PROBLEMS IN
Calculus and Analysis

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Preface

This book is intended as a companion to *Introduction to Calculus and Analysis* by Courant and John. At the time this textbook was undertaken, it became clear that the inclusion of more exercises and problems plus the expansion of solutions would require a book too bulky for convenient use. For that reason the *Examples* of the original text were divided into two classes: the *Problems* which are kept in the text for their special interest or difficulty, and the *Exercises* appearing here which are of more routine character and are intended primarily to improve skill through practice. Hints and solutions for both *Exercises* and *Problems* are included here. For clarity and for ease of reference, the *Problems* are reprinted from the original text.

The amplification of material is intended to meet the demands for more practice exercises and for a fuller and clearer exposition of the solutions of the more difficult problems. Some additional problems have been provided.

The core of this book is an outgrowth of the *Examples* from the original text. I am grateful to Richard Courant and Fritz John, who read much of this material and gave me the benefit of their valuable counsel. I am also deeply indebted to Alan Solomon, who contributed to every aspect of its production. Brigitte Hildebrandt kindly read the solutions of the problems of the first five chapters and eliminated some errors and vagueness. Although these kind and conscientious friends have done much to help correct mistakes, errors will inevitably remain. I shall appreciate being told of them and any other suggestions for the improvement of this work.

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Note to the Reader

In the statements of the *Exercises* and *Problems* and in their solutions, it is assumed that the functions involved are continuous and have all the required derivatives unless there is an indication to the contrary.

The solutions presented range from complete explanation to the merest hint. In some cases no answer is given.

If, upon careful check, you are unable to verify a solution, investigate the possibility that a mistake was made in the formulation.

Contents

1 *Introduction*

EXERCISES

SECTION 1.1a, page 2

1. (*a*) For any fixed integer $q > 1$, prove that the set of points $x = p/q^s$, p,s ranging over all positive integers, is dense on the number line.

(*b*) Show that if p is required to range only over a finite interval, $p \leq M$ for some fixed M, the set of all x is not dense on any interval.

(*c*) Prove that if we demand only that $p < q^s$, then the set of all x is dense on the interval $0 \leq x \leq 1$.

2. Prove that for n,p ranging over all positive integer values, the irrational numbers $x = p/(\sqrt{2})^{2n+1}$ are dense on the real line.

PROBLEMS

SECTION 1.1a, page 2

1. (*a*) If a is rational and if x is irrational, prove that $a + x$ is irrational, and if $a \neq 0$, that ax is irrational.

(*b*) Show that between any two rational numbers there exists at least one irrational number and, consequently, infinitely many.

2. Prove that the following numbers are not rational: (*a*) $\sqrt{3}$. (*b*) $\sqrt{n}$, where the integer n is not a perfect square, that is, not the square of an integer. (*c*) $\sqrt[3]{2}$. (*d*) $\sqrt[p]{n}$, where n is not a perfect pth power.

***3.** (*a*) Prove for any rational root of a polynomial with integer coefficients,

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad (a_n \neq 0),$$

if written in lowest terms as p/q, that the numerator p is a factor of a_0 and the denominator q is a factor of a_n. (This criterion permits us to obtain all rational real roots and hence to demonstrate the irrationality of any other real roots.)

(*b*) Prove the irrationality of $\sqrt{2} + \sqrt[3]{2}$ and $\sqrt{3} + \sqrt[3]{2}$.

(*Hint:* For each of these numbers obtain a polynominal which has the given number at a root. Then apply the result of part *a*.)

Answers to Exercises

SECTION 1.1a, page 2

1. (*a*) Show for any s that it is possible to find a point x closer to any given P than $1/q^s$.

(*b*) Show that if zero is outside any interval, then that interval contains only a finite number of points of the set.

Solutions and Hints to Problems

SECTION 1.1a, page 2

1. (*a*) Assume that $a + x$ and ax are rational and obtain a contradiction.

(*b*) If a and b are rational, $a < b$, then $a + (b - a)/\sqrt{2}$ is irrational and lies between the two.

2. (*a*) See proof of irrationality for $\sqrt{2}$ in the text (p. 5).

(*b*) If $\sqrt{n} = p/q$, where p and q are relatively prime and $q \geq 1$, then $nq^2 = p^2$; that is, q^2 is a factor of p^2. But q has no prime factor in common with p; hence q^2 has no prime factor in common with p^2. It follows that $q^2 = 1$ and therefore that $n = p^2$.

3. (*a*) If p/q is a rational root of the given polynomial, written in lowest terms, then

$$a_np^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + \cdots + a_1pq^{n-1} + a_0q^n = 0.$$

Since all the terms but the last on the left side have the factor p, it follows that p divides a_0q^n as well; but p and q^n have no common factor and so p divides a_0. Similarly, we conclude that q divides a_np^n and hence is a factor of a_n.

(*b*) Show that these numbers satisfy, respectively, the equations

$$x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4 = 0,$$

and

$$x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23 = 0,$$

and apply the preceding result.

EXERCISES

SECTION 1.1b, page 7

1. Prove that the Postulate of Nested Intervals on p. 8, cannot be true for the "punctured" real line $x < a$, $x > a$, excluding the point $x = a$.

Answers to Exercises

SECTION 1.1b, page 7

1. Consider the nested set of intervals $a - 1/n \leq x \leq a + 1/n$ for $n = 1, 2, 3, \ldots$.

EXERCISES

SECTION 1.1c, page 9

1. We can estimate $\sqrt{2}$ to one decimal place as follows. First, $1^2 = 1 < 2$ and $2^2 = 4 > 2$; therefore $1 < \sqrt{2} < 2$. Next, $1.1^2 = 1.21 < 2, \ldots, 1.4^2 = 1.96 < 2$, and $1.5^2 = 2.25 > 2$; therefore $1.4 < \sqrt{2} < 1.5$.

(*a*) Continue this process one step further.

(*b*) Calculate $\sqrt{7}$ to two decimal places by the same method.

(*c*) Calculate the real root of $x^3 + x + 1$ accurately to two decimal places.

2. (*a*) Give the representations in the base 3 of 12; of $\frac{1}{12}$.

(*b*) Give the binary representation of 156; the representation to the base 4 of the same number.

(*c*) Express the following decimal numbers in the base 12: (i) 10,000, (ii) 20,736, (iii) $\frac{1}{6}$, (iv) $\frac{1}{64}$, (v) $\frac{1}{5}$.

3. (*a*) Let p be any integer, $p > 1$. Let the representation of an integer N to the base p be

$$N = d_k d_{k-1} \cdots d_1 d_0,$$

and to the base p^3,

$$N = e_i e_{i-1} \cdots e_1 e_0.$$

Show that:

$$\begin{aligned} e_0 &= d_0 + pd_1 + p^2 d_2, \\ e_1 &= d_3 + pd_4 + p^2 d_5, \\ e_2 &= d_6 + pd_7 + p^2 d_8, \ldots. \end{aligned}$$

(*b*) Let $N = 100111$ to the base 2. Find the representation of N to the base 8 (the octal representation).

PROBLEMS

SECTION 1.1c, page 9

1. Let $[x]$ denote the integer part of x; that is, $[x]$ is the integer satisfying

$$x - 1 < [x] \leq x.$$

Set $c_0 = [x]$, and $c_n = [10^n(x - c_0) - 10^{n-1}c_1 - 10^{n-2}c_2 - \cdots - 10c_{n-1}]$ for $n = 1, 2, 3, \ldots$. Verify that the decimal representation of x is

$$x = c_0 + 0 \cdot c_1c_2c_3 \cdots$$

and that this construction excludes the possibility of an infinite string of 9's.

2. Define inequality $x > y$ for two real numbers in terms of their decimal representations (see Supplement, p. 92).

***3.** Prove if p and q are integers, $q > 0$, that the expansion of p/q as a decimal either terminates (all the digits following the last place are zeros) or is periodic; that is, from a certain point on the decimal expansion consists of the sequential repetition of a given string of digits. For example, $\frac{1}{4} = 0.25$ is terminating, $\frac{1}{11} = 0.090909 \cdots$ is periodic. The length of the repeated string is called the *period* of the decimal; for $\frac{1}{11}$ the period is 2. In general, how large may the period of p/q be?

Answers to Exercises

SECTION 1.1c, page 9

1. (*a*) $1.41 < \sqrt{2} < 1.42$.
(*b*) $2.64 < \sqrt{7} < 2.65$.
(*c*) -0.68.

2. (*a*) 110; $0.00\overline{202}$.
(*b*) 10011100; 2130.
(*c*) (i) 5954, (ii) 10,000, (iii) 0.2, (iv) 0.023, (v) $0.\overline{2497}$.

3. (*a*) The number $d_{\nu-1}d_{\nu-2} \cdots d_0$ is the remainder in the division of N by p^ν. Apply this observation repeatedly.
(*b*) 47.

Solutions and Hints to Problems

SECTION 1.1c, page 9

1. Set $x_n = c_0 + \dfrac{c_1}{10} + \cdots + \dfrac{c_{n-1}}{10^{n-1}}$ and $y_n = x_n + \dfrac{1}{10^{n-1}}$. We have from the definition of c_n

$$(1) \qquad 10^n(x - x_n) - 1 < c_n \le 10^n(x - x_n).$$

First we establish that c_n is a digit for $n \ge 1$. We set $\epsilon_0 = x - c_0$ and $\epsilon_k = 10^k(x - x_k) - c_k$ for $k \ge 1$. From (1) we conclude that $0 \le \epsilon_k < 1$.

Entering this result in (1) for $k = n - 1$, we obtain $-1 \leq 10\epsilon_{n-1} - 1 < c_n \leq 10\epsilon_{n-1} < 10$. Since c_n is an integer and $-1 < c_n < 10$, it follows that c_n must be one of the digits 0, 1, 2, . . . , 9.

Next we observe from (1) that

$$x_n + \frac{c_n}{10^n} \leq x < x_n + \frac{c_n}{10^n} + \frac{1}{10^n}$$

whence

(2) $$x_n \leq x < y_n.$$

Furthermore, from $c_n \geq 0$ we have

(3a) $$x_{n+1} = x_n + \frac{c_n}{10} \geq x_n$$

and from $c_n \leq 9$

(3b) $$y_{n+1} = x_n + \frac{c_n}{10^n} + \frac{1}{10^n} \leq x_n + \frac{1}{10^{n-1}} = y_n.$$

From (2), (3*a*), and (3*b*) we conclude that the closed intervals $[x_n, y_n]$ constitute a nested sequence. Since x belongs to all the intervals of the sequence and the length $y_n - x_n = \dfrac{1}{10^{n-1}}$ can be made arbitrarily small, it follows that x is uniquely determined by this representation.

Let $x = c_0 \cdot c_1c_2 \cdots c_n\, 9999 \ldots$, where we may assume that either $n = 0$ or $c_n \neq 9$. The decimal

$$y = c_0 \cdot c_1c_2 \cdots (c_n + 1)\, 000 \cdots$$

also represents the number x since

$$y \geq x \geq c_0 + \frac{c_1}{10} + \cdots + \frac{c_n}{10^n} + \frac{9}{10^{n+1}}\left(1 + \frac{1}{10} + \cdots + \frac{1}{10^k}\right)$$

$$\geq y - \frac{1}{10^n} + \frac{1}{10^n}\left(1 - \frac{1}{10^{k+1}}\right)$$

$$= y - \frac{1}{10^{n+k+1}}.$$

It follows that $|x - y| \leq \dfrac{1}{10^{n+k+1}}$. Since the difference between x and y is smaller than any positive quantity, it follows that $x = y$. However, if the digits were chosen by the prescribed method, clearly for the nth digit we obtain from (1) $10^n(x - x_n) = c_n$ and the infinite string of 9's could not occur.

2. Let x and y be given by the decimal representations

$$x = a_0 . a_1a_2 \ldots, \qquad y = b_0 . b_1b_2 \ldots;$$

we say that $x < y$ if and only if $a_n < b_n$ at the first place where the two representations differ; that is, if there exists an n such that $a_n < b_n$ and $a_k = b_k$ whenever $k < n$.

3. Let the rational number be represented in lowest terms by p/q. We write $p/q = \alpha/10^\nu\beta$ where β is prime to both α and 10. Let the decimal expansion defined by the method of Problem 1 be given by

$$\frac{p}{q} = c_0 \cdot c_1c_2\cdots$$

and set

$$z_\mu = 10^{\nu+\mu}(c_0 \cdot c_1c_2\cdots c_{\nu+\mu})$$

If $\beta = 1$, we have

$$\frac{p}{q} = \frac{p}{10^\nu} = c_0 \cdot c_1c_2\cdots c_\nu\bar{0},$$

where $\bar{0}$ indicates that the digit 0 is repeated from the $(\nu + 1)$th place on. Suppose $\alpha \neq 1$. From (2) in the solution of Problem 1

$$z_\mu \leq \frac{10^\mu\alpha}{\beta} < z_\mu + 1,$$

whence

$$0 \leq 10^\mu\alpha - \beta z_\mu < \beta.$$

Thus the remainder on division of $10^\mu\alpha$ by β is the integer $R_\mu = 10^\mu\alpha - \beta z_\mu$. Clearly, $R_\mu \neq 0$ since β shares no factors with $10^\mu\alpha$. Thus R_μ can only be one of the $\beta - 1$ integers between 0 and β. It follows that at least two of the values R_μ $(\mu = 1, \ldots, \beta)$ must be the same. If $R_s = R_t$ $(s > t)$, it follows that the decimal expansion can be written in the form

$$\frac{p}{q} = c_0 \cdot c_1c_2\cdots\overline{c_{q+t+1}c_{q+t+2}\cdots c_{q+s}}$$

where the $s - t$ digits $c_{q+t+1}\cdots c_{q+s}$ repeat periodically. For proof we show first that $R_\mu = R_{\mu+s-t}$. We have

$$\begin{aligned} R_{\mu+1} &= 10^{\mu+1}\,\alpha - \beta z_{\mu+1} \\ &= 10^{\mu+1}\,\alpha - \beta(10z_\mu + c_{\nu+\mu+1}) \\ &= 10R_\mu - \beta c_{\nu+\mu+1}. \end{aligned}$$

But

$$\begin{aligned} c_{\nu+\mu+1} &= \left[10^{\nu+\mu+1}\left(\frac{p}{q} - c_0 \cdot c_1c_2\cdots c_{\nu+\mu}\right)\right] \\ &= \left[10\left(\frac{10^\mu\alpha}{\beta} - z_\mu\right)\right] \\ &= [10R_\mu/\beta], \end{aligned}$$

whence

$$R_{\mu+1} = 10R_\mu - \beta[10R_\mu/\beta].$$

It follows that $R_{\mu+1}$ can be expressed in terms of R_μ alone. Hence, since $R_s = R_t$, $R_{s+1} = R_{t+1}$, $R_{s+2} = R_{t+2}$, ... Finally, since $s - t \leq \beta - 1$, we have obtained an upper bound on the period. This is a best upper bound in the sense that it is actually attained (for example, for $p/q = 1/7$).

EXERCISES

SECTION 1.1e, page 12

1. Show for each of the following statements whether it is true or false:

(*a*) $\sqrt{2} > 1.41$, (*b*) $\sqrt{2} > 1.414214$,
(*c*) $\sqrt[3]{4} > 1.59$, (*d*) $15 < \sqrt{240} < 16$,
(*e*) $2\sqrt{2} > 1 + \sqrt{3}$, (*f*) $10(7.1)(7.2) - 491 \geq \frac{1}{100}$,
(*g*) $100\sqrt{7} - (16)^2 < 8$, (*h*) $(42)^2 > (12)^3$.

2. On the number axis depict those parts which satisfy the following inequalities.

(*a*) $|x| \leq 2$, (*b*) $|x - 1| < 1$,
(*c*) $|x + 1| < 1$, (*d*) $x > 5$,
(*e*) $|x| > 5$, (*f*) $1 \leq x \leq 3$,
(*g*) $|x - 2| \leq 1$, (*h*) $(x^2 - 1) > 0$,
(*i*) $x(x^2 - 1) > 0$, (*j*) $|x - 1| < |x - 3|$,
(*k*) $\dfrac{2x + 1}{x - 1} + \dfrac{x + 1}{x - 2} > 3$, (*l*) $\sqrt{x - 1} + \sqrt{2x - 1} \geq \sqrt{3x - 2}$
(*m*) $x^3 + x \geq 1$.

3. Prove
(*a*) $|x - 1| + |x - 2| \geq 1$,
(*b*) $|x - 1| + |x - 2| + |x - 3| \geq 2$.
When does equality hold?

4. (*a*) If $x > 0$, $b > 0$, and $a \neq b$, prove that $(a + x)/(b + x)$ lies between 1 and a/b.

(*b*) More generally, if a/b and c/d are distinct fractions with $b > 0$ and $d > 0$, prove that $(a + c)/(b + d)$ lies between a/b and c/d.

5. If (a, b) and (c, d) are neighborhoods of α, β respectively, prove:
(*a*) $(a + c, b + d)$ is a neighborhood of $\alpha + \beta$,
(*b*) $(\lambda a, \lambda b)$ is a neighborhood of $\lambda\alpha$ for all positive numbers λ,
(*c*) $(a - d, b - c)$ is a neighborhood of $\alpha - \beta$,
(*d*) (ac, bd) is a neighborhood of $\alpha\beta$, if $c > 0$.
(*e*) If $c > 0$, then $(a/d, b/c)$ is a neighborhood of α/β.

6. Prove that for a given number $a \geq 0$ and any $x > 0$,

$$\sqrt{a} \leq \frac{ax^2 + 1}{2x}.$$

For what value of x is equality attained?

Use this inequality for $x = \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}$ to obtain an upper bound for $\sqrt{2}$.

7. Prove that the set of all rational numbers of the form p^2/q^2 for p, q integers is dense in the set of all positive real numbers.

PROBLEMS

SECTION 1.1e, page 12

1. Using signs of inequality alone (not using signs of absolute value), specify the values of x which satisfy the following relations. Discuss all cases.

(*a*) $|x - a| < |x - b|$.
(*b*) $|x - a| < x - b$.
(*c*) $|x^2 - a| < b$.

2. An interval (see definitions in text) may be defined as any connected part of the real continuum. A subset S of the real continuum is said to be *connected* if with every pair of points a, b in S, the set S contains the entire closed interval $[a, b]$. Aside from the open and closed intervals already mentioned, there are the "half-open" intervals $a \leq x < b$ and $a < x \leq b$ (sometimes denoted by $[a, b]$ and $(a, b]$, respectively) and the unbounded intervals that may be either the whole real line or a ray, that is, a "half-line" $x \leq a$, $x < a$, $x > a$, $x \geq a$ (sometimes denoted by $(-\infty, \infty)]$ and $(-\infty, a]$, $(-\infty, a)$, (a, ∞), $[a, \infty)$, respectively) (see also footnote, p. 22).

*(*a*) Prove that the cases of intervals specified above exhaust all possibilities for connected subsets of the number axis.

(*b*) Determine the intervals in which the following inequalities are satisfied.

(i) $x^2 - 3x + 2 < 0$.
(ii) $(x - a)(x - b)(x - c) > 0$, for $a < b < c$.
(iii) $|1 - x| - x \geq 0$.
(iv) $\dfrac{x - a}{x + a} \geq 0$.
(v) $\left|x + \dfrac{1}{x}\right| \leq 6$.
(vi) $[x] \leq x/2$. See Problem 1, Section 1.1c.
(vii) $\sin x \geq \sqrt{2}/2$.

(c) Prove if $a \leq x \leq b$, then $|x| \leq |a| + |b|$.

3. Derive the inequalities

(*a*) $x + \frac{1}{x} \geq 2$, for $x > 0$,

(*b*) $x + \frac{1}{x} \leq -2$, for $x < 0$,

(*c*) $\left|x + \frac{1}{x}\right| \geq 2$, for $x \neq 0$.

4. The harmonic mean ξ of two positive numbers a, b is defined by

$$\frac{1}{\xi} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

Prove that the harmonic mean does not exceed the geometric mean; that is, that $\xi \leq \sqrt{ab}$. When are the two means equal?

5. Derive the following inequalities:

(*a*) $x^2 + xy + y^2 \geq 0$,

*(*b*) $x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + \cdots + y^{2n} \geq 0$,

*(*c*) $x^4 - 3x^3 + 4x^2 - 3x + 1 \geq 0$.

When does equality hold?

***6.** What is the geometrical interpretation of Cauchy's inequality for $n = 2, 3$?

7. Show that the equality sign holds in Cauchy's inequality if and only if the a_ν are proportional to the b_ν: that is, $ca_\nu + db_\nu = 0$ for all ν where c and d do not depend on ν and are not both zero.

8. (*a*) $|x - a_1| + |x - a_2| + |x - a_3| \geq a_3 - a_1$, for $a_1 < a_2 < a_3$.
For what values of x does equality hold?

*(*b*) Find the largest value of y for which for all x

$$|x - a_1| + |x - a_2| + \cdots + |x - a_n| \geq y,$$

where $a_1 < a_2 < \cdots < a_n$. Under what conditions does equality hold?

9. Show that the following inequalities hold for positive a, b, c.

(*a*) $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(*b*) $(a + b)(b + c)(c + a) \geq 8abc$.

(*c*) $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$.

10. Assume that the numbers x_1, x_2, x_3, and a_{ik} $(i, k = 1, 2, 3)$ are all positive, and in addition, $a_{ik} \leq M$ and $x_1^2 + x_2^2 + x_3^2 \leq 1$. Prove that

$$a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{33}x_3^2 \leq 3M.$$

***11.** Prove the following inequality and give its geometrical interpretation for $n \leq 3$.

$$\sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2} \leq \sqrt{(a_1^2 + \cdots + a_n^2)} + \sqrt{(b_1^2 + \cdots + b_n^2)}.$$

12. Prove, and interpret geometrically for $n \le 3$,

$$\sqrt{(a_1 + b_1 + \cdots + z_1)^2 + \cdots + (a_n + b_n + \cdots + z_n)^2}$$
$$\le \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2} + \cdots + \sqrt{z_1^2 + \cdots + z_n}.$$

13. Show that the geometric mean of n positive numbers is not greater than the arithmetic mean; that is, if $a_i > 0$ $(i = 1, \ldots, n)$, then

$$\sqrt{a_1 a_2 \cdots a_n} \le \frac{1}{n}(a_1 + a_2 + \cdots + a_n).$$

(*Hint:* Suppose $a_1 \le a_2 \le \cdots \le a_n$. For the first step replace a_n by the geometric mean and adjust a_1 so that the geometric mean is left unchanged.)

Answers to Exercises

SECTION 1.1e, page 12

1. (*a*) True; (*b*) False; (*c*) False; (*d*) True; (*e*) True; (*f*) True; (*g*) False; (*h*) True.

3. (*a*) $|x - 1| + |x - 2| \ge |(x - 1) - (x - 2)| = 1$; $\quad 1 \le x \le 2$.
(*b*) $|x - 1| + |x - 2| + |x - 3|$
$$\ge |(x - 1) - (x - 3)| + |x - 2| \ge 2, \; x = 2.$$

4. The two differences,

$$\frac{a}{b} - \frac{a + x}{b + x} = \frac{x}{b}\frac{a - b}{b + x},$$
$$\frac{a + x}{b + x} - 1 = \frac{a - b}{b + x},$$

have the same sign.

5. (*a*) Apply the additive property of inequalities to $a < \alpha < b$ and $c < \beta < d$.

6. If $x > 0$, the stated inequality is equivalent to

$$(\sqrt{a}x - 1)^2 \ge 0.$$

Equality holds for $x = 1/\sqrt{a}$.
Use $x = \frac{6}{9}$ to obtain $\sqrt{2} < 1.417$.

7. For a given q choose p so that $p^2 \le xq^2 < (p + 1)^2$. Then

$$0 \le x - \frac{p^2}{q^2} < \frac{2p + 1}{q^2} \le \frac{2\sqrt{x}}{q} + \frac{1}{q^2}.$$

Consequently, for fixed x the difference between x and p^2/q^2 can be made arbitrarily small by choosing q large enough.

Solutions and Hints to Problems

SECTION 1.1e, page 12

1. (*a*) Not possible for $a = b$.

For $a < b$, $x < \dfrac{(a + b)}{2}$.

For $a > b$, $\dfrac{(a + b)}{2} < x$.

(*b*) Not possible for $a \leq b$.

For $a > b$, $x > \dfrac{(a + b)}{2}$.

(*c*) Not possible for $b \leq 0$.
For $b > 0$, $a \geq b$, $\sqrt{a - b} < |x| < \sqrt{a + b}$.
For $b > 0$, $-b < a < b$, $|x| < \sqrt{a + b}$.
For $b > 0$, $a < -b$, it is impossible.

2. (*a*) Let S be a connected set of the number axis. If there exists any number m such that $x \geq m$ for all x in S, then S has a unique "lower end point" a such that for all x in S, $x \geq a$ and such that every neighborhood of a contains points of S. We determine a by giving a nested sequence of intervals which contains it. Let β_0 be any point of S and take $\alpha_0 = m$. We divide $[\alpha_0, \beta_0]$ into equal halves. We take as $[\alpha_1, \beta_1]$ the leftmost of the halves which contains points of S. Similarly, we define $[\alpha_{k+1}, \beta_{k+1}]$ as the leftmost of the halves of $[\alpha_k, \beta_k]$ which contains points of S. Any neighborhood of the point a defined by this nested sequence contains points of S. Furthermore, since $x \geq \alpha_k$ for all points of S, it follows that $x \geq a$.

Similarly, if there exists a number M such that $x \leq M$ for all x in S, then x has a unique "upper end point" b such that for all x in S, $x \leq b$ and such that every neighborhood of b contains points of S.

The separate cases correspond to the existence of both end points (bounded intervals), one end point (rays), no end point (whole line), end points in S (closed), end points not in S (open).

(*b*) (i) $(1, 2)$.
(ii) (a, b) and (c, ∞).
(iii) $(-\infty, \frac{1}{2}]$.
(iv) $(-\infty, -|a|)$, $(|a|, \infty)$.

Include end point $x = a$ in the appropriate interval when $a \neq 0$.

(v) $(3 - \sqrt{8}, 3 + \sqrt{8}), (-3 - \sqrt{8}, -3 + \sqrt{8})$.
(vi) $(-\infty, 1)$.
(vii) $(\frac{1}{4}\pi + 2n\pi, \frac{3}{4}\pi + 2n\pi), \qquad n = 0, \pm 1, \pm 2, \ldots$.

(*c*) We have

$$|a| \leq |a| + |b| \qquad \text{and} \qquad |b| \leq |a| + |b|,$$

whence

$$-|a| - |b| \leq a \leq x \leq b \leq |a| + |b|$$

and

$$-|a| - |b| \leq -b \leq -x \leq -a \leq |a| + |b|,$$

from which the result follows.

3. (*a*) Use the relation

$$(x - 1)^2 \geq 0.$$

(*b*) Use the fact that

$$(x + 1)^2 \geq 0.$$

(*c*) This follows from (*a*) and (*b*).

4. Square both sides. Equality only if $a = b$.

5. (*a*) $x^2 + xy + y^2 = [x + (y/2)]^2 + 3y^2/4$. Equality only if $x = y = 0$.

$$(b)\ x^{2n} + x^{2n-1}y + \cdots + y^{2n} = \frac{x^{2n+1} - y^{2n+1}}{x - y}, \qquad (y \neq x)$$
$$= (2n + 1)x^{2n} \qquad (y = x).$$

Equality only if $x = y = 0$.

$$(c)\ x^4 - 3x^3 + 4x^2 - 3x + 1$$
$$= (x^2 - x + 1)(x^2 - 2x + 1)$$
$$= [(x - \tfrac{1}{2})^2 + \tfrac{3}{4}](x - 1)^2.$$

Equality only if $x = 1$.

6. Compare the law of cosines for the triangle with vertices $(0, 0, 0)$, (a_1, a_2, a_3), (b_1, b_2, b_3).

7. If $ca_\nu + db_\nu = 0$ where $c \neq 0$, say, then $a_\nu = -db_\nu/c$ and

$$B^2 = (\sum a_\nu b_\nu)^2 = \frac{d^2}{c^2}(\sum b_\nu{}^2)^2 = \sum a_\nu{}^2 \sum b_\nu{}^2$$
$$= AC.$$

Conversely, if $AC = B^2$, then

$$\sum (ca_\nu + db_\nu)^2 = c^2A + 2cd\sqrt{AC} + d^2C = 0,$$

where c and d are chosen so that $cB + dC = 0$.

8. (*a*) Equality holds for $x = a_2$.

(*b*) If n is even, $n = 2m$, then

$$y = (a_n + a_{n-1} + \cdots + a_{m+1}) - (a_1 + \cdots + a_m),$$

and equality holds for $a_m \leq x \leq a_{m+1}$.

When n is odd, $n = 2m - 1, y = (a_n + a_{n-1} + \cdots + a_{m+1}) - (a_1 + \cdots + a_{m-1})$, and equality holds for $x = a_m$.

9. (*a*) Add $a^2 + b^2 \geq 2ab$, $b^2 + c^2 \geq 2bc$, $c^2 + a^2 \geq 2ca$.

(*b*) Multiply $a + b \geq 2\sqrt{ab}$, $b + c \geq 2\sqrt{bc}$, $c + a \geq 2\sqrt{ca}$.

(*c*) Add the three inequalities of the type $a^2b^2 + b^2c^2 \geq 2b^2ac$.

10. Observe that

$$a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{33}x_3^2$$
$$\leq M(x_1^2 + x_2^2 + x_3^2) + 2M(x_1x_2 + x_2x_3 + x_3x_1)$$

and apply Cauchy's inequality to the numbers (x_1, x_2, x_3) and (x_2, x_3, x_1).

11. Square both sides and use Cauchy's inequality. For the triangle with vertices $0 = (0, 0, \ldots, 0)$, $A = (a_1, a_2, \ldots, a_n)$, $B = (b_1, b_2, \ldots, b_n)$, the sum of two sides OA and OB of the triangle is greater than or equal to the third side.

12. Note first that

$$[(a_1 + b_1)^2 + (c_2 + b_2)^2 + \cdots + (a_n + b_n)^2]^{1/2}$$
$$\leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}$$

follows by squaring and apply Cauchy's inequality. We then have

$$\begin{aligned}
&[(a_1 + b_1 + c_1)^2 + \cdots + (a_n + b_n + c_n)^2]^{1/2} \\
&\quad = \{[(a_1 + b_1) + c_1]^2 + \cdots + [(a_n + b_n) + c_n]^2\}^{1/2} \\
&\quad \leq [(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2]^{1/2} + \sqrt{c_1^2 + \cdots + c_n^2} \\
&\quad \leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b} + {}_n{}^2\sqrt{c_1^2 + \cdots + c_n^2}.
\end{aligned}$$

The general result follows by repeated application of this scheme.

For the points, $0 = (0, 0, 0)$, $A = (a_1, a_2, a_3)$, $B = (a_1 + b_1, a_2 + b_2, a_3 + b_3), \ldots,$ $Z = (a_1 + b_1 + \cdots + z_1, a_2 + b_2 + \cdots + z_2, a_3 + b_3 + \cdots + z_3)$, this inequality means that the segment OZ is shorter than the polygonal path joining the successive vertices $O, A, B, \ldots, Z$,

13. Set $\alpha = \sqrt[n]{a_1 a_2 \cdots a_n}$ and suppose $a_1 \leq a_2 \leq \cdots \leq a_n$. Replace a_n by $a_n{}^* = \alpha$ and a_1 by $a_1{}^* = a_1a_n/\alpha$ so that the geometric mean of the n quantities is unchanged. We have $a_1 \leq \alpha \leq a_n$ and therefore

$$\begin{aligned}
a_1{}^* + a_n{}^* &= \alpha + \frac{a_1a_n}{\alpha} \\
&= a_1 + a_n + \frac{\alpha^2 - \alpha(a_1 + a_n) + a_1a_n}{\alpha} \\
&= a_1 + a_n - \frac{(\alpha - a_1)(a_n - \alpha)}{\alpha} \\
&\leq a_1 + a_n.
\end{aligned}$$

Thus the effect of the replacement is to leave the geometric mean unchanged without increasing the arithmetic mean. Repeating this process until all the values a_i have been replaced by α, we obtain the conclusion.

