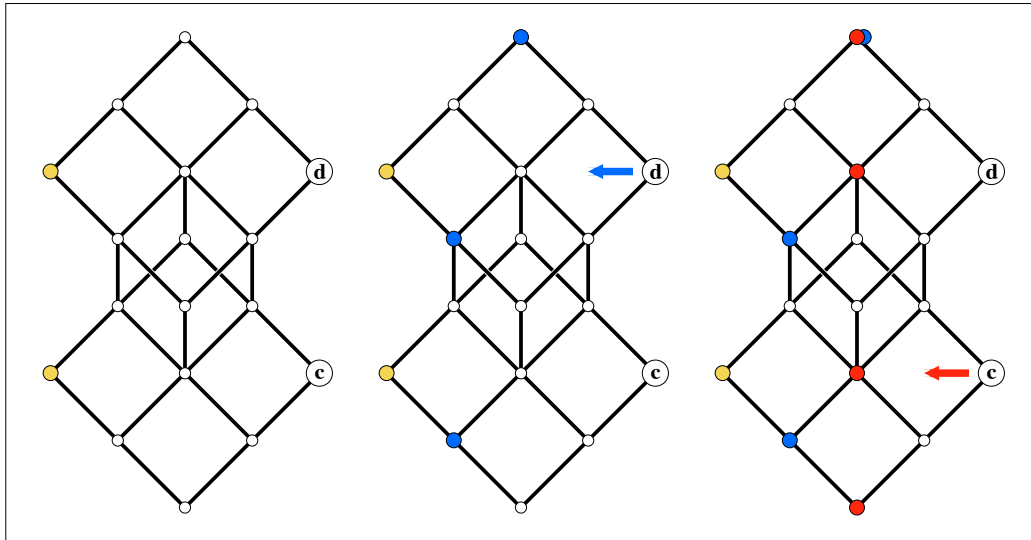
FIGURE 18. The product $c \to (d \to (a < b))$.

When this work is combined with the extraordinary synthesis [6] of Clifford algebra and Grassmann-Cayley algebra, all made super-symmetric, already achieved by Andrea Brini, Paolo Bravi, and Francesco Regonati, here present, you have finally the makings of a banquet that can truly be termed *geometric algebra*.

10. HIGHER ORDER SYZYGIES

Before closing, we should take a quick look at the *higher order syzygies* that Gian Carlo mentioned in his email messages concerning the new Whitney algebra.

Given a configuration C of n points in projective space of rank k, certain subsets $A \subseteq C$ will be dependent. The minimal dependent sets (the *circuits* of the corresponding matroid $M(C)$) will have dependencies that are uniquely defined up to an overall scalar multiple. They thus form, in themselves, a configuration of rank $n-k$ of projective points in a space of rank n. We call this the first *derived configuration*, $C^{(1)}$, and denote the associated matroid $M^{(1)}(C)$, the *derived matroid*.

In the same way, the circuits of $C^{(1)}$ form a new projective configuration, which we denote $C^{(2)}$, with matroid $M^{(2)}(C)$, and so on. Thus, any matroid represented as a configuration in projective space automatically acquires an infinite sequence of derived matroids. In classical terminology, $C^{(k)}$ is the configuration of kth-*order syzygies* of C.

This derived information is *not*, however, fully determined by the matroid itself. The simplest example of interest is given by the uniform matroid $U_{3,6}$ of six points $\{a, b, \ldots, f\}$ in general position in the projective plane. In Figure 19, we show two representations of the matroid $U_{3,6}$ in the plane. In the example on the left, the three lines ab, cd, ef do not meet, and the circuits $abcd, abef, cdef$ are independent, spanning the space of first order syzygies among the six points. On