EXERCISES

SECTION 1.2a, page 18

1. Show for the mapping $a = x/(x^2 + y^2)$, $b = y/(x^2 + y^2)$ that every circle or straight line in the x,y plane is mapped into either a circle or a straight line in the a,b plane.

2. Give the analytic formula for the function which maps a point on the x-axis onto a point of the y-axis by projection through an arbitrary point (a, b).

3. Consider the ellipse $x^2/p^2 + y^2/q^2 = 1, p > q$. Let P be the perimeter of the ellipse, A its area, e its eccentricity, and $(F, 0)$, $(-F, 0)$ the focii. Which of the following statements are true?

(*a*) A is a function of p.
(*b*) A is a function of p and q.
(*c*) p is a function of A.
(*d*) e is a function of A.
(*e*) e is a function of p, q.
(*f*) P is a function of p, q.
(*g*) P is a function of A.
(*h*) A is a function of P.
(*i*) p is a function of A and q.

Answers to Exercises

SECTION 1.2a, page 18

1. Note that $a^2 + b^2 = 1/(x^2 + y^2)$. A circle in the (x, y) plane has the equation $(x - x_0)^2 + (y - y_0)^2 = r^2$, or

$$x^2 + y^2 = 2xx_0 + 2yy_0 + r^2 - x_0^2 - y_0^2.$$

Divide by $x^2 + y^2$ to obtain

$$1 = 2ax_0 + 2by_0 + (r^2 - x_0^2 - y_0^2)(a^2 + b^2).$$

This is the equation of a circle (unless $x_0^2 + y_0^2 = r^2$):

$$(a - a_0)^2 + (b - b_0)^2 = \rho^2$$

where
$$\rho^2 = \frac{r^2}{(r^2 - x_0^2 - x_0^2)^2}$$

$$a_0 = \frac{-x_0}{r^2 - x_0^2 - y_0^2},$$

and
$$b_0 = \frac{-y_0}{r^2 - x_0^2 - y_0^2}.$$

If the original circle passes through the origin ($r^2 = x_0^2 + y_0^2$), the equation represents a straight line.

For a straight line in the (x, y) plane given in the form

$$Ax + By + C = 0,$$

we obtain on dividing by $x^2 + y^2$

$$Aa + Bb + + C(a^2 + b^2) = 0,$$

which, if $C \neq 0$, is the equation of a circle through the origin with center $a_0 = -A/2C$, $b_0 = -B/2C$. If the original line passes through the origin ($C = 0$), then its image is the line in the a,b plane which passes through the origin and has the same direction.

2. $y = \dfrac{bx}{x - a}$.

3. (*b*), (*e*), (*f*), (*i*).

EXERCISES

SECTION 1.2b, page 21

1. Given $f(x) = 2x$, $F(x) = 1/x$, $g(x) = x^2$, $G(x) = 1/(1 - x)$. Find
(*a*) $f(0)$, $F(1)$, $g(2)$, $G(3)$;
(*b*) $f(F(2))$, $F(f(2))$, $F(g(2))$, $g(F(2))$, $1 - F(G(x))$, $1 - G(F(x))$;
(*c*) $\dfrac{f(x + h) - f(x)}{h}$, $\dfrac{F(x + h) - F(x)}{h}$, $\dfrac{g(x + h) - g(x)}{h}$.

2. Which of the following relations define y as a function of x? In those cases which do, give the domain and range of the function.
(*a*) $|x - y| = 0$, (*b*) $|x^2 - y^2| = 0$,
(*c*) $|x - y| = 1$, (*d*) $x^2 + y^2 = 1$,
(*e*) $y = -\sqrt{1 - x^2}$, (*f*) $x = \sqrt{1 - y^2}$.
(*g*) $ax^2 - y^2 = b$, $a,b > 0$, (*h*) $x^2 + y^2 = -1$, x,y real numbers.

3. Let $f(x) = ax + b$, $F(x) = Ax + B$. Find $f(F(2))$, $F(f(2))$, $f(1/F(1/A))$, $f((F(x))^2)$.

Let $H(x) = (1/F(x)) + 1$. Find $H(0)$, $H(1)$, $H(2)$, $f(h(2))$, $f(H(x))$, $f((H(x))^2)$.
What is the domain and range of f, F, and H?

Answers to Exercises

SECTION 1.2b, page 21

1. (*a*) $0, 1, 4, -\frac{1}{2}$.

(*b*) $1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, x, \dfrac{1}{1-x}$,

(*c*) $2, \dfrac{-1}{x(x+h)}, \quad 2x + h$.

2. (*a*) All x, (*e*) $-1 \leq x \leq 1$.

EXERCISES

SECTION 1.2c, page 24

1. Plot the graph of $y = x^3$ from calculated values. Without further calculation obtain the graph of $y = \sqrt[3]{x}$.

2. Sketch the graphs of the following functions where defined and state whether the functions are (1) monotonic or not, (2) even or odd. Which two of these functions are identical?

(*a*) $y = x^2$, (*b*) $y = x^2$ on $[0, 1]$,
(*c*) $y = x$ on $[-1, 1]$, (*d*) $y = |x|$ on $[-1, 1]$,
(*e*) $y = \sqrt{x^2}$ on $[-1, 1]$, (*f*) $y = |x - 1|$,
(*g*) $y = ||x| - 1|$. (*h*) $y = x\,|x|$,
(*i*) $y = \dfrac{x}{|x|}$ (*j*) $y = x + |x|$,
(*k*) $y = |x| + |x - 1|$, (*l*) $y = |x^2 + 4x + 2|$ on $[-4, 3]$,
(*m*) $y = [x]$, where $[x]$ denotes the integer part of x
(*n*) $y = x - [x]$, (*o*) $y = \sqrt{x - [x]}$,
(*p*) $y = x + \sqrt{x - |x|}$,
(*q*) $y = |x - 1| + |x + 1| - 2$ on $(-5 \leq x \leq 5)$,
(*r*) $y = |x - 1| - 2\,|x| + |x + 1|$, (*s*) $y = x(x^2 - 1)$,
(*t*) $y = \dfrac{1}{x(x^2 - 1)}$, (*u*) $y = \dfrac{x^2 - 1}{x}$,

3. Sketch the graph of $y = (x - a)(x - b)(x - c)$, where $a < b < c$. Show what forms the graph takes if one or both of the signs of inequality are replaced by signs of equality.

4. Sketch the following graphs, and state whether the functions are even or odd.

(*a*) $y = \sin 2x$. (*b*) $y = 5 \cos x$.
(*c*) $y = \sin x + \cos x$. (*d*) $y = 2 \sin x + \sin 2x$.
(*e*) $y = \sin (x + \pi)$. (*f*) $y = 2 \cos \left(x + \frac{\pi}{3}\right)$.
(*g*) $y = \tan x - x$.

5. What can be said in general concerning the evenness or oddness of the sums and products of even and odd functions? What functions are both even and monotonic? Justify your answers.

6. A body dropped from rest falls approximately $16t^2$ ft in t sec. If a body falls out of a window 25 ft above ground, plot its height above ground as a function of t for the first 4 sec after it starts to fall.

7. Sketch the graphs of the following inequalities in the x,y plane.

(*a*) $y > x$, (*b*) $y > |x|$,
(*c*) $|y| > |x|$, (*d*) $y > 1$,
(*e*) $|x - 2| < 1$, (*f*) $x + y < 1$,
(*g*) $x + |y| < 1$, (*h*) $|x| + |y| < 1$,
(*i*) $|x + y| < 1$, (*j*) $y < x^2$,
(*k*) $|y| < x^2$, (*l*) $x - y > |y| - |x|$,
(*m*) $0 < y - x < 1$, (*n*) $x < \frac{1}{2}(|y| - y)$,
(*o*) $0 < y < \frac{1}{x}$, (*p*) $xy > 1$.

Answers to Exercises

SECTION 1.2c, page 24

2. (*a*) Even, (*b*) monotonic, (*c*) monotonic odd, (*d*) and (*e*) same function, even, (*g*) even, (*h*) monotonic odd, (*i*) odd, (*p*) monotonic, (*q*) even, (*r*) even, (*s*) odd, (*t*) odd; (*u*) odd.

4. (*a*) Odd, (*b*) even, (*d*) odd, (*e*) odd, (*g*) odd.

5. Sums of even functions are even; sums of odd functions are odd. The product of an odd function and an even function is odd; of two even functions even; of two odd functions, even.

6. After $\frac{5}{4}$ sec, the motion of the body is not described.

EXERCISES

SECTION 1.2d, page 31

1. (*a*) Let $f(x) = 6x$. Find a δ which may depend on ξ, so small that $|f(x) - f(\xi)| < \epsilon$ whenever $|x - \xi| < \delta$, where (1) $\epsilon = \frac{1}{10}$; (2) $\epsilon = \frac{1}{100}$; (3) $\epsilon = \frac{1}{1000}$.

Do the same for

(*b*) $f(x) = x^2 - 2x$, (*c*) $f(x) = \sqrt{x^2}$,

(*d*) $f(x) = 3x^4 + x^2 - 7$, (*e*) $f(x) = \sqrt{x}$.

2. (*a*) Let $f(x) = 6x$ in the interval $0 \le x \le 10$. Find a uniform modulus δ so small that $|f(x_1) - f(x_2)| < \epsilon$, whenever $|x_1 - x_2| < \delta$, where (1) $\epsilon = \frac{1}{100}$, 2) $\epsilon > 0$ is arbitrary.

Do the same for

(*b*) $f(x) = x^2 - 2x$, $-1 \le x \le 1$,

(*c*) $f(x) = \sqrt{x^2}$, $-2 \le x \le 2$,

(*d*) $f(x) = 3x^4 + x^2 - 7$, $2 \le x \le 4$,

(*e*) $f(x) = \sqrt[3]{x}$, $0 \le x \le 1$.

3. Determine which of the following functions are continuous. For those which are discontinuous find the points of discontinuity.

(*a*) $\dfrac{1}{1 - x^2}$, (*b*) $\dfrac{1}{1 + x^2}$,

(*c*) $\dfrac{x}{1 - |x|}$, (*d*) $\dfrac{x}{1 + |x|}$,

(*e*) $\dfrac{x}{1 + x}$, (*f*) $\dfrac{x^3 + 3x + 7}{x^2 - 6x + 8}$,

(*g*) $\dfrac{x^3 + 3x + 7}{x^2 - 6x + 9}$, (*h*) $\dfrac{x^3 + 3x + 7}{x^2 - 6x + 10}$,

(*i*) $f(x) = \begin{cases} \sin(1/x), & x \ne 0 \\ 0, & x = 0. \end{cases}$ (*j*) $f(x) = \begin{cases} x\sin(1/x), & x \ne 0 \\ 0, & x = 0. \end{cases}$

*Prove your assertions concerning the functions given in parts *i* and *j*.

4. (*a*) Find a uniform modulus of continuity for $f(x)$ on the entire real axis for each of the following cases:

$$f(x) = x, f(x) = \frac{1}{1 + |x|}, f(x) = \frac{x}{1 + x^2}.$$

(*b*) Prove that it is impossible to find a uniform modulus of continuity in each of the following cases:

$$f(x) = \frac{1}{1 + x}, f(x) = x^3, f(x) = \frac{x^3}{1 + |x|}.$$

PROBLEMS

SECTION 1.2d, page 31

1. If $f(x)$ is continuous at $x = a$ and $f(a) > 0$ show that the domain of f contains an open interval about a where $f(x) > 0$.

2. In the definition of continuity show that the centered intervals

$$|f(x) - f(x_0)| < \epsilon \quad \text{and} \quad |x - x_0| < \delta$$

may be replaced by an arbitrary open interval containing $f(x_0)$ and a sufficiently small open interval containing x_0, as indicated on p. 33.

3. Let $f(x)$ be continuous for $0 \leq x \leq 1$. Suppose further that $f(x)$ assumes rational values only and that $f(x) = \frac{1}{2}$ when $x = \frac{1}{2}$. Prove that $f(x) = \frac{1}{2}$ everywhere.

4. (*a*) Let $f(x)$ be defined for all values of x in the following manner:

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational.} \end{cases}$$

Prove that $f(x)$ is everywhere discontinuous.

(*b*) On the other hand, consider

$$g(x) = \begin{cases} 0, & x \text{ irrational} \\ \dfrac{1}{q}, & x = \dfrac{p}{q} \text{ rational in lowest terms.} \end{cases}$$

(The rational number p/q is said to be in lowest terms if the integers p and q have no common factor larger than 1, and $q > 0$. Thus $f(16/29) = 1/29$.) Prove that $g(x)$ is continuous for all irrational values and discontinuous for all rational values.

***5.** If $f(x)$ satisfies the functional equation

$$f(x + y) = f(x) + f(y)$$

for all values of x and y, find the values of $f(x)$ for rational values of x and prove if $f(x)$ is continuous that $f(x) = cx$ where c is a constant.

6. (*a*) If $f(x) = x^n$, find a δ which may depend on ξ such that

$$|f(x) - f(\xi)| < \epsilon$$

whenever

$$|x - \xi| < \delta.$$

*(*b*) Do the same if $f(x)$ is any polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$.

Answers to Exercises

SECTION 1.2d, page 31

1. and **2.**

(*a*) $\delta = \dfrac{\epsilon}{6}$.

(*b*) $\delta = \sqrt{\epsilon + (\xi - 1)^2} - |\xi - 1|$.

For the interval in question, $\delta = \sqrt{\epsilon + 4} - 2$. More simply,

$$|(x^2 - 2x) - (\xi^2 - 2\xi)| \le |x - \xi| \cdot |x + \xi - 2|$$
$$< \delta\{|x| + |\xi| + 2\} \le 4\delta = \epsilon$$

when

$$\delta = \tfrac{1}{4} \cdot \epsilon$$

(*d*) $|f(x) - f(\xi)| \le \delta\,|x + \xi|\,|3x^2 + 3\xi^2 + 1| \le \epsilon$

for $\delta = \epsilon/776$, in the interval [2, 4].

(*e*) Observe that

$$(x^{1/3} - \xi^{1/3})^2 \le 4(x^{2/3} + x^{1/3}\,\xi^{1/3} + \xi^{2/3}).$$

Consequently, on multiplying by $|x^{1/3} - \xi^{1/3}|$,

$$|x^{1/3} - \xi^{1/3}|^3 \le 4\,|x - \xi|$$

and we may take $\delta = \epsilon^3/4$.

3. (*a*) Discontinuous, 1, −1.

(*b*) Continuous; (*c*) discontinuous, 1, −1.

(*d*) Continuous; (*e*) discontinuous, −1.

(*f*) Discontinuous, 2,4; (*g*) discontinuous, 3.

(*h*) Continuous; (*i*) discontinuous, 0.

(*j*) Continuous.

For the function given in (*i*) observe that $|f(1/[n + \frac{1}{2}]\pi) - f(0)| = 1$. For the function given in (*j*) observe that $|f(x) - f(0)| \le |x|$.

4. (*a*) ϵ, ϵ, ϵ.

(*b*) For $\xi = -1 + 1/n$, $x = -1 + 2/n$,

$$\left|\frac{1}{1+x} - \frac{1}{1+\xi}\right| = \frac{n}{2}.$$

For $x = \xi + 1/n$

$$|x^3 - \xi^3| \geq \frac{3\xi^2}{n}.$$

For $x = \xi + 1/n$, $\xi \geq 1$,

$$\left|\frac{x^3}{1+|x|} - \frac{\xi^3}{1+|\xi|}\right| = \frac{(x^3 - \xi^3) + x\xi(x^2 - \xi^2)}{(1+x)(1+\xi)}$$

$$\geq \frac{3\xi^2 + 2\xi^3}{n(2+\xi)(1+\xi)} \geq \frac{2\xi^3}{n(3\xi)(2\xi)} \geq \frac{\xi}{3n}.$$

Solutions and Hints to Problems

SECTION 1.2d, page 31

1. For $\epsilon = \frac{1}{2}f(a)$ there exists a δ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. Consequently, in the δ-neighborhood of a,

$$-\tfrac{1}{2}f(a) < f(x) - f(a)$$

whence

$$f(x) > \tfrac{1}{2}f(a) > 0.$$

2. *Hint:* Any open interval about x_0 contains a centered interval.

3. *Hint:* Every interval contains irrational points.

4. (*a*) *Hint:* Every interval contains both rational and irrational points.

(*b*) For any rational number $x = p/q$, every interval about x contains irrational points ξ, so that $|f(x) - f(\xi)| = 1/q$. For an irrational value of x there exist only finitely many rational points ξ such that $f(x) > \epsilon$ and $|x - \xi| < 1$; choose δ as the lesser of 1 and the distance of the closest such point.

5. For any natural number n,

$$\begin{aligned} f(n) &= f(n-1) + f(1) \\ &= f(n-2) + 2f(1) \\ &= \cdots \\ &= nf(1). \end{aligned}$$

Furthermore, since $f(0) = f(0) + f(0)$, we have $f(0) = 0$; hence from

$$f(0) = f(n) + f(-n)$$

that

$$f(-n) = -f(n) = -nf(1).$$

Finally, for every rational number $r = p/q$

$$qf\left(\frac{p}{q}\right) = f\left(q \cdot \frac{p}{q}\right) = f(p) = pf(1).$$

Consequently, f is a linear function on the rational numbers

$$f(r) = ar,$$

where $a = f(1)$.

If f is continuous, it follows that $f(x) = ax$ for all x, since the difference $|f(x) - ar|$ can be made arbitrarily small by taking r close enough to x.

6. (*a*) $|x^n - \xi^n| = |x - \xi| \cdot |x^{n-1} + \xi x^{n-2} + \cdots + \xi^{n-1}|$
$\leq \delta \cdot |x^{n-1} + \xi x^{n-2} + \cdots + \xi^{n-1}|.$

Choose $\delta < 1$, so that $|x| < |\xi| + 1$;

$$|x^n - \xi^n| < \delta\{(|\xi| + 1)^{n-1} + |\xi| \cdot (|\xi| + 1)^{n-2} + \cdots\}$$
$$< \delta\{(|\xi| + 1)^{n-1} + (|\xi| + 1)^{n-1} + \cdots\},$$

or

$$|x^n - \xi^n| < \delta n(|\xi| + 1)^{n-1}.$$

It is sufficient to choose

$$\delta = \frac{\epsilon}{n(|\xi| + 1)^{n-1}},$$

where $\epsilon < 1$.

(*b*) $|f(x) - f(\xi)| = |a_n(x^n - \xi^n)a_{n-1}(x^{n-1} - \xi^{n-1}) + \cdots|$
$\leq |a_n|\,|x^n - \xi^n| + \cdots + |a_1|\,|x - \xi|.$

From the preceding results we conclude that

$$|f(x) - f(\xi)| < A\delta\{n(|\xi| + 1)^{n-1} + (n - 1)(|\xi| + 1)^{n-2} + \cdots\},$$

where A is the largest of the $|a_\nu|$, $\nu = 1, \ldots, n$. Consequently,

$$|f(x) - f(\xi)| < A\delta\{n(|\xi| + 1)^{n-1} + n(|\xi| + 1)^{n-1} + \cdots\}.$$

It is sufficient to take

$$\delta < \frac{\epsilon}{An^2(|\xi| + 1)^{n-1}}$$

or $\delta < 1$, whichever is smaller.

EXERCISES

SECTION 1.2e, page 44

1. Demonstrate that the following functions are monotonic on their domains and give expressions for their inverses on the appropriate domains.

(*a*) $f(x) = 3x - 4$.

(*b*) $f(x) = 3 - 4x$.

(*c*) $f(x) = \sqrt{x - 1}$, $x \geq 1$.

(*d*) $f(x) = x^2 + 2x + 2$, $x \geq 1$.

(*e*) $f(x) = x + \frac{1}{x}, x \geq 1.$

(*f*) $f(x) = \begin{cases} x, & x \leq 1, \\ x^2, & x \geq 1. \end{cases}$

2. In each of the following cases demonstrate that $f(x)$ is monotonic on the whole real axis. In which cases does $f(x)$ have a real root, that is, a number ξ such that $f(\xi) = 0$?

(*a*) $f(x) = x^3.$

(*b*) $f(x) = 1 + \dfrac{x}{1 + |x|}\,.$

(*c*) $f(x) = \dfrac{x^3 + 2x}{1 + x^2}\,.$

PROBLEMS

SECTION 1.2e, page 44

1. Prove that if $f(x)$ is monotonic on $[a, b]$ and satisfies the intermediate value property, then $f(x)$ is continuous. Can you draw the same conclusion if f is not monotonic?

2. (*a*) Show that x^n is monotonic for $x > 0$. As a consequence, show for $a > 0$ that $x^n = a$ has a unique positive solution $\sqrt[n]{a}$.

(*b*) Let $f(x)$ be a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad (a_n \neq 0).$$

Show (i) if n is odd, then $f(x)$ has at least one real root, (ii) if a_n and a_0 have opposite signs, then $f(x)$ has at least one positive root, and, in addition, if n is even, $n \neq 0$, then $f(x)$ has a negative root as well.

***3.** (*a*) Prove that there exists a line in each direction which bisects any given triangle, that is, divides the triangle into two parts of equal area.

(*b*) For any pair of triangles prove that there exists a line which bisects them simultaneously.

Answers to Exercises

SECTION 1.2e, page 44

1. The inverses are given by

(*a*) $\dfrac{x + 4}{3}$; (*b*) $\dfrac{3 - x}{4}$;

(*c*) $1 + x^2, x \geq 0$; (*d*) $-1 + \sqrt{x - 1}, x \geq 1$;

(*e*) $\dfrac{x + \sqrt{x^2 - 4}}{2}, \quad x \geq 2$; (*f*) $g(x) = \begin{cases} x, & x \leq 1 \\ \sqrt{x}, & x \geq 1. \end{cases}$

2. Assume $x < y$. Then

$$\begin{aligned}(a)\ y^3 - x^3 &= (y - x)(x^2 + xy + y^2)\\ &= (y - x)\left[\left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4}\right]\\ &> 0.\end{aligned}$$

Root at $x = 0$.

(*b*) Since $f(x) - 1$ is an odd function, it is sufficient to establish the result for $x \geq 0$.

For $y > x \geq 0$,

$$f(y) - f(x) = \frac{y}{1 + y} - \frac{x}{1 + x} = \frac{y - x}{(1 + x)(1 + y)} \geq 0.$$

root.

$$(c)\ \frac{y^3 + 2y}{1 + y^2} - \frac{x^3 + 2x}{1 + x^2} = \frac{(y - x)[x^2y^2 + x^2 - xy + y^2 + 2]}{(1 + y^2)(1 + x^2)} \geq 0.$$

Root at $x = 0$.

Solutions and Hints to Problems

SECTION 1.2e, page 44

1. If f is not continuous at ξ, then for some ϵ and every $\delta > 0$ there exist points x such that $|f(x) - f(\xi)| > \epsilon$ and $|x - \xi| < \delta$. It follows that every neighborhood of ξ contains infinitely many such values, and so for the two sets of intervals $(\xi - a, \xi)$ or $(\xi, \xi + a)$ every interval of at least one of the two sets contains such values of x. Suppose $f(x)$ is increasing and every interval $(\xi, \xi + a)$ contains a point η such that $|f(\eta) - f(\xi)| > \epsilon$. Then

$$f(\xi + a) \geq f(\eta) > f(\xi) + \epsilon > f(\xi).$$

The value $f(\xi) + \epsilon$ cannot be taken on, in contradiction to the assumption of the intermediate value property.

No, for consider $f(x) = \sin(1/x)$ for $x \neq 0$, $f(0) = 0$.

2. (*a*) If $x > y > 0$, then $x^2 > y^2 > 0$, and upon repeated multiplication, $x^n > y^n > 0$. If $0 < a \leq 1$, then a is intermediate between 0 and 1^n. If $a \geq 1$, then a is intermediate between 1 and a^n. In either case, the existence of a positive root is assured. Since x^n is monotone, there cannot be two distinct positive roots.

(*b*) (i) If n is odd, then we may set

$$\begin{aligned}f(x) &= a_{2n-1}x^{2n-1} + a_{2n-2}x^{2n-2} + a_{2n-3}x^{2n-3} + \cdots\\ &= a_{2n-1}x^{2n-1}\left(1 + \frac{a_{2n-2}}{xa_{2n-1}} + \frac{a_{2n-3}}{x^2a_{2n-1}} + \cdots\right).\end{aligned}$$

If $|x| > nA + 1$, where A is the largest of the numbers $|a_\nu/a_{2n-1}|$, $\nu = 0, 1, \ldots, 2n-2$, then

$$\left| \frac{a_{2n-2}}{xa_{2n-1}} + \frac{a_{2n-3}}{x^2 a_{2n-1}} + \cdots \right| < 1,$$

and we conclude that $f(x)$ has the same sign as $a_{2n-1}x^{2n-1}$. Thus for positive and negative values of x of large magnitude, $f(x)$ takes opposite signs. The existence of a root follows by the intermediate value property.

(ii) If a_n and a_0 have opposite signs, then for large positive values of x, $f(x)$ has the sign of a_n and since $f(0) = a_0$, it follows by the intermediate value property that $f(x)$ has a positive root. If n is even, then $f(x)$ has the sign of a_n for negative and absolutely large values of x, and it follows similarly that $f(x)$ has a negative root.

3. (*a*) It is known from elementary geometry that every line through the intersection point of medians has this property.

From the intermediate value theorem, it also follows that in each direction there is a line which bisects a triangle (or any other area). Consider the lines $L(x)$ of a parallel family, where x is the distance measured along a common perpendicular. Let $f(x)$ be the difference between the areas of the parts of the triangle on opposite sides of $L(x)$. Since there exist lines on either side of the triangle which do not intersect it, it follows that $f(x)$ takes on both negative and positive values; hence, by the intermediate value property, $f(x) = 0$ for some value of x.

(*b*) For each direction θ, choose the line $L(\theta)$ which divides the first triangle into two equal parts. Let $L(\theta)$ be oriented positively in the direction θ. Let $g(\theta) = \alpha(\theta) - \beta(\theta)$, where $\alpha(\theta)$ and $\beta(\theta)$ are those parts of the area of the second triangle which lie on the right and left of $L(\theta)$. If $g(\theta) > 0$, then $g(\pi + \theta) = \beta(\theta) - \alpha(\theta) < 0$ and by the intermediate value property it follows that $g(\theta) = 0$ for some value of θ.

EXERCISES

SECTION 1.3a, page 47

1. Draw the graphs of the rational functions:

(*a*) $y = \dfrac{1}{x^2+1}$, (*b*) $y = \dfrac{1}{x^2-1}$, (*c*) $y = x + \dfrac{1}{x}$,

(*d*) $y = \dfrac{x^3+1}{x^2-1}$, (*e*) $y = \dfrac{x^2+1}{x^2-1}$, (*f*) $y = \dfrac{x^2-1}{x^2+1}$.

EXERCISES

SECTION 1.3b, p. 49

1. Draw the graphs of the following algebraic functions:

(*a*) $y = (x^2 + 1)^{1/2}$, (*b*) $y = \dfrac{1}{(x^2 - 1)^{1/2}}$, (*c*) $y = \dfrac{1}{(x^2 + 1)^{1/2}}$,

(*d*) $y = (x + 1)^{1/3}$, (*e*) $y = (x^2 + 1)^{1/3}$, (*f*) $y = (x^2 - 1)^{1/3}$.

PROBLEMS

SECTION 1.3b, page 49

1. (*a*) Prove that $\sqrt{x}$ is not a rational function. (*Hint:* Examine the possibility of representing $\sqrt{x}$ as a rational function for $x = y^2$. Use the fact that a nonzero polynomial can have at most finitely many roots.)

(*b*) Prove $\sqrt[n]{x}$ is not a rational function.

Solutions and Hints to Problems

SECTION 1.3b, page 49

1. If

$$\sqrt{x} = \frac{a_0 + a_1x + + \cdots + a_nx^n}{b_0 + b_1x + \cdots + b_mx^m},$$

then, for $x = \nu^2$, where ν is any natural number,

$$P(\nu) = a_0 - b_0\nu + a_1\nu^2 - b_1\nu^3 \cdots = 0.$$

Thus the polynomial $P(\nu)$ has infinitely many distinct roots. We conclude that $P(\nu)$ is identically zero, or that

$$a_0 = b_0 = a_1 = b_1 = \cdots = 0;$$

hence that $\sqrt{x}$ cannot be represented as a rational function.

EXERCISES

SECTION 1.3c, p. 49

1. Find the radian measures of the angles:

(*a*) 10°, (*b*) 25°,
(*c*) 93°, (*d*) 173°,
(*e*) 453°, (*f*) 769°

2. Draw the graphs of the functions

(*a*) $y = \sin x \cdot \cos x$,
(*b*) $y = \sin (x + \pi)$,
(*c*) $y = \tan x \cdot \sin x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$.

PROBLEMS

SECTION 1.3c, page 49

1. (*a*) Show that a straight line may intersect the graph of a polynomial higher than first degree in at most finitely many points.

(*b*) Obtain the same result for general rational functions.

(*c*) Verify that the trigonometric functions are not rational.

Answers to Exercises

SECTION 1.3c, page 49

1. (*a*) 0.17453, (*b*) 0.43633, (*c*) 1.62316, (*d*) 3.01942, (*e*) 7.90634, (*f*) 13.42158.

Solutions and Hints to Problems

SECTION 1.3c, page 49

1. (*a*) The graphs $y = \alpha x + \beta$ and $y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ intersect at points for which

$$\alpha x + \beta = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

that is,

$$(1) \qquad (a_0 - \beta) + (a_1 - \alpha)x + a_2x^2 + a_3x^3 + \cdots + a_nx^n = 0.$$

If the number of intersections is infinite, then the polynomial of (1) must be identically zero, and

$$a_0 = \beta,\ a_1 = \alpha,\ a_2 = 0, \ldots, a_n = 0,$$

contradicting the assumption $n > 1$.

(*b*) If

$$\alpha x + \beta = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + \cdots + a_nx^n}{b_0 + b_1x + \cdots + b_mx^m},$$

for infinitely many values x, then by the argument used above,

$$b_0\beta = a_0,\ b_0\alpha + b_1\beta = a_1, \ldots,$$
$$b_k\alpha + b_{k+1}\beta = a_{k+1}$$

(where, for convenience, we set $a_k = 0$ for $k > n$ and $b_k = 0$ for $k > m$). Thus

$$\begin{aligned} P(x) &= a_0 + a_1x + \cdots + a_nx^n \\ &= b_0\beta + (b_0\alpha + b_1\beta)x + (b_1\alpha + b_2\beta)x^2 + \cdots \\ &= b_0(\beta + \alpha x) + b_1(\beta + \alpha x)x + \cdots \\ &= (\beta + \alpha x)Q(x). \end{aligned}$$

It follows that [except for zeros of $Q(x)$] $P(x)/Q(x)$ is precisely the same linear polynomial.

(*c*) Because of its periodicity, the horizontal line $y = 1$ has infinitely many intersections with the graph of any trigonometrical function.

EXERCISES

SECTION 1.3d, p. 51

1. Assuming the exponential function $y = a^x$ to be defined as indicated on p. 52, draw the graphs of:

(*a*) $y = (0.5)^x, \qquad x = \log_{0.5}y;$

(*b*) $y = 2^x, \qquad x = \log_2 y.$

EXERCISES

SECTION 1.3e, page 52

1. Find the domain and range of each of the following functions, and draw their graphs:

(*a*) $y = \sin(x^2)$, (*b*) $y = (\sin x)^2$,
(*c*) $y = \sin x \cos x$, (*d*) $y = (\sin x)^{1/2}$,
(*e*) $y = (1 + \cos x)^{1/2}$, (*f*) $y = \cos[(1 - x^2)^{1/2}]$,
(*g*) $y = \dfrac{1}{(\sin x)^{1/2}}$, (*h*) $y = \sin(2^x)$,
(*i*) $y = 2^{\sin x}$, (*j*) $y = \sin[\sin x + 1]$.

2. Show that the function $y = 2x^2 - 3x + 1$ is monotonic increasing for $x \geq 3$. Draw the graph of this function and of its inverse for $x \geq 3$.

3. (*a*) Let

$$g(x) = \begin{cases} x - \pi, & x \leq 0, \\ x + \pi, & x > 0, \end{cases}$$

and $h(u) = \sin u$. Prove that the symbolic product hg is continuous for all x, although g is not.

(*b*) Show that if g is a continuous function whose range contains a neighborhood of a discontinuity of h, then hg is discontinuous.

(*c*) Using the continuity of the identity mapping I, produce a pair of discontinuous functions g, h whose symbolic product hg is continuous.

Answers to Exercises

SECTION 1.3e, page 52

1. (*a*) All x, $-1 \leq y \leq 1$.
(*b*) All x, $0 \leq y \leq 1$.
(*c*) All x, $-\frac{1}{2} \leq y \leq \frac{1}{2}$.
(*d*) $2n\pi \leq x \leq (2n+1)\pi$, $0 \leq y \leq 1$.
(*e*) All x, $0 \leq y \leq \sqrt{2}$.
(*f*) $-1 \leq x \leq 1$, $\cos 1 \leq y \leq 1$.
(*g*) $2n\pi < x < (2n+1)\pi$, $1 \leq y$.
(*h*) All x, $-1 \leq y \leq 1$.
(*i*) All x, $\frac{1}{2} \leq y \leq 2$.
(*j*) All x, $0 \leq y \leq 1$.

2. If $x > \xi \geq \frac{3}{4}$, then the difference in the corresponding function values is

$$(x - \xi)[2(x + \xi) - 3] > 0.$$

3. (*a*) $\sin g(x) = -\sin x$, for all x.

(*b*) For some $\epsilon > 0$, every neighborhood of a point of discontinuity u contains points v such that $|f(v) - f(u)| > \epsilon$.

(*c*) Take g as the function of part *a*,

$$h(u) = \begin{cases} u + \pi, & u \leq -\pi, \\ u - \pi, & u > \pi, \\ 1, & -\pi < u < \pi. \end{cases}$$

EXERCISES

SECTION 1.5, p. 57

1. Using mathematical induction, derive the formulas for the partial sums of the following arithmetic and geometric progressions:

(*a*) $a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d]$
$$= \tfrac{1}{2}n[(n - 1)d + 2a],$$

(*b*) $a + ar + ar^2 + \cdots + ar^{n-1} = a\left[\dfrac{(r^n - 1)}{(r - 1)}\right].$

2. Give inductive proofs of the following formulas:

(*a*) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n \cdot (n + 1)} = \dfrac{n}{n + 1},$

(*b*) $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots n)^2$
$$= \left[\frac{n(n + 1)}{2}\right]^2.$$

3. Find formulas for the following sums and prove them by mathematical induction:

(*a*) $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n + 1),$

(*b*) $\dfrac{1}{1 + x^2} + \dfrac{1}{(1 + x^2)^2} + \cdots + \dfrac{1}{(1 + x^2)^n}.$

4. Prove by mathematical induction, that for every positive integer n, $2n \leq 2^n$.

5. Using mathematical induction, prove that for any integer $k > 1$ and for all positive integers n,

(*a*) $\dfrac{n^{k+1}}{(k + 1)} \geq 1 + 2^k + 3^k + \cdots + (n - 1)^k,$

(*b*) $\dfrac{n^{1-1/k}}{1 - 1/k} \geq 1 + 2^{-1/k} + 3^{-1/k} + \cdots + n^{-1/k}.$

6. With the aid of the binomial theorem, prove that the number $2^{1/n}$ is irrational for all integers $n \geq 2$.

7. Using the binomial theorem, prove that the decimal number $n = n_k n_{k-1} \cdots n_0 = n_0 + 10n_1 + \cdots + 10^k n_k$, $n_0, \ldots, n_k < 10$ is divisible by 11 if and only if $n_0 - n_1 + n_2 + \cdots + (-1)^k n_k$ is divisible by 11.

PROBLEMS

SECTION 1.5, page 57

1. Prove the following properties of the binomial coefficients.

(*a*) $1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$

(*b*) $1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0.$

(*c*) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n(2^{n-1}).$ (*Hint:* Represent the binomial coefficients in terms of factorials.)

(*d*) $1 \cdot 2\binom{n}{2} + 2 \cdot 3\binom{n}{3} + \cdots + (n-1)n\binom{n}{n} = n(n-1)2^{n-2}.$

(*e*) $1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$

*(*f*) $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$. (*Hint:* Consider the coefficient of x^n in $(1 + x)^{2n}$.)

*(*g*)
$$S_n = \binom{n}{0} - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \frac{1}{7}\binom{n}{3} + \cdots + \frac{(-1)^n}{2n+1}\binom{n}{n} = \frac{4^n(n!)^2}{(2n+1)!}.$$

$$\left(\textit{Hint: } \text{Prove } \frac{2n+2}{2n+3} S_n = S_{n+1}.\right)$$

2. Prove $(1 + x)^n \geq 1 + nx$, for $x > -1$.

3. Prove by induction that $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$.

***4.** Prove by induction the following:

(*a*) $1 + 2q + 3q^2 + \cdots + nq^{n-1} = \frac{1 - (n+1)q^n + nq^{n+1}}{(1-q)^2}.$

(*b*) $(1 + q)(1 + q^2) \cdots (1 + q^{2^n}) = \frac{1 - q^{2^{n+1}}}{1 - q}.$

5. Prove for all natural numbers n greater than 1 that n is either a prime or can be expressed as a product of primes. (*Hint:* Let A_{n-1} be the assertion for all integers k with $k \leq n$ that k is either prime or a product of primes.)

***6.** Consider the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \ldots, \frac{p_n}{q_n}, \ldots,$$

where $p_{n+1} = p_n + 2q_n$ and $q_{n+1} = p_n + q_n$.

(*a*) Prove for all n that p_n/q_n is in lowest terms.

(*b*) Show that the absolute difference between p_n/q_n and $\sqrt{2}$ can be made arbitrarily small. Prove also that the error of approximation to $\sqrt{2}$ alternates in sign.

7. Let a, b, a_n and b_n be integers such that

$$(a + b\sqrt{2})^n = a_n + b_n\sqrt{2},$$

where a is the integer closest to $b\sqrt{2}$. Prove that a_n is the integer closest to $b_n\sqrt{2}$.

***8.** Let a_n and b_n be defined by

$$a_1 = 3, \qquad a_{n+1} = 3^{a_n}, \qquad \text{and} \quad b_1 = 9, \qquad b_{n+1} = 9^{b_n}.$$

For each value of n, determine the minimum value m such that $a_m \geq b_n$.

9. If n is a natural number, show that

$$\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n\sqrt{5}}$$

is a natural number.

10. Determine the maximum number of pieces into which a plane may be cut by n straight lines. Show that the maximum occurs when no two of the lines are parallel and no three meet in a common point, and determine the number of pieces when concurrences and parallelisms are permitted.

11. Prove for each natural number n that there exists a natural number k such that

$$(\sqrt{2} - 1)^n = \sqrt{k} - \sqrt{k - 1}.$$

12. Prove Cauchy's inequality inductively.

Answers to Exercises

SECTION 1.5, page 57

1 and **2.** Take the formula for the sum to n terms; add the $(n + 1)$th term to obtain the formula for the sum to $n + 1$ terms.

3. (*a*) $\dfrac{n(n + 1)(n + 2)}{3}$.

(*b*) $\dfrac{1}{x^2} - \dfrac{1}{x^2(1 + x^2)^n}$.

4. True for $n = 1$. If true for $n = k$, then

$$2^{k+1} \geq 2(2k) \geq 2(k + k) \geq 2(k + 1).$$

5. (*a*) True for $n = 1$. If true for n, then

$$\sum_{\nu=1}^{n} \nu^k \leq \frac{n^{k+1}}{k + 1} + n^k \leq \frac{(n + 1)^{k+1}}{k + 1},$$

since the middle expression consists of the first two terms of the binomial expansion of $(n + 1)^k/k(k + 1)$.

(*b*) The result is certainly true for $n = 1$. If the result holds for a given n, it will hold for $n + 1$, provided that

$$\frac{n^{1-1/k}}{1 - \dfrac{1}{k}} + (n + 1)^{-1/k} \leq \frac{(n + 1)^{1-1/k}}{1 - \dfrac{1}{k}}.$$

This inequality is equivalent to

$$n^{1-1/k} \leq (n + 1)^{-1/k}\left[n + \frac{1}{k}\right];$$

hence, to

$$1 \leq \left(\frac{n}{n + 1}\right)^{1/k}\left[1 + \frac{1}{nk}\right],$$

and, for the kth powers, to

$$1 \leq \left(\frac{n}{n + 1}\right)\left[1 + \frac{1}{nk}\right]^k.$$

But

$$\left(\frac{n}{n + 1}\right)\left[1 + \frac{1}{nk}\right]^k \geq \frac{n}{n + 1}\left(1 + \frac{1}{n}\right) = 1,$$

so that the last inequality is valid.

6. If $2^{1/n}$ is rational, then set

$$2^{1/n} = \left(1 + \frac{p}{q}\right),$$

where p/q is given in lowest terms. Take the nth power to obtain

$$2q^n = p^n + \binom{n}{1}p^{n-1}q + \cdots + q^n.$$

Since q is a factor on the left, it must be a factor on the right; this implies that q divides p^n and contradicts the fact that q and p have no common factors.

7. $$\begin{aligned} n &= n_0 + 10n_1 + 100n_2 + \cdots + 10^k n_k \\ &= n_0 + (11 - 1)n_1 + \cdots + (11 - 1)^k n_k \\ &= [n_0 - n_1 + \cdots + (-1)^k n_k] + 11m. \end{aligned}$$

Solutions and Hints to Problems

SECTION 1.5, page 57

1. (*a*) Expand $(1 + 1)^n$.

(*b*) Expand $(1 - 1)^n$.

(*c*)
$$\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$$
$$= n\binom{n-1}{0} + n\binom{n-1}{1} + \cdots + n\binom{n-1}{n-1}$$
$$= n2^{n-1},$$
by part *a*.

(*d*) Use
$$(k-1)k\binom{n}{k} = \frac{(k-1)k(n!)}{k!\,(n-k)!} = \frac{n!}{(k-2)!\,(n-k)!}$$
$$= n(n-1)\binom{n-2}{k-2}.$$

(*e*) Use
$$\frac{1}{k+1}\binom{n}{k} = \frac{n!}{(k+1)!\,(n-k)!} = \frac{1}{n+1}\binom{n+1}{k+1}.$$

(*f*) The coefficient of x^n in $(1 + x)^{2n} = (1 + x)^n(1 + x)^n$ is
$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}$$
$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

(*g*) Use the relation
$$\frac{2n+2}{2n+3}\cdot S_n = \sum_{k=0}^{n}\frac{(-1)^k(2n-2k+2)}{(2n+3)(2k+1)}\binom{n+1}{k}$$
$$= \sum_{k=0}^{n}(-1)^k\,\frac{(2n+3)-(2k+1)}{(2n+3)(2k+1)}\binom{n+1}{k}$$
$$= \sum_{k=0}^{n+1}\frac{(-1)^k}{2k+1}\binom{n+1}{k} - \frac{1}{2n+3}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}$$
$$= S_{n+1} - \frac{1}{2n+3}(1-1)^{n+1}$$
$$= S_{n+1}.$$

2. For $n = 1$, the statement is trivially true. If, for $n = k$, $(1 + x)^k \geq 1 + kx$, then
$$(1+x)^{k+1} \geq (1+kx)(1+x)$$
$$\geq 1 + (k+1)x.$$

3. True for $n = 1$. If true for $n = k$, then

$$\begin{aligned} 1 + 2 + \cdots + k + (k + 1) &= (k + 1) + \tfrac{1}{2}k(k + 1) \\ &= (k + 1)(\tfrac{1}{2}k + 1) \\ &= \tfrac{1}{2}(k + 1)(k + 2). \end{aligned}$$

4. (*a*) True for $n = 1$. If true for $n = k$, then

$$(1 + 2q + \cdots + kq^{k-1}) + (k + 1)q^k$$

$$= \frac{1 - (k + 1)q^k + kq^{k+1}}{(1 - q)^2} + (k + 1)q^k$$

$$= \frac{1 - (k + 1)q^k + kq^{k+1} + (k + 1)(q^k - 2q^{k+1} + q^{k+2})}{(1 - q)^2}$$

$$= \frac{1 - (k + 2)q^{k+1} + (k + 1)q^{k+2}}{(1 - q)^2},$$

which proves the result for $n = k + 1$.

(*b*) True for $n = 1$. If true for $n = k$, then

$$[(1 + q) \cdots (1 + q^{2^k})](1 + q^{2^{k+1}})$$

$$= \frac{(1 - q^{2^{k+1}})(1 + q^{2^{k+1}})}{1 - q}$$

$$= \frac{1 - (q^{2^{k+1}})^2}{1 - q}$$

$$= \frac{1 - q^{2^{k+2}}}{1 - q}.$$

5. Let A_n be the statement, "If $1 < m \leq n + 1$, then m is either prime or can be expressed as a product of primes." The statement is obviously true for $n = 1$. If $k + 2$ is not prime, then it has at least two proper divisors: $k + 2 = pq$, where $1 < p < k + 2$ and $1 < q < k + 2$. It follows that $p \leq k + 1$ and $q \leq k + 1$. If A_k is true, then p and q are either primes or can be expressed as products of primes; consequently, $k + 2 = pq$ can be expressed as a product of primes and A_{k+1} is true.

6. (*a*) True for $n = 1$. If p_{k+1} and q_{k+1} had a common factor other than 1, then so would $q_k = p_{k+1} - q_{k+1}$, $p_k = 2q_{k+1} - p_{k+1}$, a contradiction.

(*b*) We show that $(1 - \sqrt{2})^n = p_n - q_n\sqrt{2}$. This is true for $n = 1$. If true for $n = k$, then

$$\begin{aligned} (1 - \sqrt{2})^k(1 - \sqrt{2}) &= (p_k - q_k \sqrt{2})(1 - \sqrt{2}) \\ &= [p_k + 2q_k - (p_k + q_k)\sqrt{2}] \\ &= (p_{k+1} - q_{k+1} \sqrt{2}). \end{aligned}$$

Furthermore, since p_1 and q_1 are positive, it follows by induction that p_n and q_n are increasing with n. Therefore

$$\frac{p_n}{q_n} - \sqrt{2} = \frac{(1 - \sqrt{2})^n}{q_n}$$

has the same sign as $(1 - \sqrt{2})^n$. Since $1 - \sqrt{2} < 0$, the sign of the error alternates. Finally, since

$$|1 - \sqrt{2}| < \tfrac{1}{2} \text{ and } q_n > 1,$$

$$\left|\frac{p_n}{q_n} - \sqrt{2}\right| < \frac{1}{2^n}.$$

7. Show that

$$a_{n+1} = aa_n + 2bb_n,$$
$$b_{n+1} = ab_n + ba_n.$$

Verify that

$$(a - b\sqrt{2})^n = (a_n - b_n\sqrt{2}).$$

The result follows from $|a - b\sqrt{2}| < \frac{1}{2}$.

8. The minimum is given by $m = n + 1$. For the proof we show that $a_{n+1} > 2b_n + 1$. This is certainly true for $n = 1$.

If for $n = k$, $a_{k+1} > 2b_k + 1$, then

$$a_{k+2} = 3^{a_{k+1}} > 3^{2b_k+1} \geq 3 \cdot 3^{2b_k}$$
$$\geq 3 \cdot 9^{b_k} \geq 3b_{k+1} \geq 2b_{k+1} + b_{k+1} > 2b_{k+1} + 1.$$

From this result if follows that no smaller value of m will suffice, and since $a_k < b_k$, the proof is complete.

9. Set $F_n = \dfrac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n\sqrt{5}}$. Then $F_n + F_{n+1} = F_{n+2}$. The numbers F_n are known as the Fibonacci numbers.

10. Let λ_n be the maximum number of pieces into which n lines cut the plane. Proceeding along the $(n + 1)$th line we cut each region containing the path into two parts and enter a new region if and only if we cross one of the preceding lines. We see then that $\lambda_{n+1} = \lambda_n + (n + 1)$ provided that the $(n + 1)$th line is not parallel to any preceding line or passes through a point where any two preceding lines intersect. By induction it follows that

$$\lambda_n = \tfrac{1}{2}(n^2 + n + 2).$$

For each pair of parallel lines λ_n is reduced by 1, since one crossing is eliminated. If there are k families of parallel lines with $p_1, p_2, \ldots, p_k$ lines in the respective families, then λ_n is reduced by

$$\sum_{\nu=1}^{k} \tfrac{1}{2}p_\nu(p_\nu - 1).$$

Similarly, if a line crosses an existing intersection of two lines the number of regions crossed is reduced by 1. If we assume j families of concurrent lines with $c_1, c_2, \ldots, c_j$ lines in the respective families, then λ_n is reduced by $\sum_{\mu=1}^{j} \frac{1}{2}(c_\mu - 1)(c_\mu - 2)$. Thus, in general, the number of regions into which the plane is divided by n lines is

$$\tfrac{1}{2}(n^2 + n + 2) - \sum_{\nu=1}^{k} \frac{p_\nu(p_\nu - 1)}{2} - \sum_{\mu=1}^{j} \tfrac{1}{2}(c_\mu - 1)(c_\mu - 2).$$

11. Define p_n and q_n as in Problem 6. We have

$$(\sqrt{2} - 1)^n = (-1)^n(p_n - q_n\sqrt{2}) = (-1)^n(\sqrt{p_n^2} - \sqrt{2q_n^2}).$$

From

$$p_{n+1} = p_n + 2q_n, \quad q_{n+1} = p_n + q_n,$$

we have

$$p_{n+1}^2 - 2q_{n+1}^2 = 2q_n^2 - p_n^2.$$

Since $2q_1^2 - p_1^2 = 1$, it follows inductively that

$$2q_n^2 - p_n^2 = (-1)^{n+1};$$

hence, for n even, $p_n^2 = 2q_n^2 + 1$, and

$$(\sqrt{2} - 1)^n = \sqrt{2q_n^2 + 1} - \sqrt{2q_n^2},$$

whereas for n odd, $2q_n^2 = p_n^2 + 1$ and

$$(\sqrt{2} - 1)^n = \sqrt{p_n^2 + 1} - \sqrt{p_n^2}.$$

12. The relation is obvious for $n = 1$. Suppose it is true for k:

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right).$$

Then, for $k + 1$,

$$\begin{aligned}
\left(\sum_{i=1}^{k+1} a_i b_i\right)^2 &= \left(\sum_{i=1}^{k} a_i b_i + a_{k+1}b_{k+1}\right)^2 \\
&= \left(\sum_{i=1}^{k} a_i b_i\right)^2 + 2a_{k+1}b_{k+1}\sum_{i=1}^{k} a_i b_i + a_{k+1}^2 b_{k+1}^2 \\
&\leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right) + 2\,|a_{k+1}b_{k+1}|\sqrt{\sum_{i=1}^{k} a_i^2}\sqrt{\sum_{i=1}^{k} b_i^2} \\
&\qquad + a_{k+1}^2 b_{k+1}^2 \\
&\leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right) + a_{k+1}^2 \sum_{i=1}^{k} b_i^2 + b_{k+1}^2 \sum_{i=1}^{k} a_i^2 + a_{k+1}^2 b_{k+1}^2 \\
&= \left(\sum_{i=1}^{k+1} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right) + b_{k+1}^2 \sum_{i=1}^{k} a_i^2 + a_{k+1}^2 b_{k+1}^2 \\
&= \left(\sum_{i=1}^{k+1} a_i^2\right)\left(\sum_{i=1}^{k+1} b_i^2\right)
\end{aligned}$$

where we have used the well-known inequality $2AB \leq A^2 + B^2$, or $(A - B)^2 \geq 0$.

EXERCISES

SECTION 1.6, page 60

1. Find the limits of the following expressions as $n \to \infty$:

(*a*) $\dfrac{3n+2}{2n+3}$,

(*b*) $\dfrac{6n^2+2n+1}{n^3+n^2}$,

(*c*) $\dfrac{6n^3+2n+1}{n^3+n^2}$

(*d*) $\dfrac{n^3}{n^2-n+1} - \dfrac{n^3+n,}{n^2+n+1}$,

(*e*) $\dfrac{a_p n^p + a_{p-1}n^{p-1} + \cdots + a_0}{b_p n^p + b_{p-1}n^{p-1} + \cdots + b_0}$, $\quad (b_p \neq 0)$,

(*f*) $\dfrac{1+4+9+\cdots+n^2}{n^3}$.

2. For each of the following expressions the limit is given as $n \to \infty$. In each case find an N such that for $n > N$ the difference between the expression and its limit is (*a*) less than $\frac{1}{10}$, (*b*) less than 1/1,000, (*c*) less than 1/1,000,000.

(i) $\lim\limits_{n\to\infty} \dfrac{5}{n} = 0.$

(ii) $\lim\limits_{n\to\infty} \dfrac{1-2n}{1+n} = -2.$

(iii) $\lim\limits_{n\to\infty} \dfrac{n^2+n-1}{3n^2+1} = \dfrac{1}{3}$

(iv) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n^2+1}} = 0.$

(v) $\lim\limits_{n\to\infty} \dfrac{n^2}{2^n} = 0.$

(vi) $\lim\limits_{n\to\infty} \sqrt[n]{n^2} = 1.$

(vii) $\lim\limits_{n\to\infty} \left(-\dfrac{9}{10}\right)^n = 0.$

3. Find the limits of the following as $n \to \infty$:

(*a*) $\dfrac{\sin(1/n)}{n}$,

(*b*) $\dfrac{\cos n}{n}$,

(*c*) $\sin\left(\dfrac{1}{n}\right)\cos n$,

(*d*) $\sin(\pi \cdot 2^{1/n-1})$,

(*e*) $\dfrac{2n-5}{n+3-5^{1/n}}$.

4. Using the binomial theorem (p. 59), prove that for any fixed $h > 0$, and any fixed integer k,

$$\lim_{n\to\infty} \frac{n^k}{(1+h)^n} = 0.$$

5. Prove that for any positive integer k,

$$\lim_{n\to\infty} (\sqrt[k]{n})^{1/n} = 1.$$

6. Prove that for any positive integer k,

$$\lim_{n\to\infty} (\sqrt{n^k+1} - \sqrt{n^k}) = 0.$$

7. Prove the following relations:

(*a*) $\lim\limits_{n\to\infty} (\sqrt{n^2+2n} - \sqrt{n^2+n}) = \frac{1}{2}$,

(*b*) $\lim\limits_{n\to\infty} (\sqrt{2n^2+3n+1} - \sqrt{n^2+5}) = \infty$,

(*c*) $\lim\limits_{n\to\infty} (\sqrt{n^2+2kn+1} - \sqrt{n^2+5}) = k$, for all $k \geq 0$.

8. For any given positive integer k, consider the difference

$$D_n = \sqrt{a_{2k}n^{2k} + a_{2k-1}n^{2k-1} + \cdots + a_1 n + a_0} - \sqrt{b_{2k}n^{2k} + b_{2k-1}n^{2k-1} + \cdots + b_1 n + b_0};$$

prove that if $a_i \neq b_i$ for any $i > k$, D_n tends to infinity as $n \to \infty$, whereas if $a_i = b_i$ for $i > k$, D_n converges to $(a_k - b_k)/2\sqrt{a_{2k}}$, as $n \to \infty$.

PROBLEMS

SECTION 1.6, page 60

1. Prove that $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n})(\sqrt{n+\frac{1}{2}}) = \frac{1}{2}$.

2. Prove that $\lim\limits_{n\to\infty} (\sqrt[3]{n+1} - \sqrt[3]{n}) = 0$.

3. Let $a_n = 10^n/n!$. (*a*) To what limit does a_n converge? (*b*) Is the sequence monotonic? (*c*) Is it monotonic from a certain n onward? (*d*) Give an estimate of the difference between a_n and the limit. (*e*) From what value of n onward is this difference less than 1/100?

4. Prove that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

5. (*a*) Prove that $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}\right) = \frac{1}{2}$.

(*b*) Prove that $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right) = 0$. (*Hint:* Compare the sum with its largest term.)

(*c*) Prove that $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}}\right) = \infty$.

*(*d*) Prove that $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1$.

6. Prove that every periodic decimal represents a rational number. (Compare Section 1.1c, Problem 3.)

7. Prove that $\lim_{n\to\infty} \frac{n^{100}}{1.01^n}$ exists and determine its value.

8. Prove that if a and $b \le a$ are positive, the sequence $\sqrt[n]{a^n + b^n}$ converges to a. Similarly, for any k fixed positive numbers $a_1, a_2, \ldots, a_k$ prove that $\sqrt[n]{a_1{}^n + a_2{}^n + \cdots + a_k{}^n}$ converges and find its limit.

9. Prove that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots,$ converges. Find its limit.

10. If $\nu(n)$ is the number of prime factors of n, prove that $\lim_{n\to\infty} \frac{\nu(n)}{n} = 0$.

11. Prove that if $\lim_{n\to\infty} a_n = \xi$, then $\lim_{n\to\infty} \sigma_n = \xi$, where σ_n is the arithmetic mean $(a_1 + a_2 + \cdots a_n)/n$.

12. Find

(*a*) $\lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}\right)$.

$\left(\textit{Hint: } \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.\right)$

(*b*) $\lim_{n\to\infty} \left(\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)}\right)$.

13. If $a_0 + a_1 + \cdots + a_p = 0$, prove that

$$\lim_{n\to\infty} (a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_n\sqrt{n+p}) = 0.$$

(*Hint:* Take $\sqrt{n}$ out as a factor.)

14. Prove that $\lim_{n\to\infty} \sqrt[2n+1]{n^2 + n} = 1$.

***15.** Let a_n be a given sequence such that the sequence $b_n = pa_n + qa_{n+1}$, where $|p| < q$, is convergent. Prove that a_n converges. If $|p| \geq q > 0$, show that a_n need not converge.

16. Prove the relation

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k = \frac{1}{k+1}$$

for any nonnegative integer k. (*Hint:* Use induction with respect to k and use the relation

$$\sum_{i=1}^{n} [i^{k+1} - (i-1)^{k+1}] = n^{k+1},$$

expanding $(i-1)^{k+1}$ in powers of i.)

Answers to Exercises

SECTION 1.6, page 60

1. (*a*) $\frac{3}{2}$, (*b*) 0, (*c*) 6, (*d*) 2, (*e*) a_p/b_p, (*f*) $\frac{1}{3}$.

2. (i) 50, 5000, 5000000.
(ii) 31, 3001, 3000001.
(iii) 4, 334, 333334.
(iv) 3, 32, 1000.
(v) 10, 19, 30.
(vi) 22, 66, 132.

3. (*a*) 0, (*b*) 0, (*c*) 0, (*d*) 1, (*e*) 2.

4. For $k \geq 0$,

$$(1+h)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} h^\nu > \binom{n}{k+1} h^{k+1}$$
$$\geq n(n-1)\cdots\frac{(n-k)h^{k+1}}{(k+1)!}$$
$$\geq \frac{(n-k)^{k+1}h^{k+1}}{(k+1)!}$$

Consequently, for $\frac{1}{2}n \geq k$

$$\frac{n^k}{(1+h)^n} \leq \frac{(k+1)!\, n^k}{(n-k)^{k+1}h^{k+1}} \leq \frac{2^{k+1}(k+1)!}{nh^{k+1}} = \frac{C}{n},$$

where C is constant.

Obvious for $k < 0$.

5. Set $1 + h = (\sqrt[k]{n})^{1/n}$. Employing the binomial theorem, we have

$$n = (1 + h)^{nk} \geq \binom{nk}{2} h^2.$$

Thus $h^2 \leq 2/k(nk - 1)$.

6. Multiply numerator and denominator by $\sqrt{n^k + 1} - \sqrt{n^k}$.

7 and **8.** In each case proceed as in Exercise 6.

Solutions and Hints to Problems

SECTION 1.6, page 60

1. $a_n = (\sqrt{n + 1} - \sqrt{n})\sqrt{n + \frac{1}{2}}$

$$= \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n + 1} + \sqrt{n}}$$

$$= \frac{\sqrt{1 + \dfrac{1}{2n}}}{\sqrt{1 + \dfrac{1}{n}} + 1}.$$

For any $\alpha > 0$ we have

$$1 < \sqrt{1 + \alpha} < \sqrt{1 + 2\alpha + \alpha^2} = 1 + \alpha,$$

whence

$$\frac{1}{2 + \dfrac{1}{n}} < a_n < \frac{1 + \dfrac{1}{2n}}{2}$$

and

$$-\frac{1}{4n} < -\frac{1}{n}\left(\frac{1}{4 + \dfrac{2}{n}}\right) < a_n - \frac{1}{2} < \frac{1}{4n}.$$

It follows that $\left|a_n - \frac{1}{2}\right| < \epsilon$ when $n > \dfrac{1}{4\epsilon}$.

2. $a_n = \sqrt[3]{n + 1} - \sqrt[3]{n}$

$$= \frac{1}{(n + 1)^{2/3} + n^{1/3}(n + 1)^{1/3} + n^{2/3}}$$

$$< \frac{1}{3n^{2/3}}$$

$$< \epsilon$$

when $n > (3\epsilon)^{-3/2}$.

3. (*a*) 0, (*b*) no, (*c*) for $n \geq 10$, (*d*) set $k = 10^{19}/19!$. For $n > 19$ we have $a_n \leq k/2^{n-19}$. (*e*) 30 ($29! = 8.8 \times 10^{30}$, $30! = 2.7 \times 10^{32}$). The estimate in (*d*) yields $n = 32$.

4. $a_n = \dfrac{n!}{n^n} = \dfrac{1}{n} \cdot \dfrac{2}{n} \cdot \dfrac{3}{n} \cdot \ldots \cdot \dfrac{n}{n}$. Set $m = \left[\dfrac{n}{2}\right]$. Then $a_n \leq \left(\dfrac{1}{2}\right)^m \leq \left(\dfrac{1}{2}\right)^{n/2}$.

5. (*a*)
$$a_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} = \frac{1 + 2 + \cdots + n}{n^2} = \frac{\frac{1}{2}n(n+1)}{n^2}.$$

(*b*) Observe that the largest term is $1/n^2$. Thus

$$\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2} < \frac{n+1}{n^2}.$$

(*c*) Observe that the smallest term is $1/\sqrt{2n}$. Thus

$$\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{2n}} \geq \frac{n+1}{\sqrt{2n}}.$$

(*d*) The largest and smallest terms are $1/\sqrt{n^2+1}$ and $1/\sqrt{n^2+n}$, respectively. Thus

$$\frac{n}{\sqrt{n^2+1}} \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq \frac{n}{\sqrt{n^2+n}}.$$

Now use the estimates for $\sqrt{1+\alpha}$ from the solution to Problem 1.

6. Consider the decimal as the sum of a geometric series.

7. Expanding $\left(1 + \dfrac{1}{100}\right)^n$ using the binomial theorem, we have on comparison with the 102-*nd* term

$$(1.01)^n > \frac{n(n-1)\cdots(n-100)}{101!} \cdot \frac{1}{100^{101}}.$$

Consequently, for $n \geq 101$

$$\frac{n^{100}}{1.01^n} < \frac{101!\,100^{101}}{n\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{100}{n}\right)} < \frac{k}{n},$$

where the constant k is

$$k = \frac{101!\,100^{101}}{\left(1 - \frac{1}{101}\right)\left(1 - \frac{2}{101}\right)\cdots\left(1 - \frac{100}{101}\right)} = 101^{101}\,100^{101}.$$

The limit is 0.

8. $a < \sqrt[n]{a^n + b^n} \le a\sqrt[n]{2}$.
Similarly, if a is the largest of the numbers $a_1, \ldots, a_k$, then

$$a \le \sqrt[n]{a_1{}^n + \cdots + a_k{}^n} \le a(\sqrt[n]{k})$$

and the limit is a.