the right, those three lines meet, the circuits $abcd, abef, cdef$ are dependent (rank 2 in the derived matroid). Those three circuits act as linear constraints on lifting of the figure of six points into 3-space. A height vector $(h(a), h(b), \ldots, h(f))$ is orthogonal to the vector $([bcd], [acd], [abd], [bcd], 0, 0)$ if and only if the four lifted points $a' = (a_1, a_2, a_3, h(a)), \ldots, d' = (d_1, d_2, d_3, h(d))$ are coplanar, if and only if *the dependency is preserved in the lifting.* If the three circuits are of rank 3, there will be $6 - 3$ choices for the lift of the six points, and they must remain coplanar. The three circuits are of rank 2 if and only if the plane figure has a polyhedral lifting, to form a true three-dimensional triangular pyramid. And this happens if and only if the three lines ab, cd, ef meet at a point, namely, the projection of the point of intersection of the three distinct planes $a'b'c'd', a'b'e'f', c'd'e'f'$.
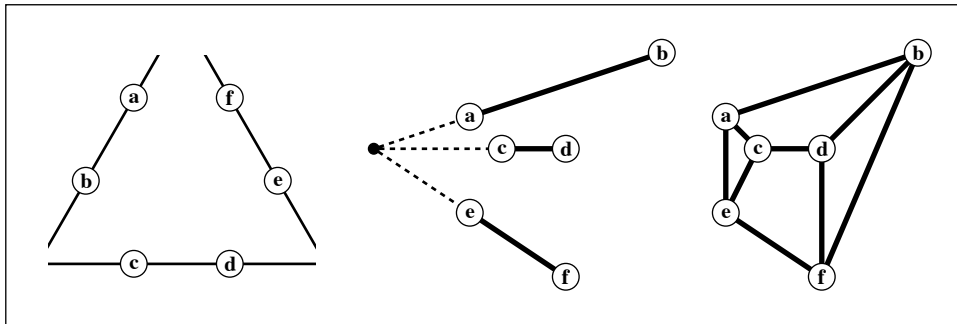

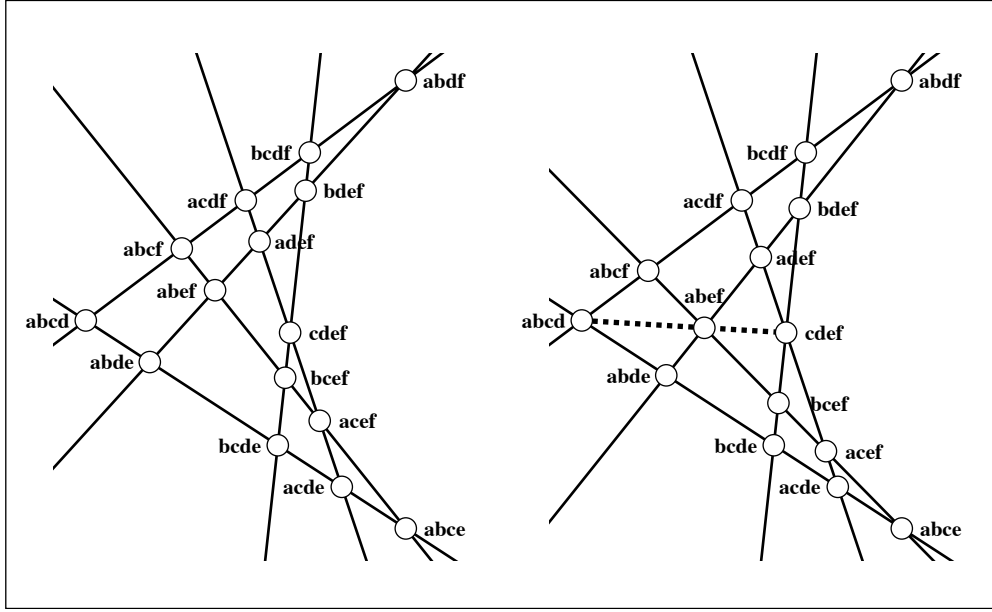FIGURE 19. Representations of the uniform matroid $U_{3,6}$.

For the figure on the left, the derived configuration of first order syzygies is of rank 3 and consists simply of the intersection of 6 lines in general position in the plane. The five circuits formed from any five of the six points have rank 2, and are thus collinear, as in Figure 20. In the special position, when the lines ab, cd, ef are concurrent, the circuits $abcd, abef, cdef$ become collinear, as on the right in Figure 20.