9. The nth term of the sequence is $2^{1-1/2^n}$.

10. The smallest prime is 2. Thus if $2^m \le n < 2^{m+1}$, we have $\nu(n) \le m$ and

$$\frac{\nu(n)}{n} \le \frac{m}{2^m}.$$

11. Since a_n is a convergent sequence, it follows that $|a_n| < A$ for all n and some suitable upper bound A (see Section 1.7a). For every positive ϵ there exists an N such that $|a_n - \xi| < \epsilon$ when $n > N$. Consequently,

$$\begin{aligned} |\sigma_n - \xi| &= \frac{1}{n}\{|a_1 - \xi| + |a_2 - \xi| + \cdots + |a_n - \xi|\} \\ &\le \frac{1}{n}\{N(A + |\xi|) + |a_{N+1} - \xi| + \cdots + |a_n - \xi|\} \\ &\le \frac{N}{n}(A + |\xi|) + \frac{\epsilon(n - N)}{n} \\ &\le \frac{N}{n}(A + |\xi|) + \epsilon \\ &\le 2\epsilon, \end{aligned}$$

provided $n > N(A + |\xi|)/\epsilon$.

12. (*a*)
$$\begin{aligned} a_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \to 1. \end{aligned}$$

(*b*)
$$\begin{aligned} a_n &= \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} \\ &= \left[\underline{\frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right)} - \left(\frac{1}{2} - \frac{1}{3}\right)\right] + \left[\underline{\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)} - \left(\frac{1}{3} - \frac{1}{4}\right)\right] \\ &\quad + \cdots + \left[\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) - \left(\frac{1}{2} + \frac{1}{n+2}\right) \\ &= \frac{1}{4} - \frac{1}{2(n+1)} - \frac{3}{2(n+2)} \to \frac{1}{4}. \end{aligned}$$

13. $c_n = a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_p\sqrt{n+p}$

$$= \sqrt{n}\left\{a_0 + a_1\sqrt{1+\frac{1}{n}} + \cdots + a_p\sqrt{1+\frac{p}{n}}\right\}$$

$$= \sqrt{n}\left\{a_0 - a_0 + a_1\left(\sqrt{1+\frac{1}{n}} - 1\right) + \cdots + a_p\left(\sqrt{1+\frac{p}{n}} - 1\right)\right\}.$$

Consequently,

$$|c_n| \le \sqrt{n}\,A\left\{\left|\sqrt{1+\frac{1}{n}} - 1\right| + \cdots + \left|\sqrt{1+\frac{p}{n}} - 1\right|\right\},$$

where A is the largest of the numbers $|a_1|, |a_2|, \ldots, |a_p|$. From the estimate $\sqrt{1+\alpha} < 1 + \alpha$ for $\alpha > 0$ (see the solution to Problem 1), we have

$$|c_n| < \sqrt{n}\,A\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{p}{n}\right)$$
$$\le \frac{Ap(p+1)}{2\sqrt{n}}.$$

14. Set $a_n = \sqrt[2n+1]{n^2 + n} = 1 + h$. We then have

$$n^2 + n = (1+h)^{2n+1} > \binom{2n+1}{3}h^3.$$

The result follows from

$$\binom{2n+1}{3} = \frac{(2n+1)(2n)(2n-1)}{6}.$$

15. Set $b = \lim_{n\to\infty} b_n$. Putting $b_n = b + \epsilon_n$, $r = p/q$, we have

$$a_{n+1} = \frac{b+\epsilon_n}{q} - \frac{p}{q}a_n$$
$$= \frac{b+\epsilon_n}{q} - ra_n,$$
$$a_{n+2} = \frac{b+\epsilon_{n+1}}{q} - ra_{n+1}$$
$$= \frac{b+\epsilon_{n+1}}{q} - \frac{r(b+\epsilon_n)}{q} + r^2a_n$$
$$= \frac{b}{q}(1-r) + \frac{1}{9}(\epsilon_{n+1} - r\epsilon_n) + r^2a_n,$$

. . . .

$$a_{n+k} = \frac{b}{q}[1 - r + r^2 + \cdots + (-r)^{k-1}]$$
$$\frac{1}{9} + [\epsilon_{n+k-1} - r\epsilon_{n+k-2} + \cdots + (-r)^{k-1}\epsilon_n] + (-r)^k a_n.$$

Note that

$$\frac{b}{q}[1 - r + r + r^2 + \cdots + (-r)^{k-1}] = \frac{b}{q}\cdot\frac{1-(-r)^k}{1+r}$$
$$= \frac{b}{p+q}[1-(-r)^k].$$

Since we can ensure $|\epsilon_\nu| < \epsilon$ for $\nu > N(\epsilon)$, we obtain

$$\left|a_{n+k} - \frac{b}{p+q}\right| < \left\{\frac{|b|}{p+q} + |a_n|\right\}|r|^k + \epsilon\frac{1-|r|^k}{1-|r|}.$$

If $|p| \geq q$ then $a_n = (-p/q)^n$ diverges, but

$$b_n = p\left(-\frac{q}{p}\right)^n + q\left(-\frac{p}{q}\right)^{n+1}$$
$$= \left(-\frac{p}{q}\right)^n\left[p - q\left(\frac{p}{q}\right)\right] = 0.$$

16. For $k = 1$,

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^n i = \lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{1}{2}.$$

Suppose the relation is true for $j \leq k$. Then for all $j \leq k$,

$$\lim_{n\to\infty}\frac{1}{n^j}\sum_{i=1}^n i^{j-1} = \frac{1}{j}.$$

Now, using the binomial theorem,

$$i^{k+1} - (i-1)^{k+1} = i^{k+1} - i^{k+1} + (k+1)i^k - \binom{k+1}{2}i^{k-1}$$
$$+ \cdots + \binom{k+1}{j}(-1)^{j+1}i^{k+1-j} + \cdots + (-1)^k,$$

and

$$n^{k+1} = \sum_{i=1}^n [i^{k+1} - (i-1)^{k+1}]$$
$$= \sum_{i=1}^n\left[(k+1)i^k - \binom{k+1}{2}i^{k-1} + \cdots + \binom{k+1}{j}(-1)^{j+1}i^{k+1-j}\right.$$
$$\left. + \cdots + (-1)^k\right]$$
$$= (k+1)\sum_{i=1}^n i^k - \binom{k+1}{2}\sum_{i=1}^n i^{k-1} + \cdots + (-1)^{j+1}\binom{k+1}{j}\sum_{i=1}^n i^{k+1-j}$$
$$+ \cdots + n(-1)^k;$$

thus

$$1 = \frac{k+1}{n^{k+1}}\sum_{i=1}^n i^k - \frac{1}{n}\binom{k+1}{2}\frac{1}{n^k}\sum_{i=1}^n i^{k-1} + \cdots + \frac{(-1)^{j+1}}{n^{j+1}}\binom{k+1}{j}$$
$$\times\frac{1}{n^{k-j}}\sum_{i=1}^n i^{k+1-j} + \cdots + \frac{(-1)^k}{n^k},$$

and so

$$1 - \frac{k+1}{n^{k+1}}\sum_{i=1}^{n} i^k = -\frac{1}{n}\binom{k+1}{2}\frac{1}{n^k}\sum_{i=1}^{n} i^{k-1}$$

$$+ \cdots + \frac{(-1)^{j+1}}{n^{j+1}}\binom{k+1}{j}\frac{1}{n^{k-j}}\sum_{i=1}^{n} i^{k-j+1} + \cdots + \frac{(-1)^k}{n^k},$$

By the induction assumption

$$\lim_{n\to\infty}\frac{1}{n}\binom{k+1}{2}\frac{1}{n^k}\sum_{i=1}^{n} i^{k-1} = \binom{k+1}{2}\frac{1}{k}\lim_{n\to\infty}\frac{1}{n} = 0,$$

. . . .

$$\lim_{n\to\infty}\frac{1}{n^{j+1}}\binom{k+1}{j}\frac{1}{n^{k-j}}\sum_{i=1}^{n} i^{k-j+1} = \binom{k+1}{j}\frac{1}{k-j}\lim_{n\to\infty}\frac{1}{n^{j+1}} = 0,$$

. . . .

$$\lim_{n\to\infty}\frac{1}{n^k} = 0,$$

and so

$$\lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k = \frac{1}{k+1}, \quad \text{for all } k.$$

EXERCISES

SECTION 1.7, page 70

1. Which of the following sequences are bounded? Monotonic? Convergent?

(*a*) $a_n = (-1)^{n+1}$.

(*b*) $a_n = \dfrac{(-1)^{n+1}}{n}$.

(*c*) $a_n = 1 + \dfrac{(-1)^n}{n}$.

(*d*) $a_n = 1 - \dfrac{1}{n}$.

(*e*) $a_n = \dfrac{[1 + (-1)^n]}{2}$.

(*f*) $s_n = 1 + \frac{1}{4} + \frac{1}{27} + \cdots + \dfrac{1}{n^n}$.

(*g*) $s_n = 1 + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \frac{5}{9} + \cdots + \dfrac{n}{2n-1}$.

(*h*) $s_n = -1 + \frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \cdots + \frac{(-1)^n n}{2n-1}$.

*(*i*) $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$;

*(*j*) $s_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$.

2. Prove that each of the following sequences is monotonically increasing:

(*a*) $a_n = 2n^2 - 3n + 5$,
(*b*) $a_n = 3n^2 - 2n - 7$,

(*c*) $a_n = \frac{n}{n+1}$,

(*d*) $a_n = \frac{\sqrt{n^2 - 1}}{n}$,

(*e*) $a_n = n - \frac{1}{n}$.

3. Prove that each of the following sequences is monotonically decreasing:

(*a*) $a_n = 2^{1/n}$,

(*b*) $a_n = \sin\left(\frac{\pi}{2n}\right)$,

(*c*) $a_n = \sqrt{n+1} - \sqrt{n}$.

4. Evaluate the limits of the following expressions as $n \to \infty$:

(*a*) $\frac{n + \sqrt{n} - 1}{2n^2 - n^{1/n}}$,

(*b*) $\frac{2^{2/n} + n^p}{n^q - 1}$, $\quad q > p > 0$,

(*c*) $\frac{n^3 - 3n^2 + 4n - 1}{2n^3 - 3^{1/n} + n}$,

(*d*) $\frac{n \sin(1/n) + n^2}{2n - 1}$,

(*e*) $\frac{n^2 \cos(1/n) + n^{1/n}}{1 + n^2}$.

5. Using the convergence of bounded monotone sequences, verify that infinite decimal fractions are convergent representations for the real numbers.

6. Use Cauchy's convergence criterion to show that the following sequences converge:

(*a*) $a_n = \frac{1}{n}$,

(*b*) $a_n = \frac{n+1}{n}$,

(*c*) $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$,

(*d*) $a_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}$,

(*e*) $a_n = 1 + \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$, for any integer $p \geq 2$.

(*Hint:* Use mathematical induction on k to show that

$$a_{n+k} - a_n \leq \frac{1}{p-1}\left[\frac{1}{n^{p-1}} - \frac{1}{(n+k)^{p-1}}\right].\Bigg)$$

(*f*) $a_n = 1 + \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n}$.

7. Give the values of the following sums:

(*a*) $\sum_{k=3}^{7} k^2$,

(*b*) $\sum_{n=0}^{5} (-1)^n$,

(*c*) $\sum_{\nu=1}^{4} \frac{1}{\nu(\nu+1)}$,

(*d*) $\sum_{p=2}^{4} \frac{1}{p^2}$,

(*e*) $\sum_{k=0}^{2n} (n-k)$,

(*f*) $\sum_{k=0}^{2n} |n-k|$,

(*g*) $\sum_{k=0}^{3} k$.

8. Verify the following formulas:

(*a*) $\sum_{k=m}^{n} x = (n-m+1)x$,

(*b*) $\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$,

(*c*) $\sum_{k=m}^{n} xa_k = x\sum_{k=m}^{n} a_k$,

(*d*) $\sum_{k=m}^{n} a_k = \sum_{j=m+p}^{n+p} a_{j-p}$,

(*e*) $\left(\sum_{p=0}^{n} a_p x^p\right)\left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{p=0}^{n+m}\left(\sum_{q=0}^{p} a_q b_{p-q}\right)x^p$,

where we define $a_k = 0$ for $k < 0$, $k > n$, whereas $b_j = 0$ for $j < 0$, $j > m$.

9. (*a*) Prove that the sequence $\{a_n\}$ defined by

$$a_{n+1} = a_n + \frac{2 - a_n^2}{2a_n},$$

with a_0 any number greater than 0, converges to $\sqrt{2}$.

(*b*) More generally, show that the sequence $\{a_n\}$ defined as

$$a_{n+1} = a_n + \frac{k - a_n^2}{2a_n},$$

with $a_0 > 0$, converges to $\sqrt{k}$, for k any positive number.

10. Prove that for any rational number $x > 0$, $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$.

11. Prove that for $a_n = (1 + 1/n^k)^n$, we have $\lim_{n\to\infty} a_n = 1$ for $k > 1$, whereas $a_n \to \infty$ as $n \to \infty$ when $k < 1$.

PROBLEMS

SECTION 1.7, page 70

***1.** Let a_1 and b_1 be any two positive numbers, and let $a_1 < b_1$. Let a_2 and b_2 be defined by the equations

$$a_2 = \sqrt{a_1 b_1}, \qquad b_2 = \frac{a_1 + b_1}{2}.$$

Similarly, let

$$a_3 = \sqrt{a_2 b_2}, \qquad b_3 = \frac{a_2 + b_2}{2},$$

and, in general,

$$a_n = \sqrt{a_{n-1} b_{n-1}}, \qquad b_n = \frac{a_{n-1} + b_{n-1}}{2}.$$

Prove (*a*) that the sequence $a_1, a_2, \ldots,$ converges, (*b*) that the sequence $b_1, b_2, \ldots,$ converges, and (*c*) that the two sequences have the same limit. (This limit is called the *arithmetic-geometric mean* of a_1 and b_1.)

***2.** Prove that the limit of the sequence

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

(*a*) exists and (*b*) it is equal to 2.

***3.** Prove that the limit of the sequence

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$$

exists. Show that the limit is less than 1 but not less than $\frac{1}{2}$.

4. Prove that the limit of the sequence

$$b_n = \frac{1}{n+1} + \cdots + \frac{1}{2n}$$

exists, and is equal to the limit of the previous example.

5. Obtain the following bounds for the limit L in the two previous examples:

$$37/60 < L < 57/60.$$

***6.** Let a_1, b_1 be any two positive numbers, and let $a_1 \leq b_1$. Let

$$a_2 = \frac{2a_1b_1}{a_1 + b_1}, \qquad b_2 = \sqrt{a_1b_1},$$

and in general

$$a_n = \frac{2a_{n-1}b_{n-1}}{a_{n-1} + b_{n-1}}, \qquad b_n = \sqrt{a_{n-1}b_{n-1}}.$$

Prove that the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ converge and have the same limit.

***7.** Show that $\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} + \cdots$. (*Hint:* Consider the product of the nth partial sums of the expansions for e and $1/e$.)

8. (*a*) Without reference to the binomial theorem show that $a_n = (1 + 1/n)^n$ is monotone increasing and $b_n = (1 + 1/n)^{n+1}$ is monotonic decreasing. (*Hint:* Consider a_{n+1}/a_n and b_n/b_{n+1}. Use the result of Section 1.5, Problem 2.)

(*b*) Which is the larger number $(1{,}000{,}000)^{1{,}000{,}000}$ or $(1{,}000{,}001)^{999{,}999}$?

9. (*a*) From the results of Problem 8*a* show that

$$\left(\frac{n}{e}\right)^n < n! < e(n + 1)\left(\frac{n}{e}\right)^n.$$

(*b*) For $n > 6$ derive the sharper inequality

$$n! < n\left(\frac{n}{e}\right)^n.$$

***10.** If $a_n > 0$, and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, then $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

11. Use Problem 10 to evaluate the limits of the following sequences:

(*a*) $\sqrt[n]{n}$, (*b*) $\sqrt[n]{n^5 + n^4}$, (*c*) $\sqrt[n]{\frac{n!}{n^n}}$.

12. Use Problem 11*c* to show

$$n! = n^n e^{-n} a_n,$$

where a_n is a number whose nth root tends to 1. (See p. 504, formula (14).)

13. (*a*) Evaluate

$$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \cdots + \frac{1}{n(n + 2)}.$$

(*Hint:* Compare Section 1.6, Problem 12*a*.)

(*b*) From the result above, prove that $\sum_{k=1}^{\infty} \frac{1}{n^2}$ converges.

14. Let p and q be arbitrary natural numbers. Evaluate

(*a*) $\displaystyle\sum_{k=1}^{n} \frac{1}{(k+p)(k+p+q)}$.

(*b*) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(k+p)(k+p+q)}$.

15. Evaluate

(*a*) $\displaystyle\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)}$

(*b*) $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)(k+3)}$.

(*c*) Evaluate the limit of each of the above expressions as $n \to \infty$.

*(*d*) Let $a_1, a_2, \ldots, a_m$ be nonnegative integers with $a_1 < a_2 < \cdots < a_m$. Show how to obtain a formula for

$$S_n = \sum_{k=1}^{n} \frac{1}{(k+a_1)(k+a_2)\cdots(k+a_m)}$$

and how to find $\lim\limits_{n\to\infty} S_n$.

16. If a_k is monotone and $\sum_{k=1}^{\infty} a_k$ converges, show that $\lim\limits_{k\to\infty} ka_k = 0$.

17. If a_k is monotone decreasing with limit 0 and $b_k = a_k - 2a_{k+1} + a_{k+2} \geq 0$ for all k, then show $\sum_{k=1}^{\infty} kb_k = a_1$.

ANSWERS TO EXERCISES

SECTION 1.7, page 70

1. (*a*) Bounded, (*b*) convergent, (*c*) convergent, (*d*) monotonic, convergent, (*e*) bounded, (*f*) monotonic, convergent, (*g*) monotonic, (*h*) bounded, $-1 < s_n < -\frac{1}{3}$.

(*i*) Note that s_n where $n = 2^\nu$ satisfies

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &\qquad + \cdots + \left(\frac{1}{2^{\nu-1}+1} + \frac{1}{2^{\nu-1}+2} + \cdots + \frac{1}{2^\nu}\right) \\ &\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = \frac{\nu+1}{2}. \end{aligned}$$

Thus s_n is monotonic and unbounded.

(*j*) S_n is monotonic increasing, and, on comparison with

$$T_n = 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n-1)} = 2 - \frac{1}{n}$$

(cf. Exercises 1.6 No. 12*a*) is seen to be bounded.

2. (*a*) $a_{n+1} - a_n = 4n - 1 > 0, \quad \text{for} \quad n \geq 1.$

(*b*) $a_{n+1} - a_n = 6n + 1.$

(*c*) $a_{n+1} - a_n = \dfrac{1}{(n+1)(n+2)} > 0.$

(*d*) $a_n = \sqrt{1 - \dfrac{1}{n^2}} \leq \sqrt{1 - \dfrac{1}{(n+1)^2}} = a_{n+1}.$

(*e*) $a_{n+1} - a_n = 1 + \dfrac{1}{n(n+1)}.$

3. (*a*) $\dfrac{a_{n+1}}{a_n} = 2^{-1/n(n+1)} < 1.$

(*b*)
$$\begin{aligned} a_n - a_{n+1} &= \sin\frac{\pi}{2n} - \sin\frac{\pi}{2n+2} \\ &= 2\cos\frac{1}{2}\left(\frac{\pi}{2n} + \frac{\pi}{2n+2}\right)\sin\frac{1}{2}\left(\frac{\pi}{2n} - \frac{\pi}{2n+2}\right) \\ &\geq 0, \end{aligned}$$

since $\frac{1}{2}\pi\left(\frac{1}{2n} + \frac{1}{2n+2}\right) < \frac{1}{2}\pi\left(\frac{1}{2} + \frac{1}{2}\right) < \frac{\pi}{2}$ for $n \geq 1$.

(*c*) $a_n = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}.$

4. (*a*) 0, (*b*) 0, (*c*) $\frac{1}{2}$, (*d*) ∞, (*e*) 1.

5. The infinite decimal fraction

$$x = c_0 c_1 c_2 c_3 \cdots$$

is the limit of the sequence

$$c_0,\ c_0 + \frac{c_1}{10},\ c_0 + \frac{c_1}{10} + \frac{c_2}{10^2}, \cdots;$$

hence is increasing since $c_k \geq 0$ for $k > 0$. On the other hand, since $c_k \leq 9$ for $k > 0$,

$$\begin{aligned} &c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \cdots + \frac{c_n}{10^n} \\ &\leq c_0 + \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} \\ &< c_0 + 1. \end{aligned}$$

6. Suppose $n, m > N$.

(*a*) $|a_n - a_m| \le \frac{1}{n} + \frac{1}{m} < \frac{1}{2N} < \epsilon$ for $N > \frac{1}{2\epsilon}$.

(*b*) $a_n = 1 + \frac{1}{n}$. Now use part *a*.

(*c*)
$$a_{n+k} - a_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+k)!}$$
$$= \frac{1}{(n+1)!}\left[1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(n+2)\cdots(n+k)}\right]$$
$$\le \frac{1}{(n+1)!}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right)$$
$$\le \frac{2}{(n+1)!} < \frac{2}{(N+1)!}.$$

(*d*) $|a_{n+k} - a_n| < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+k)!}$.

Now use part *c*.

(*e*) For $k = 1$,
$$\frac{1}{p-1}\left[\frac{1}{n^{p-1}} - \frac{1}{(n+1)^{p-1}}\right]$$
$$= \frac{1}{p-1}\left[\frac{(n+1)^{p-1} - n^{p-1}}{n^{p-1}(n+1)^{p-1}}\right]$$
$$\ge \frac{1}{p-1}\left[\frac{\{n^{p-1} + (p-1)n^{p-2}\} - n^{p-1}}{n^{p-1}(n+1)^{p-1}}\right]$$
$$\ge \frac{1}{n(n+1)^{p-1}} \ge \frac{1}{(n+1)^p} = a_{n+1} - a_n.$$

If the inequality holds for some value of k, then
$$a_{n+k+1} - a_n \le \frac{1}{p-1}\left[\frac{1}{n^{p-1}} - \frac{1}{(n+k)^{p-1}}\right] + \frac{1}{(n+k+1)^p}$$
$$\le \frac{1}{p-1}\left[\frac{1}{n^{p-1}} - \frac{(n+k+1)^p - (p-1)(n+k)^{p-1}}{(n+k)^{p-1}(n+k+1)^p}\right]$$
$$\le \frac{1}{p-1}\left[\frac{1}{n^{p-1}} - \frac{(n+k)^p + (n+k)^{p-1}}{(n+k)^{p-1}(n+k+1)^p}\right]$$
$$\le \frac{1}{p-1}\left[\frac{1}{n^{p-1}} - \frac{1}{(n+k+1)^{p-1}}\right].$$

In general then,

$$a_{n+k} - a_n \le \frac{1}{(p-1)n^{p-1}}.$$

(*f*) $$a_{n+k} - a_n = \frac{1}{(n+1)^{n+1}} + \frac{1}{(n+2)^{n+2}} + \cdots + \frac{1}{(n+k)^{n+k}}$$

$$\le \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^{n+k}}$$

$$\le \frac{1}{2^n}.$$

7. (*a*) 135, (*b*) 0, (*c*) $\frac{4}{5}$, (*d*) $\frac{61}{144}$, (*e*) 0, (*f*) $n(n+1)$, (*g*) 6.

9. (*a*) Set $e_n = a_n - \sqrt{2}$. Then

$$e_{n+1} = a_{n+1} - \sqrt{2} = e_n\left(1 - \frac{a_n + \sqrt{2}}{2a_n}\right) = \frac{e_n^{\,2}}{2a_n}.$$

We may assume the error e_n is nonnegative, for if $e_n \ge 0$, then $e_{n+1} \ge 0$, and if $a_0 > 0$, then e_1 and all succeeding terms are positive. Since $a_n > e_n$, we have

$$e_{n+1} < \frac{e_n}{2} < \frac{e_{n-1}}{4} < \cdots < \frac{e_1}{2^n} = \frac{e_0^{\,2}}{2^{n+1}a_0}.$$

Thus the sequence a_n converges to $\sqrt{2}$ for any choice of $a_0 > 0$.

(*b*) Proceed as in part *a*.

10. Let $x = p/q$ for p,q integers. Clearly, $e = \lim_{n\to\infty} (1 + 1/n)^n$, if we stipulate that n ranges over integer multiples mq of q, $m = 1, 2, \ldots$, or

$$e = \lim_{m\to\infty}\left(1 + \frac{1}{mq}\right)^{mq}.$$

Now

$$e^x = e^{p/q} = \lim_{m\to\infty}\left(1 + \frac{1}{mq}\right)^{mpq/q}$$

$$= \lim_{m\to\infty}\left(1 + \frac{1}{mq}\right)^{mp}$$

Let $r = mp$. Then as $m \to \infty$, we have $r \to \infty$, and $m = r/p$. Thus

$$e^x = \lim_{r\to\infty}\left(1 + \frac{p}{rq}\right)^r = \lim_{r\to\infty}\left(1 + \frac{x}{r}\right)^r.$$

11. For $k < 1$,

$$\left(1 + \frac{1}{n^k}\right)^n \ge 1 + \frac{n}{n^k} \to \infty.$$

For $k > 1$,

$$
\begin{aligned}
1 &< \left(1 + \frac{1}{n^k}\right)^n \\
&\le 1 + \frac{n}{n^k} + \frac{1}{2!}\frac{n}{n^k}\frac{n-1}{n^k} + \cdots \\
&\quad + \frac{1}{\nu!}\frac{n}{n^k}\frac{n-1}{n^k}\cdots\frac{n-\nu+1}{n^k} + \cdots \\
&\quad + \frac{1}{n!}\frac{n}{n^k}\frac{n-1}{n^k}\cdots\frac{1}{n^k} \\
&\le 1 + \frac{n}{n^k}\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) \\
&\le 1 + \frac{n}{n^k}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right) \\
&\le 1 + 2\frac{n}{n^k}.
\end{aligned}
$$

Since $n/n^k \to 0$, the result follows.

Solutions and Hints to Problems

SECTION 1.7, page 70

1. From the inequality between the geometric and arithmetic means we have for all n, $a_n \le b_n$. Consequently,

$$a_{n+1} = \sqrt{a_n b_n} \ge a_n, \qquad b_{n+1} = \frac{a_n + b_n}{2} \le b_n,$$

and the sequences a_n and b_n are bounded increasing and bounded decreasing, respectively. Furthermore,

$$
\begin{aligned}
b_{n+1} - a_{n+1} &= \frac{a_n + b_n}{2} - \sqrt{a_n b_n} \\
&= \frac{b_n - a_n}{2} - \sqrt{a_n}\sqrt{b_n} + a_n \\
&= \frac{b_n - a_n}{2} - \sqrt{a_n}(\sqrt{b_n} - \sqrt{a_n}) \\
&\le \frac{b_n - a_n}{2}.
\end{aligned}
$$

It follows that

$$b_{n+1} - a_{n+1} \le \frac{b_1 - a_1}{2^n}$$

and hence that a_n and b_n have the same limit.

2. We have $a_{n+1} = \sqrt{2 + a_n}$. Furthermore, if $a_n < 2$ then $a_{n+1} < 2$ and the sequence is bounded; hence the sequence is monotone increasing. Finally, if $\epsilon_n = 2 - a_n$, then

$$a_{n+1} = \sqrt{4 - \epsilon_n}$$

where $\epsilon_n \leq 2 - a_1 < 2 - \sqrt{2} < 1$. It follows that

$$\begin{aligned}\epsilon_{n+1} &= 2\left(1 - \sqrt{1 - \frac{\epsilon_n}{4}}\right)\\ &= \frac{\epsilon_n}{2(1 + \sqrt{1 - \epsilon_n/4})} \leq \frac{\epsilon_n}{2}.\end{aligned}$$

Consequently, $\epsilon_{n+1} < \epsilon_1/2^n$.

3. $$\begin{aligned}a_{n+1} - a_n &= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n}\\ &< \frac{1}{2n} + \frac{1}{2n} - \frac{1}{n} = 0.\end{aligned}$$

It follows that the sequence is monotone decreasing. Furthermore, $a_n \leq (n + 1)/n = 1 + 1/n$, and

$$\begin{aligned}a_n &= \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}\\ &\geq \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{n+1}{2n} > \frac{1}{2}.\end{aligned}$$

4. $b_n = a_n - 1/n$; consequently, b_n has the same limit as a_n. Furthermore,

$$\begin{aligned}b_{n+1} - b_n &= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}\\ &> \frac{1}{2(n+1)} + \frac{1}{2(n+1)} - \frac{1}{n+1} = 0\end{aligned}$$

so that b_n is monotone increasing.

5. We have

$$b_n < L < a_n.$$

Take $n = 3$.

6. From the inequality between the arithmetic and geometric means, we have

$$\frac{2a_nb_n}{a_n + b_n} = \left(\frac{2\sqrt{a_nb_n}}{a_n + b_n}\right)\sqrt{a_nb_n} \leq \sqrt{a_nb_n},$$

or

$$a_{n+1} \leq b_{n+1}.$$

Furthermore,

$$a_{n+1} = \frac{2}{a_n^{-1} + b_n^{-1}} \geq \frac{2}{2a_n^{-1}} = a_n$$

and similarly, $b_{n+1} \leq b_n$. Thus a_n is bounded increasing and b_n is bounded decreasing. Finally,

$$\begin{aligned} b_{n+1} - a_{n+1} &= \sqrt{a_n b_n} - \frac{2a_n b_n}{a_n + b_n} \\ &= \frac{2\sqrt{a_n b_n}}{a_n + b_n}\left(\frac{a_n + b_n}{2} - \sqrt{a_n b_n}\right) \\ &\leq \frac{2\sqrt{a_n b_n}}{a_n + b_n} \cdot \frac{b_n - a_n}{2} \\ &\leq \frac{b_n - a_n}{2}, \end{aligned}$$

where in the last two steps we have used successively the estimate of Problem 1 for the difference between the arithmetic and geometric means and the inequality between the arithmetic and geometric means.

7. Set $\alpha_n = \sum_{\nu=0}^{n} \frac{1}{\nu!}$, $\beta_n = \sum_{\nu=0}^{n} \frac{(-1)^\nu}{\nu!}$. The sequences α_n and β_n both converge by the Cauchy condition:

$$\begin{aligned} |\alpha_{n+k} - \alpha_n|, |\beta_{n+k} - \beta_n| &\leq \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+k)!} \\ &\leq \frac{1}{(n+1)!}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}}\right) \\ &< \frac{2}{(n+1)!}. \end{aligned}$$

$$\begin{aligned} \text{We have } \alpha_n\beta_n &= \sum_{\mu+\nu \leq n}\sum \frac{(-1)^\nu}{\nu!\,\mu!} + \sum_{\substack{\mu+\nu > n \\ \mu,\nu \leq n}}\sum \frac{(-1)^\nu}{\nu!\,\mu!} \\ &= \sum_{k=0}^{n} \sum_{\mu=0}^{k} \frac{(-1)^{k-\mu}}{(k-\mu)!\,\mu!} + \sum_{k=n+1}^{2n} \sum_{\mu=n+1}^{k} \frac{(-1)^{k-\mu}}{(k-\mu)!\,\mu!} \\ &= \sum_{k=0}^{n} \sum_{\mu=0}^{k} \frac{(-1)^{k-\mu}}{(k-\mu)!\,\mu!} + E, \end{aligned}$$

where we have set $\nu = k - \mu$.

For $k > 0$,

$$\begin{aligned} \sum_{\mu=0}^{k} \frac{(-1)^{k-\mu}}{(k-\mu)!\,\mu!} &= \frac{1}{k!}\sum_{\mu=0}^{k}\binom{k}{\mu}(-1)^{k-\mu} \\ &= \frac{(1-1)^k}{k!} = 0. \end{aligned}$$

Thus $\alpha_n\beta_n = 1 + E$ and we shall show that E tends to zero with n:

$$|E| \leq \sum_{k=n+1}^{2n} \sum_{\mu=n+1}^{k} \frac{1}{(k-\mu)!} \cdot \frac{1}{\mu!}.$$

From the inequality $m! \geq 2^{m-1}$ we have

$$\begin{aligned} |E| &\leq \sum_{k=n+1}^{2n} \sum_{\mu=n+1}^{k} \frac{1}{2^{k-2}} \\ &\leq \sum_{k=n+1}^{2n} \frac{k-n}{2^{k-2}} \leq \frac{1+2+\cdots+n}{2n-1} \\ &\leq \frac{n(n+1)}{2n}. \end{aligned}$$

It follows that $\lim_{n\to\infty} \alpha_n\beta_n = e \lim_{n\to\infty} \beta_n = 1$.

8. (*a*)
$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \left(\frac{n+2}{n+1}\right)^{n+1} \Big/ \left(\frac{n+1}{n}\right)^n \\ &= \left[1 - \frac{1}{(n+1)^2}\right]^n \left(\frac{n+2}{n+1}\right) \\ &\geq \left[1 - \frac{n}{(n+1)^2}\right]\left(\frac{n+2}{n+1}\right) \geq \frac{(n+1)^3+1}{(n+1)^3} > 1. \end{aligned}$$

(From the inequality $(1+h)^n \geq 1 + nh$ for $h > -1$, which is easily proved by induction. See Problems 1.5, No. 2.)

Similarly,

$$\begin{aligned} \frac{b_n}{b_{n+1}} &= \left[\frac{(n+1)^2}{n(n+2)}\right]^{n+1}\left(\frac{n+1}{n+2}\right) \\ &= \left[1 + \frac{1}{n(n+2)}\right]^{n+1}\left(\frac{n+1}{n+2}\right) \\ &\geq \left[1 + \frac{n+1}{n(n+2)}\right]\left(\frac{n+1}{n+2}\right) \\ &\geq \frac{n^3+4n^2+4n+1}{n^3+4n^2+4n} > 1. \end{aligned}$$

(*b*) $$\frac{(n+1)^{n-1}}{n^n} = \left(1 + \frac{1}{n}\right)^n \frac{1}{n+1} < \frac{e}{n+1} < 1, \qquad \text{for} \quad n \geq 2.$$

9. (*a*) On observing that a_n of Problem 8*a* increases monotonically to the limit e, we have

$$\begin{aligned} e^{n-1} &> a_1 a_2 \cdots a_{n-1} \\ &= \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \cdots \left(\frac{n}{n-1}\right)^{n-1} \\ &= \frac{n^{n-1}}{(n-1)!} = \frac{n^n}{n!}, \end{aligned}$$

whence $n! > n^n/e^{n-1} = e(n/e)^n$.

Similarly, since b_n is monotonically decreasing,

$$\begin{aligned} e^{n-1} &< b_1 b_2 \cdots b_{n-1} \\ &= \left(\frac{2}{1}\right)^2 \left(\frac{3}{2}\right)^2 \cdots \left(\frac{n}{n-1}\right)^n = \frac{n^n}{(n-1)!}, \end{aligned}$$

whence

$$n! < en\left(\frac{n}{e}\right)^n.$$

(*b*) If the result is true for $n = k$, then

$$k!\, e < k\left(\frac{k}{e}\right)^k e < k\left(\frac{k}{e}\right)^k \left(\frac{k+1}{k}\right)^{k+1},$$

whence

$$(k+1)! < (k+1)\left(\frac{k+1}{e}\right)^{k+1}.$$

To complete the proof we must find the least n for which

$$\frac{(n-1)!\, e^n}{n^n} < 1.$$

With allowance for round-off error, we have

$$\begin{aligned} \frac{5!\, e^6}{6^6} &> \frac{5!\,(403.3)}{6^6} = \frac{5 \times (403.3)}{9 \times 6^3} \\ &> \frac{2016}{1944} > 1 \end{aligned}$$

and

$$\begin{aligned} \frac{6!\, e^7}{7^7} &< \frac{6! \times 1098}{7 \times (49)^3} \leq \frac{720 \times 1098}{7 \times (50-1)^3} \\ &< \frac{792 \times 10^3}{7 \times 117 \times 10^3} < \frac{792}{819} < 1. \end{aligned}$$

Consequently, the inequality holds for $n > 6$.

10. If $L \neq 0$, choose $\epsilon > 0$, so that $\epsilon < L$. Then for $n \geq N(\epsilon)$

$$\left|\frac{a_{n+1}}{a_n} - L\right| < \epsilon,$$

whence

$$(L - \epsilon)a_n < a_{n+1} < (L + \epsilon)a_n$$

and consequently,

$$(L - \epsilon)^{n-N} a_N < a_n < (L + \epsilon)^{n-N} a_N.$$

It follows that

$$(L - \epsilon)\left[\sqrt[n]{\frac{a_N}{(L-\epsilon)^N}}\right] < \sqrt[n]{a_n} < (L + \epsilon)\left[\sqrt[n]{\frac{a_N}{(L+\epsilon)^N}}\right];$$

since the quantities in the radicals on the left and on the right are constants, the radicals both approach 1 and the result follows.

If $L = 0$, the preceding inequality is replaced by

$$0 < \sqrt[n]{a_n} < (\epsilon^{1-n/N})\sqrt[n]{a_N}$$

11. (*c*) $$\frac{(n+1)!}{(n+1)^{n+1}} \Big/ \frac{n!}{n^n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \to \frac{1}{e}.$$

13. (*a*) $$S_n = \sum_1^n \frac{1}{k(k+2)} = \frac{1}{2}\sum_1^n \left(\frac{1}{k} - \frac{1}{k+2}\right)$$
$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right).$$

(*b*) From the preceding result

$$\lim_{n\to\infty} S_n = \tfrac{3}{4}.$$

The sequence $T_n = \sum_1^n \frac{1}{k^2}$ is monotone increasing. Furthermore, for $k \geq 1$ we have

$$k^2 \geq \frac{k(k+2)}{3},$$

whence

$$T_n \leq 3S_n \leq \tfrac{9}{4},$$

(since S_n is monotonically increasing). Since T_n is increasing and bounded, it follows that T_n converges.

14. (*a*) Set $S_n = \sum_{k=1}^n \frac{1}{(k+p)(k+p+q)}$.

From $$\frac{1}{k+p} = \frac{1}{q}\left(\frac{1}{k+p} - \frac{1}{k+p+q}\right)$$ we have

$$S_n = \frac{1}{q}\left(\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{p+q} - \frac{1}{n+p+1} - \frac{1}{n+p+2} - \cdots - \frac{1}{n+p+q}\right)$$

(*b*) $$\lim_{n\to\infty} S_n = \frac{1}{q}\left(\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{p+q}\right)$$

15. (*a*) Same as Problems 1.6, No. 12*b*.

(*b*) $$\frac{1}{k(k+1)(k+3)} = \frac{1}{k(k+3)} - \frac{1}{(k+1)(k+3)}.$$

Consequently, by Problem 14*a*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+3)} = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$
$$- \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right).$$

(*c*) Limit in Problem 15*a* is $\frac{1}{4}$; in 15*b*, $\frac{7}{36}$.

(*d*) Reduce to sums of the form in 14*a* by a sequence of steps for which the first is

$$\frac{1}{(k+a_1)(k+a_2)\cdots(k+a_m)} = \frac{1}{a_2 - a_1}\left(\frac{1}{k+a_1} - \frac{1}{k+a_2}\right) \times \frac{1}{(k+a_3)\cdots(k+a_m)}$$
$$= \frac{1}{a_2 - a_1}\left[\frac{1}{(k+a_1)(k+a_3)} - \frac{1}{(k+a_2)(k+a_3)}\right]\frac{1}{(k+a_4)\cdots(k+a_m)}.$$

16. Since a_n is monotone, we see that the terms have a constant sign for sufficiently large n. Suppose, for example, that $0 \le a_{n+1} \le a_n$ for sufficiently large n. Then by the Cauchy criterion,

for sufficiently large m,

$$a_{m+1} + a_{m+2} + \cdots + a_{m+k} < \epsilon,$$

independently of k. From

$$a_{m+1} + a_{m+2} + \cdots + a_{m+k} \ge k a_{m+k}$$

we have

$$k a_{m+k} < \epsilon.$$

Similarly, for sufficiently large k

$$m a_{m+k} < \epsilon.$$

Adding, we observe that $(m + k)a_{m+k}$ can be made arbitrarily small by taking m and k large enough.

17. By induction

$$\sum_{n=1}^{k-1} n b_n = a_1 - k a_k + (k-1)a_{k+1}$$
$$= a_1 - a_{k+1} - k(a_k - a_{k+1}).$$

Now we use Problem 16. Alternatively, since $b_k \ge 0$, we have

$$a_k - a_{k+1} \ge a_{k+1} - a_{k+2}.$$

It follows that

$$a_k - a_{k+m} = \sum_{j=1}^{m} (a_{k-1+j} - a_{k+j})$$
$$\ge m(a_{k+m-1} - a_{k+m}).$$

Entering this result in

$$\sum_{n=1}^{k+m-2} nb_n = a_1 - a_{k+m} - (k + m - 1)(a_{k+m-1} - a_{k+m}),$$

we obtain

$$\left|\sum_{n=1}^{k+m-2} nb_n - a_1\right| \le a_{k+m} + \frac{k + m - 1}{m}(a_k - a_{k+m})$$

$$\le a_k + 2\frac{k + m - 1}{m} a_k$$

$$\le 2a_k\left(\frac{k + 3m - 1}{m}\right).$$

For any ν choose k and m so that $m + k - 2 = \nu$ and $\frac{\nu}{2} + 1 \le m \le \frac{\nu}{2} + 2$. Then $\frac{\nu}{2} < k \le \frac{\nu}{2} + 1$. It follows that

$$\left|\sum_{n=1}^{k+m-2} nb_n - a_1\right| \le 8a_k,$$

from which the conclusion follows.

EXERCISES

SECTION 1.8, page 82

1. Find the following limits, giving at each step the theorem on limits which justifies it.

(*a*) $\lim_{x\to 2} 3x$, (*b*) $\lim_{x\to 3} 4x + 3$,

(*c*) $\lim_{x\to 1} \frac{x^2 + 2x - 1}{2x + 2}$, (*d*) $\lim_{x\to 2} \sqrt{5 + \sqrt[3]{2x^5}}$,

(*e*) $\lim_{x\to 0} \cos(\sin x)$, (*f*) $\lim_{x\to 0} \cos\left(\pi\frac{\sin x}{x}\right)$.

2. Prove that

(*a*) $\lim_{x\to 1} \frac{x^n - 1}{x - 1} = n$,

(*b*) $\lim_{x\to \pi} \frac{\sin x}{\pi - x} = 1$,

(*c*) $\lim_{x\to 0} \frac{\sin(x^2)}{x} = 0$.

3. Determine whether or not the following limits exist, and if they do exist find their values:

(*a*) $\lim\limits_{x\to 0} \dfrac{\sqrt{1-x}}{x}$,

(*b*) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}}{x}$,

(*c*) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

4. Prove that

$$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0 .$$

5. Prove that

(*a*) $\lim\limits_{x\to \alpha} \dfrac{\sin (x-\alpha)}{x^2-\alpha^2} = \dfrac{1}{2\alpha}$,

(*b*) $\lim\limits_{x\to \infty} \dfrac{x + \cos x}{x+1} = 1$,

(*c*) $\lim\limits_{x\to \infty} \cos \dfrac{1}{x} = 1$.

6. (*a*) Let $f(x)$ be defined by the equation $y = 6x$. Find a δ, depending on ξ, so small that $|f(x) - f(\xi)| < \epsilon$ whenever $|x - \xi| < \delta$, where (i) $\epsilon = \frac{1}{10}$; (ii) $\epsilon = \frac{1}{100}$; (iii) $\epsilon = \frac{1}{1000}$. Do the same for

(*b*) $f(x) = x^2 - 2x$,
(*c*) $f(x) = 3x^4 + x^2 - 7$,
(*d*) $f(x) = \sqrt{x}$, $x \geq 0$,
(*e*) $f(x) = \sqrt{x^2}$.

7. (*a*) Let $f(x) = 6x$ in the interval $0 \leq x \leq 10$. Find a δ so small that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$, where (1) $\epsilon = \frac{1}{100}$; (2) ϵ is arbitrary, > 0.
Do the same for

(*b*) $f(x) = x^2 - 2x$, $-1 \leq x \leq 1$,
(*c*) $f(x) = 3x^4 + x^2 - 7$, $2 \leq x \leq 4$,
(*d*) $f(x) = \sqrt{x}$, $0 \leq x \leq 4$,
(*e*) $f(x) = \sqrt{x^2}$, $-2 \leq x \leq 2$.

8. Determine which of the following functions are continuous. For those which are discontinuous, find the points of discontinuity.

(*a*) $x^2 \sin x$,
(*b*) $x \sin^2 (x^2)$,
(*c*) $\dfrac{1}{x} \sin x$,
(*d*) $\dfrac{\sin x}{\sqrt{x}}$,

(*e*) $\dfrac{x^3 + 3x + 7}{x^2 - 6x + 8}$, (*f*) $\dfrac{x^3 + 3x + 7}{x^2 - 6x + 9}$,

(*g*) $\dfrac{x^3 + 3x + 7}{x^2 - 6x + 10}$, (*h*) $\tan x$,

(*i*) $\dfrac{1}{\sin x}$, (*j*) $\cot x$,

(*k*) $\dfrac{1}{\cos x}$, (*l*) $x \cot x$,

(*m*) $(\pi - x)\tan x$. (*n*) $\operatorname{sgn} x$.

9. Prove that $\lim_{x\to 0} \dfrac{x+2}{x+1} = 2$. Find a δ such that for $|x| < \delta$ the difference between 2 and $\dfrac{x+2}{x+1}$ is, in absolute value, (*a*) less than $\frac{1}{10}$, (*b*) less than $\frac{1}{1000}$, (*c*) less than ϵ, $\epsilon > 0$.

10. (*a*) Prove that $\lim_{x\to 1} \dfrac{x+2}{x+1} = \dfrac{3}{2}$. Find a δ such that for $|1 - x| < \delta$ the difference between $\dfrac{3}{2}$ and $\dfrac{x+2}{x+1}$ is, in absolute value, less than ϵ, $\epsilon > 0$.

Do the same for

(*b*) $\lim_{x\to 2} \sqrt{1 + x^3}$,

(*c*) $\lim_{x\to 0} \dfrac{\sin x}{x}$.

11. Prove that

(*a*) $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1}{x} = \dfrac{1}{2}$,

(*b*) $\lim_{x\to\infty} \sqrt{x + \frac{1}{2}}\ \ (\sqrt{x+1} - \sqrt{x}) = \frac{1}{2}$.

12. Prove that the elementary trigonometric functions $\sin x$, $\cos x$, are continuous.

PROBLEMS

SECTION 1.8, page 82

1. (*a*) Prove that $\lim_{m\to\infty} (\cos \pi x)^{2m}$ exists for each value of x and is equal to 1 or 0 according to whether x is an integer or not.

(*b*) Prove that $\lim_{n\to\infty} [\lim_{m\to\infty} (\cos n!\, \pi x)^{2m}]$ exists for each value of x and is equal to 1 or 0 according to whether x is rational or irrational.

(*c*) Discuss the continuity of these limit functions.

2. Let $f(x)$ be continuous for $0 \leq x \leq 1$. Suppose further that $f(x)$ assumes rational values only, and that $f(x) = \frac{1}{2}$ when $x = \frac{1}{2}$. Prove that $f(x) = \frac{1}{2}$ everywhere.

Answers to Exercises

SECTION 1.8, page 82

1. (*d*)

$$\lim_{x\to 2} \sqrt{5 + \sqrt[3]{2x^5}} = \sqrt{\lim_{x\to 2} (5 + \sqrt[3]{2x^5})} \qquad \text{(limit of continuous function)}$$

$$= \sqrt{5 + \lim_{x\to 2} \sqrt[3]{2x^5}} \qquad \text{(limit of a sum)}$$

$$= \sqrt{5 + \sqrt[3]{\lim_{x\to 2} 2x^5}} \qquad \text{(limit of continuous function)}$$

$$= \sqrt{5 + \sqrt[3]{2^6}} \qquad \text{(limit of product)}$$

$$= 3.$$

2. (*a*) $$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + 1.$$

(*b*) $$\lim_{x\to\pi} \frac{\sin x}{\pi - x} = \lim_{z\to 0} \frac{\sin (\pi - z)}{z} = \lim_{z\to 0} \frac{\sin z}{z} = 1.$$

(*c*) $$\lim_{x\to 0} \frac{\sin (x^2)}{x} = \left(\lim_{x\to 0} x\right) \lim_{x\to 0} \frac{\sin (x^2)}{x^2}.$$

3. Limits (*a*) and (*b*) do not exist; limit (*c*) exists and is equal to 1.

4. $$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} \lim_{x\to 0} x \sin \frac{1}{x}$$

Now, $\lim_{x\to 0} \frac{x}{\sin x} = 1$, and since for $|x| < \epsilon$,

$$\left| x \sin \frac{1}{x} \right| = |x| \cdot \left| \sin \frac{1}{x} \right| \leq |x| < \epsilon,$$

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0.$$

5. (*a*) $\lim_{x\to\alpha} \dfrac{\sin(x-\alpha)}{x^2-\alpha^2} = \lim_{x\to\alpha} \dfrac{\sin(x-\alpha)}{x-\alpha} \lim_{x\to\alpha} \dfrac{1}{x+\alpha}$.

(*b*) $\lim_{x\to\infty} \dfrac{x+\cos x}{x+1} = \lim_{x\to\infty} \dfrac{1+(\cos x)/x}{1+1/x}$

(*c*) $\lim_{x\to\infty} \cos\dfrac{1}{x} = \cos\left(\lim_{x\to\infty}\dfrac{1}{x}\right)$.

6. (*a*) $\dfrac{1}{60}, \dfrac{1}{600}, \dfrac{1}{6000}$.

(*b*) $\dfrac{1}{10}(1+2\,|\xi-1|)^{-1}$, etc.

(*c*) $\dfrac{1}{120}(1+|\xi|)^{-3}$, etc.

(*d*) $\dfrac{1}{100}, \dfrac{1}{10000}, \dfrac{1}{1000000}$. (*e*) $\dfrac{1}{10}, \dfrac{1}{100}, \dfrac{1}{1000}$.

7. (*a*) $\dfrac{1}{600}, \dfrac{\epsilon}{6}$. (*b*) $\dfrac{1}{400}, \dfrac{\epsilon}{4}$. (*c*) $\dfrac{1}{77600}, \dfrac{\epsilon}{776}$. (*d*) $\dfrac{1}{10000}, \epsilon^2$. (*e*) $\dfrac{1}{100}, \epsilon$.

8. (*a*), (*b*), (*c*), (*d*), (*g*) continuous, or have only removable discontinuities.
(*e*) Discontinuous at $x = 2, 4$.
(*f*) Discontinuous at $x = 3$.
(*h*), (*k*), (*m*) Discontinuous at $x = (n + \frac{1}{2})\pi$.
(*i*), (*j*) Discontinuous at $x = n\pi$.
(*l*) Discontinuous at $x = n\pi$. Removable discontinuity at $x = 0$.
(*n*) Discontinuous at $x = 0$.

9. (*a*) $\dfrac{1}{11}$, (*b*) $\dfrac{1}{1001}$, (*c*) $\dfrac{\epsilon}{1+\epsilon}$.

10. (*a*) $\dfrac{4\epsilon}{1+2\epsilon}$, (*b*) $\dfrac{\epsilon}{7}$, (*c*) $\arccos(1-\epsilon)$.

11. (*a*) $\dfrac{\sqrt{1+x}-1}{x} = \dfrac{1}{\sqrt{1+x}+1}$.

(*b*) $\sqrt{x+\frac{1}{2}}\,(\sqrt{x+1}-\sqrt{x}) = \dfrac{\sqrt{x+\frac{1}{2}}}{\sqrt{x+1}+\sqrt{x}} = \dfrac{\sqrt{1+1/2x}}{\sqrt{1+1/x}+1}$.

12. $|\sin(x+\alpha) - \sin x|$

$$= |\sin x(\cos\alpha - 1) + \cos x \sin\alpha|$$
$$\le |\cos\alpha - 1| + |\sin\alpha|$$
$$\le \left|2\sin^2\frac{\alpha}{2}\right| + |\sin\alpha|.$$

From the inequality,

$$\frac{\alpha}{\sin \alpha} = \left| \frac{\alpha}{\sin \alpha} \right| > 1,$$

it follows that

$$|\sin (x + \alpha) - \sin x| \le \frac{\alpha^2}{2} + |\alpha|.$$

Solutions and Hints to Problems

SECTION 1.8, page 82

1. (*a*) If x is an integer, then $y = (\cos \pi x)^2 = 1$ and $y^m \to 1$. If x is not an integer, then $0 \le y < 1$ and $y^m \to 0$.

(*b*) If x is rational, $x = p/q$ for integers p, q, then for $n \ge q$ we have $z_n = (\cos n!\, \pi x)^2 = 1$ and $\lim_{n\to\infty} w_n = 1$, where $w_n = \lim_{m\to\infty} z_n{}^m$. If x is irrational, then $n!\, x$ is irrational and by (*a*) for fixed n, $w_n = \lim_{m\to\infty} z_n{}^m = 0$, hence $\lim_{n\to\infty} w_n = 0$.

(*c*) The function of 1*a* is continuous except for integral values of x. The function of 1*b* is nowhere continuous.

2. If $f(x)$ assumed any value $y \ne \frac{1}{2}$, then $f(x)$ would take on all values between $\frac{1}{2}$ and y and would therefore take on irrational values. (See Problems 1.1*a*, No. 1*b*.)

PROBLEMS

SECTION 1.S.1, page 89

1. Let $r = p/q$, $s = m/n$ be arbitrary rational numbers where p, q, m, n are integers and q, n are positive. In terms of the integers p, q, m, n, define

$$(a)\ r + s, \quad (b)\ r - s, \quad (c)\ rs, \quad (d)\ \frac{r}{s}, \quad (e)\ r < s.$$

2. Prove for nested sequences of rational numbers $[a_n, b_n]$ and $[a_n', b_n']$ that each of the following conditions is necessary and sufficient for equivalence:

(*a*) $a_n' - a_n$ is a null sequence,

(*b*) $a_n \le b_n'$ and $a_n' \le b_n$.

3. Given $x \sim \{[a_n, b_n]\}$, $y \sim \{[\alpha_n, \beta_n]\}$, (*a*) verify that the definitions of addition and subtraction,

$$x + y = \{[a_n + \alpha_n, b_n + \beta_n]\}, \qquad x - y = \{[a_n - \beta_n, b_n - \alpha_n]\},$$

are meaningful. Specifically, verify that

(i) the given representations are, in fact, nested sets for $x + y$ and $x - y$ when x and y are rational;

(ii) if $x < y$, then $x + z < y + z$, where z is an arbitrary real number.

(*b*) Define the product xy and verify specifically that your definition of product is meaningful.

(i) that the given nested set is, in fact, a nested set for xy when x and y are rational.

(ii) that if $x < y$ and $z > 0$, then $xz < yz$.

4. Prove that the following principles are equivalent in the sense that any one can be derived as a consequence of any other.

(*a*) Every nested sequence of intervals with real end points contains a real number.

(*b*) Every bounded monotone sequence converges.

(*c*) Every bounded infinite sequence has at least one accumulation or limit point.

(*d*) Every Cauchy sequence converges.

(*e*) Every bounded set of real numbers has an infimum and a supremum.

Solutions and Hints to Problems

SECTION 1.S.1, page 89

1. (*a*) $r + s = \dfrac{np + mq}{nq}$.

(*b*) $r - s = \dfrac{np - mq}{nq}$.

(*c*) $rs = \dfrac{mp}{nq}$.

(*d*) $\dfrac{r}{s} = \dfrac{np}{mq}$.

(*e*) $r < s$ if and only if $np < mq$.

2. (*a*) If $x < a_N$ for some N, then choose $n > N$ so large that

$$|a_n - a_n'| < a_N - x;$$

it follows that

$$a_n - a_n' < a_N - x,$$

whence

$$a_n' > x + (a_n - a_N) \geq x.$$

Thus, if x is an element of the first class for $[a_n, b_n]$, it is an element of the first class for $[a_n', b_n']$.

If $x > b_N$ for some N, then choose $n > N$ so large that

$$b_n - a_n < \frac{\epsilon}{3},$$

$$|a_n - a_n'| < \frac{\epsilon}{3},$$

$$b_n' - a_n' < \frac{\epsilon}{3},$$

where $\epsilon = x - b_N$. From the last two of these inequalities we have

$$|b_n' - a_n| < \frac{2\epsilon}{3}$$

and, thence, from the first,

$$b_n' - b_n < \epsilon$$

or

$$b_n' < x + (b_n - b_N) \leq x.$$

Thus, if x is a member of the third class for $[a_n, b_n]$, it is a member of the third class for $[a_n', b_n']$.

Conversely interchanging the roles of a and a', b and b', we see that the first and third classes are the same for the two nested sets of intervals. It follows that the second classes are also the same.

(*b*) If $a_n \leq b_n'$ and $x \leq a_n$ for all sufficiently large n, then $x \leq b_n'$ for sufficiently large n. Consequently, if x is in the first or second class for $[a_n, b_n]$ it is in the first or second class for $[a_n', b_n']$. Conversely, on interchanging the roles of a_n and a_n', b_n and b_n', we see that if x is in the first or second class for $[a_n', b_n']$, then it is in the first or second class for $[a_n, b_n]$. It follows that the two third classes are the same.

Similarly, we can prove that the two first classes are the same.

It follows that the two second classes also are the same.

3. (*a*) The sequences $[a_n + \alpha_n, b_n + \beta_n]$ and $[a_n - \beta_n, b_n - \alpha_n]$ are nested, for $a_n + \alpha_n < b_n + \beta_n$ and $a_n - \beta_n < b_n - \alpha_n$ and since $b_n - a_n$ and $\beta_n - \alpha_n$ are null sequences, then so also are

$$(b_n + \beta_n) - (a_n + \alpha_n) = (b_n - a_n) + (\beta_n - \alpha_n)$$

and

$$(b_n - \alpha_n) - (a_n - \beta_n) = (b_n - a_n) + (\beta_n - \alpha_n).$$

Finally, if $[a_n', b_n']$ is equivalent to $[a_n, b_n]$ and $[\alpha_n', \beta_n']$ is equivalent to $[\alpha_n, \beta_n]$, then so are the corresponding sequences for the sum and difference; thus from $a_n' < b_n$, $\alpha_n' < \beta_n$ and $a_n < b_n'$, $\alpha_n < \beta_n'$ we have

$$a_n' + \alpha_n' < b_n + \beta_n, \qquad a_n + \alpha_n < b_n' + \beta_n'$$

and

$$a_n - \beta_n < b_n' - \alpha_n', \qquad a_n' - \beta_n' < b_n - \alpha_n.$$

(i) If x and y are rational, then from

$$a_n \leq x \leq b_n \quad \text{and} \quad \alpha_n \leq y \leq \beta_n$$

we have

$$a_n + \alpha_n \leq x + y \leq b_n + \beta_n$$

and

$$a_n - \beta_n \leq x - y \leq b_n - \alpha_n.$$

(ii) Let $z = \{[A_n, B_n]\}$. Since $x < y$, there exists an N such that $b_N < \alpha_N$. Choose $n > N$ so large that

$$B_n - A_n < \alpha_N - b_N.$$

Using this inequality, we find

$$\begin{aligned} b_n + B_n &\leq b_N + B_n \\ &< \alpha_n + A_n. \end{aligned}$$

Thus the condition for the inequality is established.

(*b*) If x and y are positive, then for sufficiently large n, $a_n > 0$ and $\alpha_n > 0$. We then define

$$xy = \{[a_n\alpha_n, b_n\beta_n]\}, \qquad (a_n > 0, \alpha_n > 0).$$

In general, we define the product of any two real numbers in terms of the product of nonzero numbers by

$$xy = |x| \cdot |y| \operatorname{sgn} x \cdot \operatorname{sgn} y,$$

where

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

and $xy = 0$ if either x or y is zero. Here $-x$ is defined by $-x = \{[-\beta_n, -\alpha_n]\}$.

In proof, for positive x and y, we first observe that since $0 < a_n < b_n$ and $0 < \alpha_n < \beta_n$,

$$a_n\alpha_n < b_n\beta_n.$$

Furthermore, since $b_n - a_n$ and $\beta_n - \alpha_n$ are null sequences, so is

$$b_n\beta_n - a_n\alpha_n = b_n(\beta_n - \alpha_n) + \alpha_n(b_n - a_n).$$

Finally, for $[a_n', b_n']$ equivalent to $[a_n, b_n]$ and $[\alpha_n', \beta_n']$ equivalent to $[\alpha_n, \beta_n]$ we have $a_n'\alpha_n' < b_n\beta_n$ and $a_n\alpha_n < b_n'\beta_n'$.

The proofs when the signs of x and y may be negative are similar.

(i) If x and y are rational and nonzero, we form $|x| \cdot |y|$. We consider only the case $x < 0 < y$ as an example. We have

$$\begin{aligned} xy = -|x| \cdot |y| &= -[-b_n\alpha_n, -a_n\beta_n] \\ &= [a_n\beta_n, b_n\alpha_n], \end{aligned}$$

where n is so large that $b_n < 0$ and $\alpha_n > 0$. From

$$a_n \leq x \leq b_n < 0 \quad \text{and} \quad 0 < \alpha_n \leq y \leq \beta_n$$

we have

$$a_n\beta_n \le x\beta_n \le xy \le b_ny \le b_n\alpha_n,$$

or, in short,

$$a_n\beta_n \le xy \le b_n\alpha_n.$$

Furthermore, it is easly verified that strictly $a_n\beta_n < b_n\alpha_n$.

(ii) Set $z = \{[A_n, B_n]\}$, where we may suppose $A_n > 0$. We consider the case $x < y < 0$. For all sufficiently large n,

$$b_n < \alpha_n < \beta_n < 0.$$

We have from (i) above

$$xz = \{[a_nB_n, b_nA_n]\}$$
$$yz = \{[\alpha_nB_n, \beta_nA_n]\};$$

however,

$$b_nA_n < \alpha_nA_n < \beta_nA_n,$$

thus proving the inequality.

The proof for the remaining cases is equally simple.

4. (i) First let us assume the nested interval principle (*a*). Let S be a bounded set, $s \le M$ for all s in S. Given any s_0 in S we define a nested sequence $[a_n, b_n]$ of real numbers as follows: $a_0 = s_0$, $b_0 = M$; given $[a_k, b_k]$, we set $c_k = \frac{1}{2}(a_k + b_k)$ and choose $[a_{k+1}, b_{k+1}]$ to be $[c_k, b_k]$ if $[c_k, b_k]$ contains any points of s and choose $[a_k, c_k]$ if not. Since

$$b_n - a_n = \frac{b_0 - a_0}{2^n},$$

the sequence of intervals $[a_n, b_n]$ is clearly nested. From (*a*) we conclude that $\{[a_n, b_n]\}$ contains a real number x. We observe further that $s \le b_j$ for all j, and all s in S: this result clearly holds for $j = 0$, $b_0 = M$, and if it holds for b_k then by our definition of the b_j it holds for b_{k+1}. We prove that x is a supremum for S. If for any s in S we had $s > x$, then, since $b_n - x \le b_n - a_n = (b_0 - a_0)/2^n$, there would be some value b_n closer to x than s, such that $s > b_n > x$, contradicting the fact that b_n is an upper bound for S. Furthermore, x is the supremum of S since every interval $[a_n, b_n]$ and therefore every interval $[a_n, x]$ contains points of S. A similar proof establishes the existence of an infimum. Thus (*a*) implies (*e*).

(ii) Next consider a bounded monotone sequence a_n, say $a_n \le a_{n+1} \le M$ for all n. From principle (*e*) we know that a_n has a supremum a. Since for every ϵ there exists an N such that

$$a - a_N < \epsilon$$

and for $n > N$

$$a - a_n \le a - a_N < \epsilon,$$

we conclude that a is the limit of a_n. A similar proof holds for a decreasing sequence. Thus (*e*) implies (*b*).