The geometric algebra can also be put to service to provide coefficients for higher-order syzygies, in much the same way that generic first-order syzygies are obtained as coproducts of dependent sets. To see how this is done, it will suffice to take another look at the present example. Check the matrix of coefficients of these first-order syzygies:

$$
\begin{array}{ccccccc}
 & a & b & c & d & e & f \\
(abcd) & abc & -acd & abd & -abc & 0 & 0 \\
(abef) & bef & -aef & 0 & 0 & abf & -abe \\
(cdef) & 0 & 0 & def & -cef & cdf & -cde
\end{array}
$$

This row space S is orthogonal to the space of *linear functions* on the set $S = \{a, b, c, d, e, f\}$, a space T spanned by the coordinate projection functions

$$
\begin{array}{ccccccc}
 & a & b & c & d & e & f \\
(1) & a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\
(2) & a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\
(3) & a_3 & b_3 & c_3 & d_3 & e_3 & f_3
\end{array}
$$

FIGURE 20. First order syzygies for the uniform matroid $U_{3,6}$.

The Grassmann coordinates of these two vector subspaces differ from one another (up to an overall "scalar" multiple) by *complementation of places* and a sign of that complementation. This algebraic operation is called the *Hodge star* operator. Concretely, where σ is the sign of the permutation merging two words into one, in a given linear order,

$$S_{abc} = \sigma(def, abc)T_{def} = -T_{def}$$
$$S_{abd} = \sigma(cef, abd)T_{cef} = T_{cef}$$
$$\cdots$$
$$S_{def} = \sigma(abc, def)T_{abc} = T_{abc}$$

The Grassmann coordinate T_{xyz} for any letters x, y, z is just the 3-extensor xyz. The "scalar" in question, which Gian Carlo and I called *the resolving bracket* [9] is obtainable by calculating any 3×3 minor of the matrix for S, and dividing by the 3-extensor obtained from the complementary set of columns in the matrix for T. If we do this in columns abc for S, we find determinant $(-bcd \otimes aef + acd \otimes bef)def$, which, when divided by $-def$ yields $bcd \otimes aef - acd \otimes bef$. It helps to recognize that this expression can be obtained by joining the meet of $ab \wedge cd = a \otimes bcd - b \otimes acd$ with ef, so this resolving bracket is equal to zero if and only if the meet of ab and cd is on the line ef, that is, if and only if the three lines ab, cd, ef are concurrent. This is the explicit synthetic condition under which the three first-order syzygies $abcd, abef, cdef$ form a dependent set in the first derived configuration.

Much work remains in order to develop an adequate set of tools for dealing with higher order syzygies in general. The concept of resolving bracket is but a first step. Gian Carlo and I spent many hours discussing these higher order syzygies, usually on the white-boards in his apartment in Boston, later in Cambridge, in his office, or in more exotic places such as Strasbourg or Firenze, during mathematical

gatherings. I think he enjoyed these discussions, in the period 1985-95, difficult as it was for him to force me to express my ideas clearly. The only major breakthrough was in Gian Carlo's very fruitful collaboration with David Anick [1, 10], where they found a resolution of the bracket ring of a free exterior algebra, bases for syzygies of all orders being represented by families of totally non-standard tableaux. In this way, you have only to deal with syzygies having *single bracket coefficients*.

11. Balls in Boxes

As a closing thought, I would like to express my conviction that Gian Carlo was rightfully fascinated by probabilistic questions arising from quantum theory, but somehow never really got a proper hold on the basic issues, despite having approached them from all quarters: via general combinatorial theory, *espèces de structure*, supersymmetric algebra, umbral calculus, probability theory, and ...philosophy.

Let me suggest that it is high time we reread what he has written here and there on *balls and boxes*, as in title of today's memorial assembly, for hints he may generously have left us.

As he wrote in the introduction to *The Power of Positive Thinking* (with Wendy Chan),

> *"The realization that the classical system of Cartesian coordinates can and should be enriched by the simultaneous use of two kinds of coordinates, some of which commute and some of which anticommute, has been slow in coming; its roots, like the roots of other overhaulings of our prejudices about space, go back to physics, to the mysterious duality that is found in particle physics between identical particles that obey or do not obey the Pauli exclusion principle."*

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E-mail address: `crapo@ehess.fr`

Centre de Recherche "Les Moutons matheux", 34520 La Vacquerie, France.