(iii) Conversely, we show that (*b*) implies (*e*). Let S be any bounded set. From the construction in (i) above, we obtain a monotonic decreasing sequence b_k of upper bounds to S, which is bounded below. Let $x = \lim_{k\to\infty} b_k$. An argument similar to that of (i) proves that x is the supremum of S. A similar argument shows the existence of an infimum.

(iv) Next, let a_n be a bounded infinite sequence. From (*e*) we know that the set of values $\{a_n\}$ has a supremum b_1. If b_1 is a limit point of the sequence, then (*c*) holds. If b_1 is not a limit point of the sequence, then let ν_1 be the largest index such that $a_{\nu_1} = b_1$ and consider the set $\{a_n\colon n > \nu_1\}$. The supremum b_2 of this set is less than b_1. If it is a limit point of the sequence, we are through; otherwise choose ν_2 as the largest index such that $a_{\nu_2} = b_2$ and repeat the process. In this way we either obtain a b_k which is a limit point of the original sequence or a new sequence $b_1, b_2, \ldots$ which is bounded and strictly decreasing. Since by (*b*) the sequence $\{b_k\}$ must converge, and since each b_k is a term in $\{a_n\}$, it follows that $\{a_n\}$ has a limit point. Thus from (*b*) and (*e*) we derive (*c*). But since (*b*) and (*e*) are equivalent, either one alone implies (*c*).

(v) Let a_n be a Cauchy sequence. For some sufficiently large n we have $|a_n - a_{n+k}| < \epsilon$ for all k. Thus

$$|a_{n+k}| < |a_n| + \epsilon = M_1$$

and, on setting

$$|a_1| + |a_2| + \cdots + |a_{n-1}| = M_2$$

we have

$$|a_\nu| \leq M_1 + M_2$$

for all ν; that is, the sequence is bounded.

It follows from (*c*) that the sequence has a limit point α. For any ϵ we can find an N so that if $n, m > N$

$$|a_n - a_m| < \frac{\epsilon}{2}\,.$$

Furthermore, since α is a limit point of the sequence, we can fix m so that

$$|a_m - \alpha| < \frac{\epsilon}{2}\,.$$

It follows that

$$|a_n - \alpha| < \epsilon$$

for all $n > N$. Thus (*c*) implies (*d*).

(vi) Now consider any nested sequence $[a_n, b_n]$. Since $a_i < b_j$ for all i and j, we have

$$0 \leq b_n - b_{n+k} < b_n - a_n.$$

It follows that b_n is a Cauchy sequence and by (*d*) has a limit x. We have $b_n \geq x$ for all n, for if $b_N < x$, then for $n > N$ we would have $b_n \leq b_N < x$ or

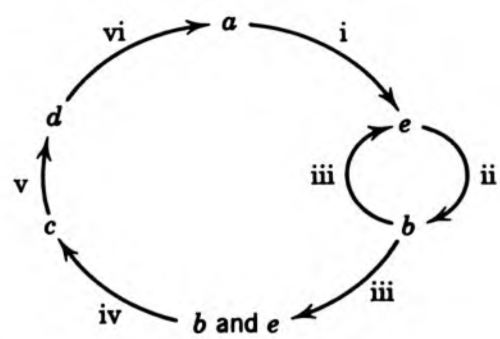
$$|b_n - x| \geq x - b_N > 0,$$

and x could not be the limit of b_n. Furthermore, $a_n \le x$ for all n, for if $a_N > x$, then $b_n \ge a_N > x$ and

$$b_n - x \ge a_N - x,$$

and again x could not be the limit of b_n. Thus (*d*) implies (*a*).

Letting an arrow denote the direction of implication, we indicate the train of arguments below. Clearly, assuming any one of the principles, we can deduce all the others.



MISCELLANEOUS PROBLEMS, Chapter 1

1. If $w_1, w_2, \ldots, w_n > 0$, prove that the weighted average

$$\frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots w_2}$$

lies between the greatest and the least of the x's.

2. Prove

$$2(\sqrt{n+1} - 1) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

3. Prove for $x, y > 0$

$$\frac{x^n + y^n}{2} \ge \left(\frac{x+y}{2}\right)^n.$$

Interpret this result geometrically in terms of the graph of x^n.

4. If $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, prove

$$n\sum_1^n a_ib_i \ge \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right).$$

5. (*a*) Show that the sequence $a_1, a_2, a_3, \ldots$ can be written as the sequence of partial sums of the series $u_1, u_2, u_3, \ldots$ where $u_n = a_n - a_{n-1}$ for $n > 1$ and $u_1 = a_1$.

(*b*) Write the sequence $a_n = n^3$ as the sequence of partial sums of a series.

(*c*) From the result obtain a formula for the nth partial sum of the series

$$1 + 4 + 9 + \cdots + n^2 + \cdots.$$

(*d*) From the formula for $1^2 + 2^2 + \cdots + n^2$, find a formula for

$$1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2.$$

6. A sequence is called an arithmetic progression of the first order if the differences of successive terms are constant. It is called an arithmetic progression of the second order if the differences of successive terms form an arithmetic progression of the first order; and, in general, it is called an arithmetic progression of order k if the differences of successive terms form an arithmetic progression of order $(k - 1)$.

The numbers 4, 6, 13, 27, 50, 84 are the first six terms of an arithmetic progression. What is its least possible order? What is the eighth term of the progression of smallest order with these initial terms?

7. Prove that the nth term of an arithmetic progression of the second order can be written in the form $an^2 + bn + c$, where a, b, c are independent of n.

***8.** Prove that the nth term of an arithmetic progression of order k can be written in the form $an^k + bn^{k-1} + \cdots + pn + q$, where $a, b, \ldots, p, q$ are independent of n.

Find the nth term of the progression of smallest order in Problem 6.

9. Find a formula for the nth term of the arithmetic progressions of smallest order for which the following are the initial terms:

(*a*) 1, 2, 4, 7, 11, 16, ...
(*b*) −7, −10, −9, 1, 25, 68, ...

***10.** Show that the sum of the first n terms of an arithmetic progression of order k is

$$a_kS_k + a_{k-1}S_{k-1} + \cdots + a_1S_1 + a_0n,$$

where S_ν represents the sum of the first n νth powers and the a_i are independent of n. Use this result to evaluate the sums for the arithmetic progressions of Problem 9.

11. By summing

$$\nu(\nu + 1)(\nu + 2) \cdots (\nu + k + 1) - (\nu - 1)\nu(\nu + 1) \cdots (\nu + k)$$

from $\nu = 1$ to $\nu = n$, show that

$$\sum_{\nu=1}^{n} \nu(\nu + 1)(\nu + 2) \cdots (\nu + k) = \frac{n(n + 1) \cdots (n + k + 1)}{k + 2}.$$

12. Evaluate $1^3 + 2^3 + \cdots + n^3$ by using the relation

$$\nu^3 = \nu(\nu + 1)(\nu + 2) - 3\nu(\nu + 1) + \nu.$$

13. Show that the function

$$f(x) = \begin{cases} \dfrac{1}{\log_2 |x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous but not Hölder-continuous. (*Hint:* Show Hölder continuity with exponent α fails at the origin by considering the values $x = 1/2^{n/\alpha}$.)

14. Let a_n be a monotone decreasing sequence of nonnegative numbers. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{\nu=0}^{\infty} 2^\nu a_{2^\nu}$ does.

15. Investigate for convergence and determine the limit when possible,

(*a*) $n!\, e - [n!\, e]$

(*b*) a_n/a_{n+1}, where $a_1 = 0$, $a_2 = 1$, and $a_{k+2} = a_{k+1} + a_k$.

Solutions and Hints to Problems

MISCELLANEOUS PROBLEMS, Chapter 1

1. Suppose $x_1 \leq x_2 \leq \cdots \leq x_n$. Then $w_k x_1 \leq w_k x_k \leq w_k x_n$, and consequently,

$$w_1x_1 + w_2x_1 + \cdots + w_nx_1 \leq w_1x_1 + w_2x_2 + \cdots + w_nx_n \leq w_1x_n + w_2x_n + \cdots + w_nx_n,$$

from which the result follows.

2. The proof is inductive. For $n = 1$ we have

$$2(\sqrt{2} - 1) < 1 < 2,$$

which is easily verified. If the result is true for any n, then

$$2(\sqrt{n+1} - 1) + \frac{1}{\sqrt{n+1}} < \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}},$$

but

$$2\sqrt{n+1} + \frac{1}{\sqrt{n+1}} = \frac{2(n + \frac{3}{2})}{\sqrt{n+1}} = \frac{2\sqrt{n^2 + 3n + \frac{9}{4}}}{\sqrt{n+1}}$$

$$> \frac{2\sqrt{(n+1)(n+2)}}{\sqrt{n+1}} = 2\sqrt{n+2},$$

whence

$$2(\sqrt{n+2} - 1) < \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}}.$$

On the other hand,

$$2\sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{2\sqrt{n(n+1)} + 1}{\sqrt{n+1}} < \frac{2\sqrt{(n+\frac{1}{2})^2} + 1}{\sqrt{n+1}}$$

$$\leq \frac{2(n+1)}{\sqrt{n+1}} = 2\sqrt{n+1};$$

hence

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} < 2\sqrt{n+1}$$

and the proof is complete.

3. For $n = 1$, the inequality is obviously true. If the inequality is true for $n = k$, then

$$\left(\frac{x+y}{2}\right)^{k+1} \leq \left(\frac{x^k + y^k}{2}\right)\left(\frac{x+y}{2}\right)$$

$$\leq \frac{x^{k+1} + y^{k+1}}{4} + \frac{xy^k + yx^k}{4}.$$

Observe that $(y - x)(y^k - x^k) \geq 0$ and $x^{k+1} + y^{k+1} \geq xy^k + yx^k$. Applying this result to the second term on the left above, we obtain the desired conclusion.

The midpoint of the chord joining two points of the graph lies on or above the graph.

4. Since $(a_i - a_j)(b_i - b_j) \geq 0$, we have

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i - a_j)(b_i - b_j)$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} [(a_ib_i + a_jb_j) - (a_ib_j + b_ia_j)]$$

$$\leq 2n \sum_{i=1}^{n} a_ib_i - 2\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right).$$

5. (*a*) $S_n = u_1 + u_2 + \cdots + u_n$

$= a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1})$

$= a_1 + (-a_1 + a_2) + (-a_2 + a_3) + \cdots + (-a_{n-1} + a_n)$

$= (a_1 - a_1) + (a_2 - a_2) + \cdots + (a_{n-1} - a_{n-1}) + a_n.$

(*b*) $u_n = n^3 - (n-1)^3 = 3n^2 - 3n + 1,$

$$n^3 = \sum_{k=1}^{n} (3k^2 - 3k + 1).$$

(*c*) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{3}\left[n^3 + \sum_{k=1}^{n} (3k - 1)\right].$

Applying the formula for the sum of an arithmetic progression, we obtain

$$\sum_{k=1}^{n} (3k - 1) = \frac{n(3n + 1)}{2},$$

and hence

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n + 1)(2n + 1)}{6}.$$

$$(d) \sum_{k=1}^{n+1} (2k - 1)^2 = 4\sum_{k=1}^{n+1} k^2 - 4\sum_{k+1}^{n+1} k + \sum_{k=1}^{n+1} 1$$
$$= \tfrac{2}{3}(n + 1)(n + 2)(2n + 3) - 2(n + 1)(n + 2) + (n + 1)$$
$$= \tfrac{1}{3}(n + 1)(2n + 1)(2n + 3).$$

6. Take the differences until one term or a sequence of constants is reached:

4	6	13	27	50	84		131	193
2	7	14	23	34		47	62	
5	7	9	11		13	15		
2	2	2		2	2			

To calculate the next terms, add back the successive differences. *Answer:* 3rd order. 193.

7. Let the terms of the arithmetic progression be denoted by a_n. Set $b_n = a_{n+1} - a_n$, $c_n = b_{n+1} - b_n = c_i$. The second differences c_n are constant. We have then

$$b_n = b_1 + c_1 + c_2 + \cdots + c_{n-1}$$
$$= b_1 + (n - 1)c_i,$$

whence

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$
$$= a_1 + (n - 1)b_1 + \frac{(n - 1)(n - 2)c_i}{2},$$

which is quadratic in n.

8. Let a_n be the arithmetic progression of kth order and define the sequence of differences $d_n^{\,1} = a_{n+1} - a_n, d_n^{\,2} = d_{n+1}^1 - d_n^{\,1}, \ldots, d_n^{\,k} = d_{n+1}^{k-1} - d_n^{k-1} =$ constant. We prove

$$(1) \qquad a_n = a_1 + \binom{n-1}{1} d_1^{\,1} + \binom{n-1}{2} d_2^{\,1} + \cdots + \binom{n-1}{k} d_k^{\,1},$$

from which the result follows, since

$$\binom{n-1}{r} = \frac{n(n - 1)\cdots(n - r + 1)}{r!}$$

is a polynomial of degree r in n.

The proof is inductive. We observe that (1) holds for $n = 1$. If the result is true for $n \le p$, then

$$\begin{aligned} a_{p+1} &= a_p + d_p^1 \\ &= \left[a_1 + \binom{p-1}{1} d_1^1 + \cdots + \binom{p-1}{k} d_k^1\right] \\ &\quad + \left[d_1^1 + \binom{p-1}{1} d_2^1 + \cdots + \binom{p-1}{k-1} d_k^1\right] \\ &= a_1 + \left[1 + \binom{p-1}{1}\right] d_1^1 + \left[\binom{p-1}{1} + \binom{p-1}{2}\right] d_2^1 \\ &\qquad + \cdots + \left[\binom{p-1}{k} + \binom{p-1}{k-1}\right] d_k^1 \\ &= a_1 + \binom{p}{1} d_1^1 + \cdots + \binom{p}{k} d_k^1, \end{aligned}$$

where we use the basic property of the binomial coefficients

$$\binom{m}{r} + \binom{m}{r+1} = \binom{m+1}{r+1}.$$

From the solution of **6** we take d_1^1, d_2^1, d_3^1 to obtain

$$\begin{aligned} a_n &= 4 + 2(n-1) + \frac{5(n-1)(n-2)}{2} + \frac{2(n-1)(n-2)(n-3)}{6} \\ &= \frac{1}{6}(2n^3 + 3n^2 - 11n + 30). \end{aligned}$$

9. (*a*) $\frac{1}{2}(n^2 - n + 2)$.

(*b*) $\frac{1}{6}(5n^3 - 18n^2 + n - 30)$.

10. Let α_n be an arithmetic progression of order k. We have proved in Problem 8 that

$$\alpha_n = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0,$$

that is, that α_n is a polynomial of degree k in n. Consequently,

$$\begin{aligned} \sum_{\nu=1}^{n} \alpha_n &= \sum_{\nu=1}^{n} (a_k \nu^k + a_{k-1} \nu^{k-1} + \cdots + a_0) \\ &= a_k S_k + a_{k-1} S_{k-1} + \cdots + a_0 n. \end{aligned}$$

To obtain the sums of the progressions of Problem 9, we take S_2 from Problem 5*c* and S_3 from Problem 12 below.

Answers: $\dfrac{n(n^2+5)}{6}$, $\dfrac{n(n-5)(5n^2+11n+26)}{24}$.

11. Setting $u_\nu = (\nu - 1)\nu(\nu + 1)(\nu + 2) \cdots (\nu + k)$, we obtain

$$\sum_{\nu=1}^{n} (u_{\nu+1} - u_\nu) = u_{n+1} - u_1 = u_{n+1}$$
$$= n(n + 1)(n + 2) \cdots (n + k + 1).$$

But

$$u_{\nu+1} - u_\nu = \nu(\nu + 1) \cdots (\nu + k)[(\nu + k + 1) - (\nu - 1)]$$
$$= \nu(\nu + 1) \cdots (\nu + k)[k + 2],$$

from which the result follows.

12. From the preceding problem

$$\sum_{\nu=1}^{n} \nu^3 = \frac{n(n + 1)(n + 2)(n + 3)}{4} - \frac{3n(n + 1)(n + 2)}{3} + \frac{n(n + 1)}{2}$$
$$= \frac{n^2(n + 1)^2}{4}.$$

13. Set $x_n = 1/2^{n/\alpha}$. Then

$$|f(x_n) - f(0)| = \left| \frac{1}{\log_2(1/2^{n/\alpha})} \right| = \frac{\alpha}{n}.$$

But

$$L\,|x_n - 0|^\alpha = \frac{L}{2^n}.$$

Since $2^n \geq [n(n - 1)]/2$ for all n, we have

$$\frac{L}{2^n} \leq \frac{2L}{n(n - 1)} < \frac{\alpha}{n}$$

for n sufficiently large ($n > (2L/\alpha) + 1$); hence

$$|f(x_n) - f(0)| > L\,|x_n - 0|^\alpha$$

and Hölder-continuity is not possible.

14. If $\sum_{\nu=0}^{\infty} 2^\nu a_{2^\nu}$ converges, then

$$S_n = \sum_{\nu=1}^{n} a_\nu = a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7)$$
$$+ \cdots + (a_{2^{k-1}} + \cdots a_{2^k-1})$$
$$+ a_{2^k} + a_{2^k+1} + \cdots + a_n,$$

where

$$2^k \leq n < 2^{k+1}.$$

Consequently, since a_n is monotone decreasing,

$$S_n \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$$
$$= \sum_{\nu=0}^{k} 2^\nu a_{2^\nu}.$$

Thus S_n is monotone increasing and bounded; hence convergent.

Conversely, if $\sum_{\nu=1}^{\infty} a_\nu$ converges, then

$$\begin{aligned} S_{2^n} &= a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{n-1}+1} + a_{2^n}) \\ &\geq \frac{a_1}{2} + a_2 + 2a_4 + \cdots + 2^{n-1}a_{2^n} \\ &\geq \frac{1}{2}\sum_{\nu=0}^{n} 2^\nu a_{2^\nu}. \end{aligned}$$

Thus $\sum_{\nu=0}^{n} 2^\nu a_{2^\nu}$ is bounded and monotone; hence convergent.

15. (*a*) Observe that

$$\begin{aligned} n!\, e - [n!\, e] &= n! \sum_{k=1}^{\infty} \frac{1}{(n+1)!} \\ &= \frac{1}{n+1}\left[1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right] \\ &< \frac{1}{n+1}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) \\ &\leq \frac{2}{n+1} \leq 1. \end{aligned}$$

The limit is 0.

(*b*) For $n \geq 4$ we have

$$\tfrac{5}{3}a_n \geq a_{n+1} \geq \tfrac{3}{2}a_n.$$

The proof is by induction. The inequality is verified for $n = 4$. If the inequality holds for $n = k$, then from $a_{k+2} = a_{k+1} + a_k$, we have

$$\tfrac{8}{5}a_{k+1} \leq a_{k+1} + \tfrac{3}{5}a_{k+1} \leq a_{k+2} \leq a_{k+1} + \tfrac{2}{3}a_{k+1} \leq \tfrac{5}{3}a_{k+1},$$

from which the inequality holds for $n = k + 1$.

From the inequality we have for $n \geq 4$

$$(a_{n+1})^2 \leq (\tfrac{5}{3}a_n)(\tfrac{2}{3}a_{n+2}) < a_n a_{n+2},$$

so that

$$\frac{a_n}{a_{n+1}} > \frac{a_{n+1}}{a_{n+2}} \geq \frac{3}{5}$$

and the sequence is both monotone decreasing and bounded below.

The limit is $(\sqrt{5} - 1)/2$. (Compare Section 1.5, Problem 7.)

2 *The Fundamental Ideas of Integral and Differential Calculus*

EXERCISES

SECTION 2.1, page 120

Approximate the area under the following curves from above and below by using circumscribed and inscribed rectangles. Obtain the area to within one decimal place accuracy and show that your estimate of error is correct.

1. $f(x) = x + 1$ on $[0, 2]$.

2. $f(x) = x^2$ on $[0, 2]$.

3. $f(x) = \dfrac{1}{x}$ on $[1, 2]$.

4. $f(x) = \sqrt{x}$ on $[0, 1]$.

PROBLEMS

SECTION 2.1, page 120

1. Let f be a positive monotone function defined on $[a, b]$, where $0 < a < b$. Let ϕ be the inverse of f and set $\alpha = f(a)$, $\beta = f(b)$. Using the interpretation of integral as area show that

$$\int_\alpha^\beta \phi(y)\,dy = b\beta - a\alpha - \int_a^b f(x)\,dx.$$

Answers to Exercises

SECTION 2.1, page 120

In each case let S be the area for circumscribed rectangles and s the area for inscribed rectangles.

1. For a subdivision into n equal parts,

$$S = \sum_{k=1}^{n} f\left(\frac{2k}{n}\right)\left(\frac{2}{n}\right),$$

$$s = \sum_{k=1}^{n} f\left(\frac{2k-2}{n}\right)\left(\frac{2}{n}\right).$$

$S - s = 4/n$. For the desired accuracy take $n > 80$.

2. Proceed as in Problem 1. $S - s = 8/n$. Take $n > 160$.

3. For a subdivision into n equal parts,

$$S = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1 + (k-1)/n},$$

$$s = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1 + k/n}.$$

$S - s = 1/2n$. Take $n > 10$.

4. Subdivide using the points $x_k = k^2/n^2$. Then

$$S = \sum_{k=1}^{n}\left(\frac{k}{n}\right)\left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right)$$

$$= \frac{1}{n^3}\sum_{k=1}^{n}(2k^2 - k)$$

$$s = \sum_{k=1}^{n}\left(\frac{k-1}{n}\right)\left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right)$$

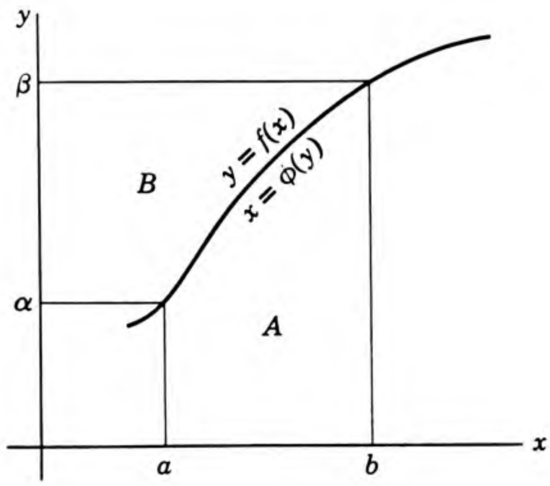
$$= \frac{1}{n^3}\sum_{k=1}^{n}[(2k^2 - k) - (2k - 1)].$$

Thus $S - s = \dfrac{1}{n^3}\displaystyle\sum_{k=1}^{n}(2k-1) = \dfrac{1}{n}$. Take $n > \dfrac{1}{20}$.

Solutions and Hints to Problems

SECTION 2.1, page 120

1. The numbers $A = \int_a^b f(x)\,dx$ and $B = \int_\alpha^\beta \phi(y)\,dy$ are represented by areas indicated in the figure on p. 84. It is geometrically obvious that $A + B = b\beta - a\alpha$ from which the result is obtained (see also p. 275).



PROBLEMS

SECTION 2.2, page 128

1. Prove for natural number p that

$$\int_a^b x^p\,dx = \frac{1}{p+1}(b^{p+1} - a^{p+1})$$

using a subdivision of $[a, b]$ into cells of equal length. Employ the techniques in Chapter 1, Miscellaneous Problems 5 to 12, to evaluate the approximating sums F_n.

2. Derive the formula for $\int_a^b x^\alpha\,dx$, a, $b > 0$, when α is rational and negative, say $\alpha = -r/s$, where r and s are natural numbers. (*Hint:* Set $q^{-1/s} = \tau$, where $q = \sqrt[n]{b/a}$.)

3. By the method used to find the integral of sin x, derive the formula

$$\int_a^b \cos x\,dx = \sin b - \sin a.$$

4. Make a general statement about $\int_{-a}^a f(x)\,dx$ when $f(x)$ is (*a*) an odd function and (*b*) an even function.

5. Calculate $\int_0^{\pi/2} \sin x\,dx$ and $\int_0^{\pi/2} \cos x\,dx$. Explain on geometrical grounds why these should be the same. Furthermore, explain why

$$\int_a^{a+2\pi} \sin x\,dx = \int_b^{b+2\pi} \cos x\,dx$$

for all values of a and b.

6. (*a*) Evaluate $I_n = \int_0^a x^{1/n}\,dx$. What is $\lim_{n\to\infty} I_n$? Interpret geometrically.

(*b*) Do the same for $I_n = \int_0^a x^n\,dx$.

7. Evaluate

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right).$$

Solutions and Hints to Problems

SECTION 2.2, page 128

1. Taking an equal subdivision of the interval into *n* cells we form the sum

$$\sum_{k=0}^{n-1}(a + kh)^p h,$$

where $h = (b - a)/n$. Using the binomial theorem, we have

$$\begin{aligned}
&[a + (k + 1)h]^{p+1} - [a + kh]^{p+1} \\
&\quad = \binom{p+1}{1} h(a + kh)^p + \binom{p+1}{2} h^2(a + kh)^{p-1} + \cdots \\
&\quad = (p + 1)h(a + kh)^p + h^2\left[\binom{p+1}{2}(a + kh)^{p-1} + \cdots\right] \\
&\quad = (p + 1)h(a + kh)^p + h^2 T_k.
\end{aligned}$$

For $h < 1$, it is easily proved that T_k is bounded independently of k,

$$\begin{aligned}
|T_k| &\le \sum_{r=2}^{p+1}\binom{p+1}{r} h^{r-2}\,|a + kh|^{p-r+1} \\
&\le \sum_{r=0}^{p+1}\binom{p+1}{r}\,|a + kh|^{p-r+1} \\
&\le (a + kh + 1)^{p+1} \\
&\le (|a| + |b| + 1)^{p+1}
\end{aligned}$$

(cf. Problems 1.1e, No. 2*c*).

It follows that

$$(a + kh)^p h = \frac{[a + (k + 1)h]^{p+1} - (a + kh)^{p+1} - h^2 T_k}{p + 1},$$

whence

$$\left| \sum_{k=0}^{n-1} (a + kh)^p h - \frac{(a + nh)^{p+1} - a^{p+1}}{p + 1} \right|$$
$$\leq h^2 \sum_{k=0}^{n-1} |T_k|$$
$$\leq nh^2(|a| + |b| + 1)^{p+1}.$$

Using $nh = b - a$ and passing to the limit as n tends to infinity, we obtain the desired result.

2. In the notation of Section 2.2, the integral is $\lim_{n\to\infty} F_n$, where

$$F_n = (b^{\alpha+1} - a^{\alpha+1})q^\alpha \frac{q - 1}{q^{\alpha+1} - 1}$$

and the problem is to evaluate

$$\lim_{q\to 1} \frac{q - 1}{q^{\alpha+1} - 1}.$$

We have $q^\alpha = \tau^r$, $q = \tau^{-s}$, $(\tau \neq 1)$, whence

$$\frac{q - 1}{q^{\alpha+1} - 1} = \frac{\tau^{-s} - 1}{\tau^{r-s} - 1}$$
$$= \frac{1 - \tau^s}{\tau^r - \tau^s}$$
$$= \begin{cases} \dfrac{1 + \tau + \cdots + \tau^{s-1}}{\tau^r(1 + \tau + \cdots + \tau^{s-r-1})}, & \text{for } s \geq r \\ \dfrac{-(1 + \tau + \cdots + \tau^{s-1})}{\tau^s(1 + \tau + \cdots + \tau^{r-s-1})}, & \text{for } s \leq r. \end{cases}$$

In either case we obtain for the limit as τ tends to 1, $\dfrac{s}{s - r} = \dfrac{1}{1 + \alpha}$.

3. In the summation formula of Section 2.2e, we apply the identities

$$\cos x = \sin\left(x + \frac{\pi}{2}\right), \sin x = -\cos\left(x + \frac{\pi}{2}\right)$$

to obtain

$$\sum_{k=1}^{n} h \cos(a + kh) = \sum_{k=1}^{n} \sin\left(a + kh + \frac{\pi}{2}\right)$$

$$= \frac{h}{2 \sin \frac{h}{2}}\left[\cos\left(a + \frac{h}{2} + \frac{\pi}{2}\right) - \cos\left(a + \frac{2n+1}{2} h + \frac{\pi}{2}\right)\right]$$

$$= \frac{h}{2 \sin \frac{h}{2}}\left[\sin\left(a + \frac{2n+1}{2} h\right) - \sin\left(a + \frac{h}{2}\right)\right]$$

$$= \frac{h}{2 \sin \frac{h}{2}}\left[\sin\left(b + \frac{h}{2}\right) - \sin\left(a + \frac{h}{2}\right)\right]$$

which yields the result on taking the limit as n tends to infinity.

4. Subdivide the interval into $2n$ equal parts of length $h = a/n$. We then have

$$\int_{-a}^{a} f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} [f(kh) + f(-kh)]h.$$

(a) The integral is zero.

(b) $\displaystyle\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$

5. $\displaystyle\int_{0}^{\pi/2} \sin x\, dx = \int_{0}^{\pi/2} \cos x\, dx = 1.$

The integrals correspond to the areas of congruent regions.

Since $\cos x = \sin(x + \pi/2)$, the region "under the curve" $y = \cos x$ between $x = b$ and $x = b + 2\pi$ is congruent to the region under the curve $y = \sin x$ between $x = b + \pi/2$ and $b + 3\pi/2$. Moreover, the area under the curve $y = \sin x$ is the same over every interval of length 2π because of the periodicity property $\sin(x + 2\pi) = \sin x$. The area over the interval $[a, a + 2\pi]$ can be decomposed into areas over the intervals $[a, 2n\pi]$ and $[2n\pi, a + 2\pi]$, where n is the unique integer for which $a/2\pi \le n < a/2\pi + 1$. These areas are congruent to the areas over the respective intervals $[a + 2\pi - 2n\pi, 2\pi]$ and $[0, a + 2\pi - 2n\pi]$, and together constitute the area over the interval $[0, 2\pi]$.

6. (a) $\displaystyle I_n = \frac{a^{1+(1/n)}}{1 + (1/n)}, \lim_{n\to\infty} I_n = a.$

The graph $y = x^{1/n}$ approaches the horizontal line $y = 1$ for $x > 0$.

(*b*) $I_n = \dfrac{a^{n+1}}{n+1}$.

$$\lim_{n\to\infty} I_n = \begin{cases} \infty, & a > 1 \\ 0, & -1 \le a \le 1 \\ \text{no limit for } a < -1. \end{cases}$$

Note the behavior of the graph $y = x^n$ (Fig. 1.35, p. 48) as n tends to infinity.

7. Compare Chapter 1, Miscellaneous Problems, No. 2, or interpret as $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. (Note that this is an "improper" integral since $1/\sqrt{x}$ is discontinuous for $x = 0$.)

EXERCISES

SECTION 2.3, page 136

1. In each of the following cases determine the area under the curve $f(x)$ on the interval $[a, b]$.

(*a*) $f(x) = 1 + 2x + 3x^2$, $a = 0, b = 1$.
(*b*) $f(x) = 1 + 2x - 3x^2$, $a = 0, b = 1$.
(*c*) $f(x) = 1 - 2x + 3x^2$, $a = 0, b = 1$.
(*d*) $f(x) = x^2 - 2x + 1$, $a = 1, b = 4$.
(*e*) $f(x) = 2x^2 + x + 1$, $a = 1, b = 3$.
(*f*) $f(x) = x^3 - 1$, $a = 1, b = 2$.
(*g*) $f(x) = \sqrt{x}$, $a = 0, b = 1$.

2. Find the area of the region bounded by the graph of $f(x) = x(1 - x)$ and the x-axis.

3. Find the area bounded by the parabola $y = \frac{1}{2}x^2 + 1$ and the straight line $y = 3 + x$.

4. Find the area bounded by the parabola $y^2 = 5x$ and the straight line $y = 1 + x$.

5. Find the area bounded by the parabola $y = x^2$ and the straight line $y = ax + b$.

6. Using the methods in the text, evaluate the integrals

(*a*) $\int_a^b (x+1)^\alpha\,dx$, (*b*) $\int_a^b \sin \alpha x\,dx$, (*c*) $\int_a^b \cos \alpha x\,dx$,

where α is an arbitrary integer.

7. Use the formulas obtained on p. 135 along with the identities $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ to prove that

$$\int_a^b \cos^2 x\, dx = \frac{b-a}{2} + \frac{\sin 2b - \sin 2a}{4},$$

$$\int_a^b \sin^2 x\, dx = \frac{b-a}{2} - \frac{\sin 2b - \sin 2\alpha}{4}.$$

8. By use of Exercises 1.5, No. 2*b*, evaluate $\int_a^b x^3\, dx$, using division into equal subintervals.

9. Evaluate $\int_0^1 (1-x)^n\, dx$ (where n is an integer).

10. In each of the following cases give the mean of $f(x)$ on the interval $[a, b]$ and a value ξ for which it is achieved.

(*a*) $f(x) = 4x - 1$, $a = 1, b = 3$.
(*b*) $f(x) = x^2 + 1$, $a = 0, b = 1$.
(*c*) $f(x) = (x+1)^2$, $a = 0, b = 1$.
(*d*) $f(x) = x^3 - 3x$, $a = -1, b = 2$.
(*e*) $f(x) = \sqrt{x}$, $a = 0, b = 1$.
(*f*) $f(x) = x^n$.

11. Let $f(x)$ and $g(x)$ be continuous on $[a, b]$. Prove if $f(x) \geq g(x)$ and definite inequality $f(\xi) > g(\xi)$ holds for at least one point ξ in the interval, then definite inequality holds for the integrals

$$\int_a^b f(x)\, dx > \int_a^b g(x)\, dx.$$

PROBLEMS

SECTION 2.3, page 136

***1.** *Cauchy's inequality for integrals.* Prove that for all continuous functions $f(x)$, $g(x)$

$$\int_a^b [f(x)]^2\, dx \int_a^b [g(x)]^2\, dx \geq \left(\int_a^b f(x)g(x)\, dx\right)^2.$$

***2.** Prove that if $f(x)$ is continuous and

$$f(x) = \int_0^x f(t)\, dt,$$

then $f(x)$ is identically zero.

***3.** Let $f(x)$ be Lipschitz-continuous on [0, 1]; that is,

$$|f(x) - f(y)| < M\,|x - y|$$

for all x, y in the interval. Prove that

$$\left|\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right| < \frac{M}{2n}.$$

Answers to Exercises

SECTION 2.3, page 136

1. (*a*) 3, (*b*) 1, (*c*) 1, (*d*) 9, (*e*) $23\frac{1}{3}$, (*f*) $2\frac{3}{4}$, (*g*) $\frac{2}{3}$.

2. $\frac{1}{6}$.

3. Required area may be regarded as the difference between the area under the line and the area under the parabola, taken between the points of intersection of curve and line: $10\sqrt{5}/3$.

4. $\dfrac{\sqrt{5}}{6}$.

5. $\frac{1}{6}(a^2 + 4b)^{3/2}$

6. (*a*) $\dfrac{(1 + b)^{1+\alpha} - (1 + a)^{1+\alpha}}{1 + \alpha}$.

(*b*) $\dfrac{-(\cos \alpha b - \cos \alpha a)}{\alpha}$.

(*c*) $\dfrac{\sin \alpha b - \sin \alpha a}{\alpha}$.

8. $\dfrac{b^4 - a^4}{4}$.

9. $\dfrac{1}{n + 1}$.

10. (*a*) 7, 2; (*b*) $\frac{4}{3}, \dfrac{1}{\sqrt{3}}$; (*c*) $\frac{7}{3}$, $\sqrt{\frac{7}{3}} - 1$.

(*d*) $-\frac{1}{4}$, at any of the three real roots of $4x^3 - 12x + 1$.

(*e*) $\frac{2}{3}, \frac{4}{9}$.

(*f*) $\mu = \dfrac{b^n + ab^{n-1} + a^2b^{n-2} + \cdots + a^n}{n + 1}$, $\xi = \sqrt[n]{\mu}$.

11. There exists a neighborhood $(\xi - \delta, \xi + \delta)$ of ξ where $f(x) \geq g(x) + \epsilon$ for $\epsilon = \frac{1}{2}[f(\xi) - g(\xi)]$. Then for $h(x) = f(x) - g(x)$

$$\int_a^b h(x)\, dx = \int_a^{\xi-\delta} + \int_{\xi-\delta}^{\xi+\delta} + \int_{\xi+\delta}^b h(x)\, dx$$
$$\geq 0 + 2\epsilon\delta + 0 > 0.$$

(If ξ is an end point the proof must be modified slightly.)

Solutions and Hints to Problems

SECTION 2.3, page 136

1. Observe that for all t,

$$\int_a^b [f(x) + tg(x)]^2\, dx \geq 0$$

and consequently,

$$\int_a^b f(x)^2\, dx + 2t \int_a^b f(x)g(x)\, dx + t^2 \int_a^b g(x)^2\, dx \geq 0.$$

Set $A = \int_a^b f(x)^2\, dx$, $B = \int_a^b f(x)g(x)\, dx$,

$$C = \int_a^b g(x)^2\, dx,$$

and follow the proof of Cauchy's inequality in Section 1.1e.

Alternatively, express the integrals as limits of sums using equal subdivisions of $[a, b]$ and apply Cauchy's inequality to these sums.

2. By the mean value theorem of integral calculus,

$$f(x) = xf(x_1) = xx_1f(x_2) = xx_1x_2f(x_3)\ldots,$$

where x_1 lies between 0 and x, x_2 between 0 and x_1, x_3 between 0 and x_2, etc. Let M be an upper bound for $|f(x)|$ on $[-1, 1]$. We have $|x_i| \leq |x|$ and consequently,

$$|f(x)| \leq |x|^n\, |f(x_n)| \leq |x|^n M.$$

Since $\lim_{n\to\infty} x^n = 0$, we have $f(x) = 0$ for all x in $[-1, 1]$, (at the end points, by continuity). Applying the same argument to

$$f(1 + x) = \int_1^{1+x} f(t)\, dt \text{ and } f(-1 - x) = \int_{-1}^{-1-x} f(t)\, dt$$

for $|x| < 1$ we can extend the result to $[-2, 2]$, and, inductively, to $[-n, n]$ for all n.

3. $$\sigma_n = \left| \int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right|$$

$$= \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{(k/n)} \left[f(x) - f\left(\frac{k}{n}\right)\right] dx \right|$$

$$\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{(k/n)} \left| f(x) - f\left(\frac{k}{n}\right)\right| dx.$$

But for x in $[(k-1)/n, k/n]$

$$\left| f(x) - f\left(\frac{k}{n}\right)\right| < M\left(\frac{k}{n} - x\right)$$

whence, on integrating,

$$\int_{(k-1)/n}^{k/n} \left| f(x) - f\left(\frac{k}{n}\right)\right| dx < M\left\{\left[\frac{k^2}{n^2} - \frac{k^2}{2n^2}\right] - \frac{(k-1)}{n}\left(\frac{k}{n} - \frac{k-1}{2n}\right)\right\}$$

$$= \frac{M}{2n^2}.$$

(Strict inequality holds because strict inequality between functions implies strict inequality between their integrals. Compare Problems 1.2*d*, No. 1.) Consequently,

$$\sigma_n \leq \frac{M}{2n}.$$

EXERCISES

SECTION 2.4, page 143

1. In each of the following cases give the definite integral $\phi(x) = \int_\alpha^x f(u)\,du$ and the value α which satisfy the stated conditions.

(*a*) $f(x) = 2$, $\phi(0) = 1$.
(*b*) $f(x) = 3x - 4$, $\phi(0) = \frac{7}{2}$.
(*c*) $f(x) = \sqrt{x}$, $\phi(1) = 1$.
(*d*) $f(x) = 12x^2$, $\int_0^1 \phi(x)\,dx = 1$.
(*e*) $f(x) = 4x^3 - 3x^2 + 2x - 1$, $\phi(\pi) = 0$.

2. In each of the following find a uniform modulus of continuity for the indefinite integrals of $f(x)$ on the interval $[a, b]$.

(*a*) $f(x) = 0$.
(*b*) $f(x) = 1$.
(*c*) $f(x) = 2x + 1$, $a = 0$, $b = 2$.
(*d*) $f(x) = x$, $a = 0$, $b = 2$.
(*e*) $f(x) = x^2 - x + 1$, $a = -1$, $b = 1$.
(*f*) $f(x) = \sqrt{x}$, $a = 0$, b arbitrary positive.
(*g*) $f(x) = 4x^3 + 3x^2 + 2x + 1$, $a = 0$, $b = \pi$.

Answers to Exercises

SECTION 2.4, page 143

1. (*a*) $2(x - \alpha)$, $\alpha = -\frac{1}{2}$.
(*b*) $\frac{3}{2}(x^2 - \alpha^2) - 4(x - \alpha)$, $\frac{1}{3}(4 \pm \sqrt{2})$
(*c*) $\frac{2}{3}(x^{3/2} - \alpha^{3/2})$, none.
(*d*) $4(x^3 - \alpha^3)$, $\alpha = 0$.
(*e*) $(x^4 - \alpha^4) - (x^3 - \alpha^3) + (x^2 - \alpha^2) - (x - \alpha)$, $\alpha = \pi$.

2. (*a*) Any positive δ, (*b*) ϵ.
(*c*) $\frac{\epsilon}{5}$, (*d*) $\epsilon/\max\{|a|,|b|\}$
(*e*) $\frac{\epsilon}{3}$, (*f*) $\epsilon/\sqrt{b}$
(*g*) $\epsilon/f(\pi)$.

EXERCISES

SECTION 2.5, page 145

1. (*a*) Sketch the function $y = (1/x)$ $(1 \le x \le 2)$ on a large scale, using graph paper, and find $\log_e 2$ by counting squares.

(*b*) Obtain upper and lower estimates for log 2 using inscribed and circumscribed rectangles. In this way obtain an estimate of how close the answer in (*a*) is to the true value.

2. Prove for all natural numbers n that

$$\log n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \log n$$

3. (*a*) Show that $\log x < x$ for all $x > 0$.

(*b*) Show that the line $x + y = 0$ and the curve $y = \log x$ have exactly one intersection.

PROBLEMS

SECTION 2.5, page 145

1. Prove

$$\log \frac{p}{q} \le \sqrt{\frac{p-q}{q}} \quad (q \le p).$$

(*Hint:* Apply Cauchy's inequality, Problem 1, Section 2.3.)

2. (*a*) Verify that $\log(1 + x) = \int_0^x \frac{1}{1+u}\,du$, where $x > -1$.

(*b*) Show for $x > 0$ that

$$x - \frac{x^2}{2} < \log(1 + x) < x.$$

*(*c*) More generally, show for $0 < x$ that

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \frac{x^{2n}}{2n} < \log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2n+1}}{2n+1}$$

(*Hint:* Compare $1/(1 + u)$ with a geometric progression.)

Answers to Exercises

SECTION 2.5, page 145

1. (*a*) $\log 2 = 0.693. \ldots$

(*b*) Compare Exercises 2.1, No. 3.

2. The sum is $1/n$ plus the total of areas of circumscribed rectangles for a subdivision into $n - 1$ equal parts of the interval $1 \le x \le n$, and is 1 plus the total of the areas of the inscribed rectangles for the curve $y = 1/x$.

3. (*a*) $x - \log x = 1 + \int_1^x \left(1 - \frac{1}{t}\right) dt \ge 1$ for all positive x.

(*b*) Since $x + \log x$ is monotone increasing, there can be at most one intersection where $x + \log x$ has a root. That $x + \log x$ has a root is clear by the intermediate value theorem from Exercise 1; $+ \log \frac{1}{2} < 0$, but $x + \log x > 0$ for $x \ge 1$.

Solutions and Hints to Problems

SECTION 2.5, page 145

1. $$\left(\int_q^p \frac{1}{x}\,dx\right)^2 \le \int_q^p dx \int_q^p \frac{1}{x^2}\,dx.$$

2. (*a*) $$\log(1+x) = \int_1^{1+x} \frac{1}{t}\,dt = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{h}{1+kh} = \int_0^x \frac{du}{1+u},$$

where for the common representation of the two integrals as the limit of a sum we use a subdivision of the interval into n equal parts of length $h = x/n$.

(*b*) Integrating the inequality

$$1 - x < \frac{1}{1+x} < 1, \qquad (x > 0),$$

we obtain the desired result.

(*c*) For $0 < x < 1$ we have

$$(1+x)[1 - x + x^2 - \cdots + (-x)^n] = 1 + (-x)^{n+1},$$

which is greater or less than 1 according as n is odd or even. Consequently,

$$\sum_{\nu=0}^{2n-1} (-x)^\nu < \frac{1}{1+x} < \sum_{\nu=0}^{2n} (-x)^\nu.$$

The result is obtained by integrating this inequality.

EXERCISES

SECTION 2.6, page 149

1. Prove for positive a, b, and real α that $(ab)^\alpha = a^\alpha b^\alpha$.

2. (*a*) Show that $e^x > x$ for all x (cf. Exercise 2.5, No. 3*a*).

(*b*) Show that the line $x + y = 0$ and curve $y = e^x$ have exactly one interaction.

3. Prove for $0 < x < n$ that

$$\left(\frac{n + x}{n}\right)^n < e^x < \left(\frac{n}{n - x}\right)^n.$$

(*Hint:* consider the logarithms of these expressions.)

PROBLEMS

SECTION 2.6, page 149

1. (*a*) Prove

$$\int_a^b e^x\,dx = e^b - e^a$$

using a subdivision of $[a, b]$ into equal cells. [*Hint:* Apply $\log \alpha = \lim_{n\to\infty} n(\sqrt[n]{\alpha} - 1)$.]

(*b*) Find $\int_a^b \log x\,dx$. (See Section 2.1, Problem 1.)

(*c*) Show for $x \geq 0$ that

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \leq e^x \leq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^x x^{n+1}}{(n + 1)!}.$$

(*Hint:* Obtain upper and lower estimates for $\int_0^x e^u\,du$ and integrate repeatedly.)

Obtain estimates of the same type for e^x when $x < 0$.

Answers to Exercises

SECTION 2.6, page 149

1. $(ab)^\alpha = e^{\alpha \log ab} = e^{\alpha(\log a + \log b)} = e^{\alpha \log a + \alpha \log b}$

$$= e^{\alpha \log a} e^{\alpha \log b} = a^\alpha b^\alpha.$$

2. (*a*) Since $\log x < x$, it follows from the monotonicity of the exponential function that $x = e^{\log x} < e^x$.

(*b*) The intersections occur at points $\log y = -y$ and by Exercises 2.5, No. 3*b* only one such value of y exists.

3. From Section 2.6c,

$n \log (1 + x/n) = x/a < x$, where $1 < a < 1 + x/n$. Similarly
$n \log (1 - x/n) = -x/b$, where $1 - x/n < b < 1$; hence
$n \log 1/(1 - x/n) = x/b > x$.

Solutions and Hints to Problems

SECTION 2.6, page 149

1. (*a*) $\int_a^b e^x\,dx = \lim_{n\to\infty} S_n$

where, for $h = (b - a)/n$,

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1} e^{a+kh}h \\ &= he^a[1 + e^h + (e^h)^2 + \cdots + (e^h)^{n-1}] \\ &= he^a\,\frac{e^{nh} - 1}{e^h - 1} \\ &= \frac{(b - a)e^a(e^{b-a} - 1)}{n(\sqrt[n]{e^{b-a}} - 1)} \end{aligned}$$

But

$$\lim_{n\to\infty} n(\sqrt[n]{e^{b-a}} - 1) = \log(e^{b-a}) = b - a.$$

Consequently,

$$\lim_{n\to\infty} S_n = e^a(e^{b-a} - 1) = e^b - e^a.$$

(*b*) From (*a*) and Problem 2.1, No. 1,

$$\begin{aligned} \int_a^b \log x\,dx &= b\log b - a\log a - \int_{\log a}^{\log b} e^y\,dy \\ &= (b\log b - b) - (a\log a - a). \end{aligned}$$

(*c*) From the monotone property of e^t we have for $0 \le t \le x$,

$$e^0 \le e^t \le e^x.$$

Integrating with respect to t we obtain

$$x \le \int_0^x e^t\,dt = e^x - 1 \le e^x x,$$

whence

$$1 + x \le e^x \le 1 + xe^x.$$

If the result holds for $n = k$, then it holds for $n = k + 1$, since

$$\int_0^x \left(\sum_{\nu=0}^{k} \frac{t^\nu}{\nu!}\right) dt \le \int_0^x e^t\,dt \le \int_0^x \left(\frac{e^x t^{k+1}}{(k + 1)!} + \sum_{\nu=0}^{k} \frac{t^\nu}{\nu!}\right) dt.$$

For $x < 0$,

$$\sum_{\nu=0}^{2n+1} \frac{x^\nu}{\nu!} \le e^x \le \frac{e^x x^{2n+1}}{(2n + 1)!} + \sum_{\nu=0}^{2n} \frac{x^\nu}{\nu!}$$

EXERCISES

SECTION 2.8a, page 156

1. In both of the following cases carefully plot the function $f(x)$ in the interval $[a, b]$, draw the tangent at the point ξ by eye, obtain the slope of the tangent as accurately as you can from the graph, and, finally, compute $f'(\xi)$ and compare it with the value found graphically.

(*a*) $f(x) = x^2$, $a = 0$, $b = 1$, $\xi = \frac{1}{4}$.

(*b*) $f(x) = \sqrt{x}$, $a = \frac{1}{4}$, $b = \frac{49}{64}$, $\xi = \frac{9}{16}$.

2. Calculate the derivatives of the following functions as the limits of their difference quotients.

(*a*) $2x - 3$. (*b*) $4x^2$.

(*c*) $4x^2 + 2x - 3$. (*d*) $\dfrac{1}{x}$.

(*e*) $\dfrac{1}{x + 1}$. (*f*) $\dfrac{1}{x^2 + 2}$.

(*g*) $\dfrac{1}{2x^2 + 1}$.

Answers to Exercises

SECTION 2.8a, page 156

1. (*a*) $f'(\xi) = \frac{1}{2}$, (*b*) $f'(\xi) = \frac{2}{3}$.

2. (*a*) 2, (*b*) $8x$, (*c*) $8x + 2$, (*d*) $-\dfrac{1}{x^2}$, (*e*) $-\dfrac{1}{(x + 1)^2}$, (*f*) $-\dfrac{2x}{(x^2 + 2)^2}$,

(*g*) $-\dfrac{4x}{(2x^2 + 1)^2}$.

EXERCISES

SECTION 2.8c, page 163

Calculate the derivatives of the following functions wherever defined, directly as the limits of their difference quotients.

1. $\sin 3x$. **2.** $\cos ax$.

3. $\sin^2 x$. **4.** $\cos^2 x$.

5. Given $f(x) = \frac{1}{3}x^3 - x^2 + 1$, find a number δ such that for every h less in absolute value than δ and every x in the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$ the following inequality holds:

$$\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{1}{100}.$$

PROBLEMS

SECTION 2.8c, page 163

Calculate the derivatives of the following functions wherever defined directly as the limits of their difference quotients.

1. $\tan x$.

2. $\sec^2 x$.

3. $\sin \sqrt{x}$.

4. $\sqrt{\sin x}$.

5. $\dfrac{1}{\sin x}$.

6. $\sin \dfrac{1}{x}$.

7. x^α, where α is rational and negative.

Answers to Exercises

SECTION 2.8c, page 163

1. $3 \cos 3x$.

2. $-a \sin ax$.

3. $\sin 2x$.

4. $-\sin 2x$.

5. $\delta = 0.006$.

Solutions and Hints to Problems

SECTION 2.8c, page 163

1. $$\frac{\tan(x+h) - \tan x}{h} = \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \frac{\sin h}{h}\left[\frac{1}{\cos(x+h)\cos x}\right].$$

The derivative is

$$\frac{1}{\cos^2 x} = 1 + \tan^2 x.$$

2. $2 \sec^2 x \tan x$.

3. $$\frac{\sin \sqrt{x+h} - \sin \sqrt{x}}{h} = \frac{2 \sin \frac{1}{2}(\sqrt{x+h} - \sqrt{x}) \cos \frac{1}{2}(\sqrt{x+h} + \sqrt{x})}{h}$$

$$= \frac{2 \sin \dfrac{1}{2}\left(\dfrac{h}{\sqrt{x+h} + \sqrt{x}}\right) \cos \dfrac{1}{2}(\sqrt{x+h} + \sqrt{x})}{h}$$

$$= \frac{\sin \dfrac{1}{2}\left(\dfrac{h}{\sqrt{x+h} + \sqrt{x}}\right)}{\dfrac{1}{2}\left(\dfrac{h}{\sqrt{x+h} + \sqrt{x}}\right)} \cdot \frac{\cos \dfrac{1}{2}(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}}$$

Taking the limit as $h \to 0$, we obtain the derivative

$$\frac{\cos \sqrt{x}}{2\sqrt{x}}.$$

4. $$\frac{\sqrt{\sin (x+h)} - \sqrt{\sin x}}{h} = \frac{\sin (x+h) - \sin x}{h} \cdot \frac{1}{\sqrt{\sin (x+h)} + \sqrt{\sin x}}.$$

The first factor on the right is simply the difference quotient for $\sin x$; therefore, on passing to the limit we obtain

$$\frac{\cos x}{2\sqrt{\sin x}}.$$

5. $\dfrac{-\cos x}{\sin^2 x}$.

6. $$\frac{1}{h}\left[\sin \frac{1}{x+h} - \sin \frac{1}{x}\right] = \frac{2}{h}\left[\sin \frac{1}{2}\left(\frac{1}{x+h} - \frac{1}{x}\right) \cos \frac{1}{2}\left(\frac{1}{x+h} + \frac{1}{x}\right)\right]$$

$$= -\frac{2}{h} \sin\left(\frac{h}{2x(x+h)}\right) \cdot \cos\left(\frac{2x+h}{2x(x+h)}\right)$$

$$= -2\,\frac{\sin [h/2x(x+h)]}{h/2x(x+h)} \cdot \frac{\cos [(2x+h)/2x(x+h)]}{2x(x+h)}.$$

Passing to the limit as $h \to 0$, we obtain

$$-\frac{1}{x^2} \cos \frac{1}{x}.$$

7. Set $\alpha = -p/q$, where p and q are natural numbers, $t = x^{-1/q}$ and $\tau = \xi^{-1/q}$. Then

$$\frac{\xi^\alpha - x^\alpha}{\xi - x} = \frac{\tau^p - t^p}{\tau^{-q} - t^{-q}}$$

$$= \frac{(\tau^p - t^p)(\tau t)^q}{t^q - \tau^9}$$

$$= -\frac{(\tau^{p-1} + \tau^{p-2}t + \tau^{p-3}t^2 + \cdots + t^{p-1})(\tau t)^q}{(\tau^{q-1} + \tau^{q-2}t + \tau^{q-3}t^2 + \cdots + t^{q-1})}.$$

Taking the limit as $\tau \to t$, we obtain

$$\frac{d}{dx}(x^\alpha) = -\frac{pt^pt^{2q}}{qt^q} = \alpha t^{p+q} = \alpha x^{-(p/q)-1}$$

$$= \alpha x^{\alpha-1}.$$

EXERCISES

SECTION 2.8d, page 165

Give $f'(x)$ and its value at $x = a$ in each of the following cases.

1. $f(x) = x^2 - x + 1,\qquad a = -1.$

2. $f(x) = 4x^3 + x^2 - 2x,\qquad a = \frac{1}{2}.$

3. $f(x) = x^2 + \dfrac{1}{x}.\qquad a = \dfrac{1}{2}.$

4. $f(x) = x(\sqrt{x} + \sqrt{2}),\qquad a = \dfrac{\sqrt{2}}{6}$

5. $f(x) = 2x - 3\sin x,\qquad a = \pi/2.$

6. $f(x) = x\,|x| + x^2,\qquad a = 0.$

Answers to Exercises

SECTION 2.8d, page 165

1. $2x - 1,\ -3.$

2. $12x^2 + 2x - 2,\ \frac{5}{4}.$

3. $2x - \dfrac{1}{x^2},\ -3.$

4. $\frac{3}{2}\sqrt{x} + \sqrt{2}, \frac{3}{2}\sqrt{\sqrt{2}/6} + \sqrt{2}.$

5. $x - 3\cos x, \dfrac{\pi}{2}$

6. $2(|x| + x)$, 0.

EXERCISES

SECTION 2.8f, page 169

1. In each of the following cases give the derivatives of $f(x)$ to all orders, and give their values at $x = a$.

(*a*) $f(x) = x^2 - x + 1, \quad a = 0.$
(*b*) $f(x) = x^3 + 3x + 1, \quad a = \frac{2}{3}.$
(*c*) $f(x) = x^n, \quad a = 1.$
(*d*) $f(x) = \dfrac{1}{x}, \quad a = 1.$
(*e*) $f(x) = \sin x, \quad a = \dfrac{\pi}{4}.$

2. Determine the value of the twelfth derivative of

$$3x^9 + 2x^4 + x$$

at $x = \frac{4}{3}$.

Answers to Exercises

SECTION 2.8f, page 169

1. (*a*) $f'(x) = 2x - 1, f''(x) = 2, f^{(2+k)}(x) = 0;\ -1, 2, 0.$
(*b*) $f'(x) = 3x^2 + 3, f''(x) = 6x, f'''(x) = 6, f^{(3+k)}(x) = 0;\ 4\frac{1}{3}, 4, 6, 0.$

(*c*) $f^{(k)}(x) = \dfrac{n!\, x^{n-k}}{(n-k)!}, k \le n, f^{(n+k)}(x) = 0;\ \dfrac{n!}{(n-k)!}, 0.$

(*d*) $f^{(k)}(x) = \dfrac{(-1)^k k!}{x^{k+1}};\ \dfrac{(-1)^k}{k!}.$

(*e*) $f^{(4k)}(x) = \sin x, f^{(4k+1)}(x) = \cos x, f^{(4k+2)}(x) = -\sin x, f^{(4k+3)}(x) = -\cos x;\ \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}.$

2. 0.

EXERCISES

SECTION 2.8h, page 173

1. Find the intermediate value ξ of the mean value theorem for the following functions, and illustrate graphically:

(*a*) $2x$, (*b*) x^2, (*c*) $5x^3 + 2x$, (*d*) $\dfrac{1}{x^2+1}$, (*e*) $x^{1/3}$.

2. (*a*) Show that the mean value theorem fails for the following functions when the two points are taken with opposite signs, for example, $x_1 = -1$, $x_2 = 1$.

(1) $\dfrac{1}{x}$, (2) $|x|$, (3) $x^{2/3}$.

(*b*) Although the following functions may fail to be continuous or to have continuous derivatives, show nonetheless that there exist values ξ satisfying the mean value theorem on $[-1, 1]$.

(1) $$f(x) = \begin{cases} (x+1)^2 & \text{for } -1 \le x \le 0, \\ x^2 + 1 & \text{for } 0 \le x \le 1, \end{cases}$$

(2) $$f(x) = \begin{cases} (x+1)^2 & \text{for } -1 \le x \le 0, \\ x^2 & \text{for } 0 \le x \le 1. \end{cases}$$

3. Prove if $f'(x)$ is positive, then $f(x)$ can have at most one root.

4. (*a*) Show that if $f(x)$ has n real zeros [values of x for which $f(x) = 0$] in $[a, b]$, then $f'(x)$ has at least $n - 1$ real zeros.

(*b*) What conclusion can you draw about the maximum number of distinct real zeros for a polynomial of degree n?

Answers to Exercises

SECTION 2.8h, page 173

1. (*a*) ξ has any value.

(*b*) $\xi = \dfrac{x_1 + x_2}{2}$.

(*c*) $\xi = \sqrt{\dfrac{x_2^2 + x_2x_1 + x_1^2}{3}}$.

(*d*) $\dfrac{2\xi}{(\xi^2+1)^2} = \dfrac{x_1 + x_2}{(x_1^2+1)(x_2^2+1)}$.

(*e*) $\xi = \left(\dfrac{x_2^{2/3} + x_2^{1/3}x_1^{1/3} + x_1^{2/3}}{3}\right)^{3/2}$.

2. (*a*) For $x_1 = -1, x_2 = 1$,

(1) Slope of chord is 1, but $f'(x) = -1/x^2 \leq 0$.

(2) Slope of chord is 0, but for $x < 0, f'(x) = -1$, for $x > 0, f'(x) = 1$, and for $x = 0$ there is no derivative.

(3) Slope of chord is zero, but for $x \neq 0$, $|f'(x)| = \frac{2}{3}|x|^{-1/3}$ and $f'(0)$ not defined.

(*b*) [1] $\xi = \pm\frac{1}{2}$; [2] $\xi = -\frac{3}{4}, \frac{1}{4}$.

3. If $f(x)$ had two roots, then by Rolle's theorem for some value ξ between them, $f'(\xi) = 0$.

4. (*a*) If the roots are given by $x_1 < x_2 < \cdots < x_n$, then by Rolle's theorem there exist values ξ_k, where $x_k < \xi_k < x_{k+1}$ such that $f'(\xi_k) = 0$.

(*b*) Since the *n*th derivative of a polynomial of degree *n* is a nonzero constant, it follows that $f^{(n-1)}(x)$ can have at most one zero; hence, $f^{(n-2)}(x)$ can have at most two zeros, etc.

PROBLEMS

SECTION 2.8i, page 175

1. Show $x > \sin x$ for positive x and $x < \tan x$ for x in $\left(0, \frac{\pi}{2}\right)$.

2. If $f(x)$ is continuous and differentiable for $a \leq x \leq b$, show that if $f'(x) \leq 0$ for $a \leq x < \xi$ and $f'(x) \geq 0$ for $\xi < x \leq b$, the function is never less than $f(\xi)$.

***3.** If the continuous function $f(x)$ has a derivative $f'(x)$ at each point x in the neighborhood of $x = \xi$, and if $f'(x)$ approaches a limit L as $x \to \xi$, then $f'(\xi)$ exists and is equal to L.

***4.** Let $f(x)$ be defined and differentiable on the entire x-axis. Show that if $f(0) = 0$ and everywhere $|f'(x)| \leq |f(x)|$, then $f(x) = 0$ identically.

Solutions and Hints to Problems

SECTION 2.8i, page 175

1. We have for $f(x) = x - \sin x$,

$$f'(x) = 1 - \cos x \geq 0.$$

Consequently, f is nondecreasing and is strictly increasing in the interval

$(0, 2\pi)$. It follows that $f(x) > f(0) = 0$. For $g(x) = \tan x - x$ we have

$$g'(x) = 1 + \tan^2 x - 1 > 0, \qquad (0 < x < \tfrac{1}{2}\pi);$$

thus g is strictly increasing for $0 \leq x < \frac{1}{2}\pi$ and $g(x) > g(0) = 0$.

2. For $a \leq x < \xi$ we have by the mean value theorem, for some u between x and ξ,

$$f(\xi) - f(x) = f'(u)(\xi - x) \leq 0.$$

For $\xi < x \leq b$ we have, for some v between ξ and x,

$$f(x) - f(\xi) = f'(v)(x - \xi) \geq 0.$$

3. By the mean value theorem

$$\frac{f(x) - f(\xi)}{x - \xi} = f'(u),$$

where $|u - \xi| < |x - \xi|$. Taking the limit as $x \to \xi$, we obtain

$$f'(\xi) = L.$$

4. Applying the mean value theorem repeatedly, we have

$$\begin{aligned} |f(x)| &= |f'(x_1)| \cdot |x|, \qquad |x_1| \leq |x| \\ &\leq |f(x_1)| \cdot |x|, \\ &\leq |f'(x_2)x_1| \cdot |x|, \qquad |x_2| \leq |x_1| \\ &\leq |f'(x_2)| \cdot |x|^2 \\ &\leq |f(x_2)| \cdot |x|^2 \\ &\ldots\ldots\ldots\ldots\ldots \\ &\leq |f(x_n)| \cdot |x|^n, \qquad |x_n| \leq |x_{n-1}|. \end{aligned}$$

Since $f(x)$ is continuous for $|x| \leq 1$, it is bounded and therefore for $|x| < 1$ we have on passing to the limit as $n \to \infty$, $|f(x)| = 0$. By continuity we also have $f(x) = 0$ for $x = \pm 1$. The argument may now be repeated for the successive new origins $x = \pm 1, \pm 2, \ldots$ to extend the result to the entire real line.

EXERCISES

SECTION 2.9, page 184

From the differentiations performed in Exercises 2.8*a*, No. 2; 2.8*c*, Nos. 1, 2, 3, 4, and 2.8*h*, No. 1, derive the corresponding integration formulas.

PROBLEMS

SECTION 2.9, page 184

***1.** If a particle traverses distance 1 in time 1, beginning and ending at rest, then at some point in the interval it must have been subjected to an acceleration equal to 4 or more.

Solutions and Hints to Problems

SECTION 2.9, page 184

1. Let the acceleration be equal to a, where $|a| < 4$, and let the velocity be v. From $a < 4$ we have on integrating from 0 to t (where $0 < t \leq 1$)

$$v < 4t,$$

and from $a > -4$ we have on integrating from t to 1,

$$v < 4 - 4t.$$

For the total displacement we have

$$\int_0^1 v\,dt < \int_0^{1/2} 4t\,dt + \int_{1/2}^1 (4 - 4t)\,dt = 1.$$

Thus the displacement cannot be as much as 1.

PROBLEMS

SUPPLEMENT, page 192. Existence of the Definite Integral

1. Let $f(x)$ be defined and bounded on $[a, b]$. We define the upper sum Σ and lower sum σ for the subdivision

$$a = x_0 < x_1 < x_2 \cdots < x_n = b$$

to be

$$\Sigma = \sum_{i=1}^{n} M_i\,\Delta x_i, \qquad \sigma = \sum_{i=1}^{n} m_i\,\Delta x_i,$$

where M_i is the least upper bound and m_i the greatest lower bound of $f(x)$ in the cell $[x_{i-1}, x_i]$.

(*a*) Show that in any refinement of a subdivision the upper sum either decreases or remains unchanged and, similarly, the lower sum increases or remains unchanged.

(*b*) Prove that each upper sum is greater than or equal to every lower sum.

(*c*) The *upper Darboux integral* F^+ is defined as the greatest lower bound of the upper sums and the *lower Darboux integral* F^- as the least upper bound of the lower sums over all subdivisions. From (*b*), $F^+ \geq F^-$. If $F^+ = F^-$ we call the common value the *Darboux integral* of f. Prove that the Darboux integral of f is actually the ordinary Riemann integral; furthermore, show that the Riemann integral exists if and only if the upper and lower Darboux integrals exist and are equal.

2. Let $f(x)$ be a monotone function defined on $[a, b]$.

(*a*) Show that the difference between the upper and lower sums for a subdivision into n equal cells is given exactly by

$$\sum - \sigma = |f(b) - f(a)|\,(b - a)/n,$$

and explain this result geometrically.

(*b*) Use the result of (*a*) to prove that the Darboux integral exists.

(*c*) Estimate $\Sigma - \sigma$ in terms of $f(a), f(b)$ and the span of the subdivision if the cells of the subdivision may be unequal.

(*d*) Usually $f(x)$, if not monotone, can be written as the sum of monotone functions, $f(x) = \phi(x) + \psi(x)$, where ϕ is nonincreasing and ψ is nondecreasing. Estimate the difference between the upper and lower sums in that case.

3. Show that if $f(x)$ has a continuous derivative in the closed interval $[a, b]$, then $f(x)$ can be written as the sum of monotone functions as in Problem 2*d*.

Solutions and Hints to Problems

SUPPLEMENT TO CHAPTER 2, page 192

1. (*a*) Let $x_{i-1} = \xi_0 < \xi_1 < \cdots < \xi_k = x_i$ be those points of the refinement of the original subdivision which lie in the cell $[x_{i-1}, x_i]$ and let α_j and A_j be the infimum and supremum, respectively, of f in the subcell $[\xi_{j-1}, \xi_j]$. If m_i and M_i are the infimum and supremum of f in the original cell $[x_{i-1}, x_i]$, then they are lower and upper bounds for f in the subcell $[\xi_{i-1}, \xi_i]$ and we have

$$\sum_{j=1}^{k} \alpha_j(\xi_j - \xi_{j-1}) \geq \sum_{j=1}^{k} m_i(\xi_j - \xi_{j-1})$$
$$= m_i(x_i - x_{i-1}).$$

Similarly,

$$\sum_{j=1}^{k} A_j(\xi_j - \xi_{j-1}) \leq M_i(x_i - x_{i-1}).$$

Summing from $i = 1$ to n, we obtain the desired results.

(*b*) Let Σ_i and σ_i denote upper and lower sums, respectively, for the subdivision S_i. Let S_1 and S_2 be any subdivisions whatever and let S_3 be the joint subdivision consisting of all the subdividing points of S_1 and S_2 taken together. S_3 is a refinement of both S_1 and S_2; consequently, from the result of *a*,

$$\sum_1 \geq \sum_3 \geq \sigma_3 \geq \sigma_2.$$

(*c*) First we prove for any positive ϵ that there exists a subdivision S for which the upper sum Σ and lower sum σ satisfy

$$\sum - \sigma < \epsilon.$$

Since $F = F^+$ is the infimum of upper sums, there exists a subdivision S_1 for which

$$\sum_1 - F < \frac{\epsilon}{2},$$

and since $F = F^-$ is the supremum of lower sums, there exists an S_2 such that

$$F - \sigma_2 < \frac{\epsilon}{2}.$$

Let S be the joint subdivision of S_1 and S_2. Since it is a refinement of both S_1 and S_2, we may apply the result of b to obtain

$$\sum_1 \geq \sum \geq F \geq \sigma \geq \sigma_2,$$

whence

$$\frac{\epsilon}{2} \geq \sum_1 - F \geq \sum - F,$$

$$\frac{\epsilon}{2} \geq F - \sigma_2 \geq F - \sigma.$$

Adding, we obtain

(1)
$$\epsilon \geq \sum - \sigma.$$

Next we prove the inequality (1) holds not just for a particular subdivision but for any subdivision which is sufficiently fine; that is, we determine a $\delta(\epsilon)$ such that (1) is satisfied for every subdivision with a span smaller than $\delta(\epsilon)$. According to the preceding result there is a particular subdivision S^* of $[a, b]$ into ν cells with upper and lower sums Σ^* and σ^* such that

$$\sum^* - \sigma^* < \frac{\epsilon}{2}.$$

Now let S be any subdivision of $[a, b]$ and denote its span by Δ. The cells of S are of two kinds: those containing dividing points of S^* and those which do not. At most $\nu - 1$ dividing points of S^* may lie in the interior of cells of the first kind. If M and m are the supremum and infimum of f in the entire interval $[a, b]$, then clearly, for the difference between the upper sum Σ_1 and the lower sum σ_1 over cells of the first kind, we have

$$\sum_1 - \sigma_1 \leq (M - m)(\nu - 1)\Delta.$$

A cell of the second kind can have no points of S^* in its interior and therefore must lie entirely within one of the subdivisions of S^*. Summing over cells of the second kind we find therefore

$$\sum_2 - \sigma_2 \leq \sum^* - \sigma^* < \frac{\epsilon}{2}.$$

Consequently, for the difference between the upper and lower sums for S, $\Sigma = \Sigma_1 + \Sigma_2$ and $\sigma = \sigma_1 + \sigma_2$, respectively, we have

$$\sum - \sigma \leq \frac{\epsilon}{2} + (M - m)(\nu - 1)\Delta.$$

To make this difference smaller than ϵ it is sufficient to take $\Delta < \delta(\epsilon)$, where $\delta(\epsilon) = \epsilon/2(M - m)(\nu - 1)$.

Finally, for any subdivision, any Riemann sum F^* lies between the upper and lower Darboux sums. Thus

$$\sum \geq F^* \geq \sigma,$$

but also

$$\sum \geq F \geq \sigma,$$

where F is the Darboux integral. It follows that $|F^* - F| \leq \Sigma - \sigma$ and from the preceding result that F is the limit of Riemann sums. Thus we have proved if the Darboux integral exists, then the Riemann integral exists and is equal to the Darboux integral.

Conversely, if the Riemann integral exists, then the Darboux integral exists and is equal to the Riemann integral. The proof follows directly from the fact that the Darboux sums can be approximated arbitrarily closely by Riemann sums. In each cell $[x_{i-1}, x_i]$ of a subdivision there exist two points α_i and β_i such that $M_i - f(\alpha_i) < \epsilon/[4(b - a)]$ and $f(\beta_i) - m_i < \epsilon/[4(b - a)]$. For the Riemann sums $A = \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1})$ and $B = \sum_{i=1}^{n} f(\beta_i)(x_i - x_{i-1})$, we then have

$$\sum - A < \frac{\epsilon}{4}, \qquad B - \sigma < \frac{\epsilon}{4}.$$

Thus

$$\sum - \sigma < \frac{\epsilon}{2} + A - B.$$

We suppose that the subdivision has been chosen so fine that $|A - F^*| < \epsilon/4$ and $|B - F^*| < \epsilon/4$, where F^* is the Riemann integral. It follows that $|A - B| < \epsilon/2$ and therefore

$$\sum - \sigma < \epsilon.$$

Since

$$\sum \geq F^+ \geq F^- \geq \sigma$$

we see that

$$F^+ - F^- < \epsilon$$

for every positive ϵ and hence that the Darboux integral $F = F^+ = F^-$ exists. Furthermore, from $\Sigma - F < \epsilon$, $\Sigma - A < \epsilon/4$ and $|A - F^*| < \epsilon/4$ we conclude that $|F - F^*| < \frac{3}{2}\epsilon$ and hence that the Darboux integral is the Riemann integral.

2. (*a*) In the notation of Problem 1, if f is monotonically increasing, then $M_i = f(x_i)$, $m_i = f(x_{i-1})$, and

$$\sum - \sigma = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \frac{b-a}{n}$$

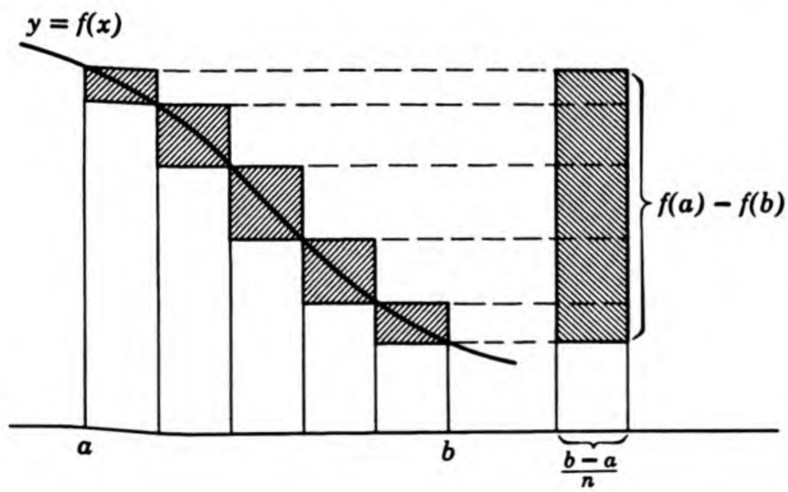
$$= [f(x_n) - f(x_0)] \frac{b-a}{n}$$

$$= [f(b) - f(a)] \frac{b-a}{n}.$$

Similarly, if f is monotonically decreasing,

$$\sum - \sigma = [f(a) - f(b)] \frac{b-a}{n}.$$

In the figure depicting a monotonically decreasing function, Σ is the total area of the outer rectangles, σ of the inner rectangles, and it is clear that $\Sigma - \sigma$ is precisely the area of the shaded rectangle on the right.



(*b*) Taking $n > \frac{|f(b) - f(a)|\,(b-a)}{\epsilon}$ above, we have

$$\sum - \sigma < \epsilon$$

and since $\Sigma \geq F^+ \geq F^- \geq \sigma$, we have $F^+ - F^- < \epsilon$, which yields the desired result.

(*c*) $\Sigma - \sigma \leq |f(b) - f(a)|\,\Delta$, where Δ is the span of the subdivision.
(*d*) $\Sigma - \sigma \leq \{[\psi(b) - \psi(a)] + [\phi(a) - \phi(b)]\}\,\Delta$.

3. $$f(x) = f(a) + \frac{1}{2}\int_a^x \{f'(t) + |f'(t)|\}\,dt$$
$$+ \frac{1}{2}\int_a^x \{f'(t) - |f'(t)|\}\,dt.$$

PROBLEMS

MISCELLANEOUS PROBLEMS, Chapter 2

1. Prove that

$$(a)\ \int_{-1}^{1}(x^2-1)^2\,dx = \tfrac{16}{15}, \qquad (b)\ (-1)^n\int_{-1}^{1}(x^2-1)^n\,dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}\,.$$

2. Prove for the binomial coefficient $\binom{n}{k}$ that

$$\binom{n}{k} = \left[(n+1)\int_0^1 x^k(1-x)^{n-k}\,dx\right]^{-1}$$

***3.** If $f(x)$ possesses a derivative $f'(x)$ (not necessarily continuous) at each point x of $a \le x \le b$, and if $f'(x)$ assumes the values m and M it also assumes every value μ between m and M.

4. If $f''(x) \ge 0$ for all values of x in $a \le x \le b$, the graph of $y = f(x)$ lies on or above the tangent line at any point $x = \xi$, $y = f(\xi)$ of the graph.

5. If $f''(x) \ge 0$ for all values of x in $u \le x \le b$, the graph of $y = f(x)$ in the interval $x_1 \le x \le x_2$ lies below the line segment joining the two points of the graph for which $x = x_1$, $x = x_2$.

6. If $f''(x) \ge 0$, then $f\left(\dfrac{x_1+x_2}{2}\right) \le \dfrac{f(x_1)+f(x_2)}{2}$.

***7.** Let $f(x)$ be a function such that $f''(x) \ge 0$ for all values of x and let $u = u(t)$ be an arbitrary continuous function. Then

$$\frac{1}{a}\int_0^a f[u(t)]\,dt \ge f\left(\frac{1}{a}\int_0^a u(t)\,dt\right).$$

8. (*a*) Differentiate directly and write down the corresponding integration formulae: (i) $x^{1/2}$, (ii) $\tan x$.

(*b*) Evaluate

$$\lim_{n\to\infty}\frac{1}{n}\left(1+\sec^2\frac{\pi}{4n}+\sec^2\frac{2\pi}{4n}+\cdots+\sec^2\frac{n\pi}{4n}\right).$$

9. Let $f(x)$ have first and second derivatives for all real values of x. Prove that if $f(x)$ is everywhere positive and concave, then $f(x)$ is constant.

Solutions and Hints to Problems

MISCELLANEOUS PROBLEMS, Chapter 2, page 199

1. (*b*) Since the integrand is an even function (See Problems 2.2, No. 4*b*), the integral equals

$$2(-1)^n \int_0^1 (x^2 - 1)^n\, dx = 2(-1)^n \int_0^1 \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^{2k}\, dx$$

$$= 2 \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{2k+1}$$

$$= 2\,\frac{4^n (n!)^2}{(2n+1)!},$$

where at the last step we use the result of Problems 1.5, No. 1*g*.

2. Set $n = j + k$, and

$$\lambda_j = \binom{j+k}{k}(j+k+1) \int_0^1 x^k (1-x)^j\, dx.$$

Clearly, $\lambda_0 = 1$. We prove that if $\lambda_\nu = 1$, then $\lambda_{\nu+1} = 1$, so that $\lambda_j = 1$ for all j. We have

$$\lambda_j = \binom{j+k}{k}(j+k+1) \int_0^1 \sum_{r=0}^{j} (-1)^r \binom{j}{r} x^{k+r}\, dx$$

or

$$\lambda_j = \binom{j+k}{k}(j+k+1) \sum_{r=0}^{j} \frac{(-1)^r \binom{j}{r}}{k+r+1}. \tag{1}$$

We rewrite this formula in the form

$$\lambda_j = \binom{j+k+1}{k} \sum_{r=0}^{j} (-1)^r \binom{j+1}{r} \frac{j-r+1}{k+r+1}. \tag{2}$$

On the other hand, replacing j by $j + 1$ in (1), we obtain

$$\lambda_{j+1} = \binom{j+k+1}{k} \sum_{r=0}^{j+1} (-1)^r \binom{j+1}{r} \frac{j+k+2}{k+r+1}. \tag{3}$$

Using (2) and (3) we form the difference

$$\lambda_{j+1} - \lambda_j = \binom{j+k+1}{k}\left\{\sum_{r=0}^{j}(-1)^r\binom{j+1}{r} + (-1)^{j+1}\right\}$$
$$= \binom{j+k+1}{k}(1-1)^{j+1} = 0.$$

It follows that $\lambda_{j+1} = \lambda_j$.

3. Let α and β be points of $[a, b]$, where $f'(x_1) = m$, $f'(x_2) = M$. Consider the continuous function

$$\phi(x) = \frac{f(x+h) - f(x)}{h}$$

for a fixed $h > 0$. If $m < \mu < M$, then it is possible to choose $\delta > 0$ so that

$$\left|\frac{f(x_1 + h_1) - f(x_1)}{h_1} - m\right| < \mu - m$$

and

$$\left|\frac{f(x_2 + h_2) - f(x_2)}{h_2} - M\right| < M - \mu$$

whenever $|h_1|$ and $|h_2|$ are less than δ. Choose $h = |h_1| = |h_2| < \delta$. Since

$$\frac{f(x_i + h_i) - f(x_i)}{h_i} = \begin{cases}\phi(x_i), & \text{for } h_i > 0\\ \phi(x_i - h), & \text{for } h_i < 0,\end{cases}$$

it follows that there exist points α, β in $[a, b]$ such that

$$\phi(\alpha) < \mu < \phi(\beta).$$

Since ϕ is continuous, we know by the intermediate value theorem that for some point ξ between α and β, $\phi(\xi) = \mu$. It follows by the mean value theorem, for some value η between ξ and $\xi + h$, that

$$f'(\eta) = \phi(\xi) = \mu.$$

4. Find the equation $y = g(x)$ of the tangent; apply the mean value theorem to $f'(x) - g'(x)$ and use the result of Problems 2.8i, No. 2.

5. Let $y = g(x)$ be the equation of the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ of the curve. For $h(x) = f(x) - g(x)$ we have $h''(x) = f''(x) \geq 0$. Furthermore, since $h(x_1) = h(x_2) = 0$, by Rolle's theorem $h'(\xi) = 0$ for some ξ in (x_1, x_2). Consequently, $h'(x) \leq 0$ for $x_1 \leq x \leq \xi$ and $h'(x) \geq 0$ for $\xi \leq x \leq x_2$. It follows that $h(x) \leq 0$ in $[x_1, \xi]$ and in $[\xi, x_2]$.

6. The midpoint of the chord in the preceding problem lies on or above the curve $y = f(x)$.

7. Let $\xi = \frac{1}{a}\int_0^a u(t)\,dt$. If $y = g(x)$ is the tangent to the graph of f at $(\xi, f(\xi))$ then, by Problem 4, $f(x) \geq g(x)$ for all values of x. Enter $x = u(t)$ in this inequality, and integrate (note that the result is independent of the sign of a).

8. $$\lim_{n\to\infty} \frac{4}{\pi}\sum_{k=0}^{n}\left[\sec^2\left(\frac{k\pi}{4n}\right)\right]\frac{\pi}{4n} = \frac{4}{\pi}\int_0^{\pi/4}\sec^2 x\,dx$$
$$= \frac{4}{\pi}\tan\frac{\pi}{4} = \frac{4}{\pi}.$$

9. If $f(x)$ is not constant, then there exist points x_1, x_2 such that $f(x_1) < f(x_2)$. Consider the case $x_1 < x_2$. By the mean value theorem, for some ξ between x_1 and x_2, we have

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

Since $f''(x) \leq 0$, it follows that $f'(x_1) \geq f'(\xi) > 0$, and for $x < x_1$, $f'(x) \geq f'(x_1)$. The curve $y = f(x)$ lies on or below the tangent line $y = f'(x_1)(x - x_1) + y_1$ which takes on negative values for x sufficiently to the left of x_1, namely

$$x < x_1 - \frac{y_1}{f'(x_1)}.$$

For such values of x, $f(x)$ must be negative, contrary to our assumption.

3 *Techniques of the Calculus*

EXERCISES

SECTION 3.1, page 201

1. Find the numerical values of all the derivatives of $x^5 - x^4$ at $x = 1$.

2. What is the numerical value of the eleventh derivative of

$$317x^9 - 202x^7 + 76 \text{ at } x = 13\tfrac{1}{2}?$$

3. Differentiate the following functions and write down the corresponding integral formulas:

(*a*) $ax + b$, (*b*) $25cx^7$,

(*c*) $a + 2bx + cx^2$, (*d*) $\dfrac{ax + b}{cx + d}$,

(*e*) $\dfrac{ax^2 + 2bx + c}{\alpha x^2 + 2\beta x + \gamma}$, (*f*) $\dfrac{1}{1 - x^2} - \dfrac{1}{1 + x^2}$,

(*g*) $\dfrac{(x^8 - \sqrt{8}x^4 + 4)(x^8 + \sqrt{8}x^4 + 4)}{x^{16} + 16}$.

4. Differentiate the following functions and write down the corresponding integral formulas.

(*a*) $2 \sin x \cos x$, (*b*) $\dfrac{1}{1 + \tan x}$,

(*c*) $x \tan x$, (*d*) $\dfrac{\sin x + \cos x}{\sin x - \cos x}$,

(*e*) $\dfrac{\sin x}{x}$.

5. Differentiate

$$\sec x = \frac{1}{\cos x} \quad \text{and} \quad \operatorname{cosec} x = \frac{1}{\sin x}.$$

6. Recalling that $\sec x = 1/\cos x$, $\operatorname{cosec} x = 1/\sin x$, find the derivatives indicated in the following:

(*a*) $\dfrac{d^2}{dx^2} \sec x$, (*b*) $\dfrac{d^3}{dx^3} \sec x \tan x$,

(*c*) $\dfrac{d^3}{dx^3} \operatorname{cosec} x$, (*d*) $\dfrac{d^4}{dx^4} \tan x \sin x$.

7. Differentiate

$$\frac{x^3 \sin x - x^5 \cos x}{x^2 \tan x}.$$

Evaluate:

8. $\int (ax + b)\, dx.$

9. $\int (ax^2 + 2bx + c)\, dx.$

10. $\int (9x^8 + 7x^6 + 5x^4 + 3x^2 + 1)\, dx.$

11. $\int \left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4}\right) dx.$

12. $\int \left(x^2 + \frac{1}{x^2}\right) dx.$

13. $\int \left(a \cos x + \frac{b}{\sin^2 x}\right) dx.$

14. $\int \left(3x + 7 \sin x + \frac{5}{x^3} - \frac{9}{\cos^2 x}\right) dx.$

15. $\int \sec x \tan x\, dx.$

16. Find the *n*th derivative of:
(*a*) $\sin x \sin 2x$,
(*b*) $\cos mx \sin kx$.

PROBLEMS

SECTION 3.1, page 201

1. Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.
(*a*) Calculate the polynomial $F(x)$ from the equation

$$F(x) - F'(x) = P(x).$$

*(*b*) Calculate $F(x)$ from the equation

$$c_0F(x) + c_1F'(x) + c_2F''(x) = P(x).$$

2. Find the limit as $n \to \infty$ of the absolute value of the *n*th derivative of $1/x$ at the point $x = 2$.

3. Prove if $f^{(n)}(x) = 0$, for all x, then f is a polynomial of degree at most $n - 1$, and conversely.

4. Determine the form of a rational function r for which

$$\lim_{x\to\infty} \frac{xr'(x)}{r(x)} = 0.$$

5. Prove by induction that the nth derivative of a product may be found according to the following rule (Leibnitz's rule):

$$\frac{d^n}{dx^n}(fg) = f\frac{d^ng}{dx^n} + \binom{n}{1}\frac{df}{dx}\frac{d^{n-1}g}{dx^{n-1}} + \binom{n}{2}\frac{d^2f}{dx^2}\frac{d^{n-2}g}{dx^{n-2}} + \cdots + \binom{n}{n-1}\frac{d^{n-1}f}{dx^{n-1}}\frac{dg}{dx} + \frac{d^nf}{dx^n}g.$$

Here $\binom{n}{1} = n$, $\binom{n}{2} = \dfrac{n(n-1)}{2!}$, etc., denote binomial coefficients.

6. Prove that $\displaystyle\sum_{i=1}^{n-1} ix^{i-1} = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}$.

Answers to Exercises

SECTION 3.1, page 201

1. $f'(1) = 1$, $f''(1) = 8$, $f'''(1) = 36$, $f^{iv}(1) = 96$, $f^{v}(1) = 120$, $f^{vi}(1) = 0$, $f^{vii}(1) = 0, \ldots$.

2. 0.

3. (*a*) a, (*b*) $175\,cx^6$, (*c*) $2(b + cx)$, (*d*) $\dfrac{ad - bc}{(cx + d)^2}$,

(*e*) $\dfrac{2x^2(a\beta - \alpha b) + 2x(a\gamma - \alpha c) + 2(b\gamma - \beta c)}{(\alpha x^2 + 2\beta x + \gamma)^2}$,

(*f*) $\dfrac{4x(1 + x^4)}{(1 - x^2)^2(1 + x^2)^2}$, (*g*) 0.

4. (*a*) $2\cos 2x$, (*b*) $\dfrac{-1}{1 + \sin 2x}$ (*c*) $\tan x + \dfrac{x}{\cos^2 x}$

(*d*) $\dfrac{-2}{(1 - \sin 2x)}$ (*e*) $-\dfrac{\sin x}{x^2} + \dfrac{\cos x}{x}$.

5. $\dfrac{d}{dx}\sec x = \sec x \tan x$,

$$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x.$$

6. (*a*) $\dfrac{d^2}{dx^2}\sec x = \dfrac{d}{dx}\sec x \tan x = 2\sec^3 x - \sec x.$

(*b*) From 6*a*,

$$\frac{d^3}{dx^3}\sec x \tan x = \frac{d^2}{dx^2}(2\sec^3 x - \sec x)$$
$$= \frac{d}{dx}(6\sec^3 x \tan x - \sec x \tan x)$$
$$= 24\sec^5 x - 20\sec^3 x + \sec x.$$

(*c*) $\operatorname{cosec} x \cot x - 6\operatorname{cosec}^3 x \cot x.$

(*d*) $\dfrac{d^4}{dx^4}\tan x \sin x = \dfrac{d^3}{dx^3}(\sec x \tan x + \sin x)$

Now applying part *b*, we obtain

$$24\sec^5 x - 20\sec^3 x + \sec x - \cos x.$$

7. $-x\sin x + \cos x + 3x^2 \sin x + x^3 \cos x - \dfrac{3x^2}{\sin x} + \dfrac{x^3 \cos x}{\sin^2 x}.$

8. $\dfrac{ax^2}{2} + bx.$

9. $\dfrac{ax^3}{3} + bx^2 + cx.$

10. $x^9 + x^7 + x^5 + x^3 + x.$

11. $-\dfrac{1}{x} - \dfrac{1}{2x^2} - \dfrac{1}{3x^3}.$

12. $\dfrac{x^3}{3} - \dfrac{1}{x}.$

13. $a\sin x - b\cot x.$

14. $\dfrac{3}{2}x^2 - 7\cos x - \dfrac{5}{2x^2} - 9\tan x.$

15. $\sec x.$

16. (*a*) $\dfrac{(-1)^m}{2}\{\cos x - 3^{2m}\cos 3x\}$, for $n = 2m$,

$\dfrac{(-1)^m}{2}\{3^{2m+1}\sin 3x - \sin x\}$, for $n = 2m + 1$.

(*b*) $\dfrac{(-1)^l}{2}[(m + k)^{2l}\sin(m + k)x - (m - k)^{2l}\sin(m - k)x]$,

for $n = 2l$;

$\dfrac{(-1)^l}{2}[(m + k)^{2l+1}\cos(m + k)x - (m - k)^{2l+1}\cos(m - k)x]$, for

$n = 2l + 1$.

Solutions and Hints to Problems

SECTION 3.1, page 201

1. Use the method of undetermined coefficients, equating the coefficients of like powers on the left and right. If $F(x) = \sum_{\nu=0}^{n} b_\nu x^\nu$, then

$$(a)\ b_k = a_k + (k+1)a_{k+1} + (k+1)(k+2)a_{k+2} + \cdots + [(k+1)(k+2)\cdots n]a_n$$

$$= \sum_{j=0}^{n-k} \frac{(k+j)!}{k!} a_{k+j}.$$

$$(b)\ b_n = \frac{a_n}{c_0},\ b_{n-1} = \frac{a_{n-1}}{c_0} - \frac{c_1}{c_0^{\,2}} n a_n,$$

$$b_{n-2} = \frac{a_{n-2}}{c_0} - \frac{c_1}{c_0^{\,2}}(n-1)a_{n-1} - \frac{c_0c_2 - c_1^{\,2}}{c_0^{\,3}} n(n-1)a_n,$$

etc.

2. By induction,

$$\frac{d^n}{dx^n}\left(\frac{1}{x}\right) = \frac{n!\,(-1)^n}{2^{n+1}},$$

$\lim n!/2^{n+1} = \infty$ (cf. Problems 1.6, No. 3).

3. *Proof by Induction.* The proposition is true for $n = 1$. Suppose it is true for $n = k$. If $f(x)$ is a polynomial of degree at most k, then $f'(x)$ is a polynomial of degree at most $k - 1$. We then have $f^{(k+1)}(x) = \dfrac{d^k}{dx^k} f'(x) = 0$. Conversely, if $f^{(k+1)}(x) = 0$ for all x, then $f^{(k)}(x) = C$, where C is constant. Consequently, for $g(x) = f(x) - Cx^k/k!$ we have $g^{(k)}(x) = 0$ for all x. We conclude that $g(x)$ is a polynomial of degree at most $k - 1$ and hence $f(x)$ is of degree at most k

4. Let $r(x) = \dfrac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0}$, where $a_n \neq 0$, $b_m \neq 0$.

$$\frac{xr'(x)}{r(x)} = \frac{(n-m)a_nb_mx^{n+m} + \cdots}{a_nb_mx^{n+m} + \cdots}.$$

Answer: numerator and denominator have the same degree.

5. *Proof by Induction.* The theorem is proved for $n = 1$. Suppose it is true for $n = k$. Then

$$\frac{d^{k+1}}{dx^{k+1}}f(x)g(x) = \frac{d}{dx}\left[\frac{d^k}{dx^k}f(x)g(x)\right]$$

$$= \frac{d}{dx}\sum_{\nu=0}^{k}\binom{k}{\nu}f^{(\nu)}(x)\,g^{(k-\nu)}(x)$$

$$= \sum_{\nu=0}^{k}\binom{k}{\nu}\left[f^{(\nu)}(x)g^{(k-\nu+1)}(x) + f^{(\nu+1)}(x)g^{(k-\nu)}(x)\right]$$

$$= f(x)g^{(k+1)}(x) + \sum_{\nu=1}^{k-1}\left[\binom{k}{\nu} + \binom{k}{\nu-1}\right]f^{(\nu)}(x)g^{(k-\nu+1)}(x) + f^{(k+1)}(x)g(x)$$

$$= \sum_{\nu=0}^{k+1}\binom{k+1}{\nu}f^{(\nu)}(x)g^{(k+1-\nu)}(x),$$

where we have used the addition property of the binomial coefficients (See Section 1.5) at the last step.

6. $$\sum_{i=1}^{n-1} ix^{i-1} = \frac{d}{dx}\sum_{i=0}^{n-1}x^i = \frac{d}{dx}\frac{x^n - 1}{x - 1}.$$

EXERCISES

SECTION 3.2, page 206

1. If $y = x^2/4$, $y = 16$ corresponds to $x = 8$. Find dy/dx for $x = 8$; solve $y = x^2/4$ for x and find dx/dy for $y = 16$, and show that the values of these derivatives are consistent with the rule for inverse functions.

2. Prove that

(*a*) $\arcsin \alpha + \arcsin \beta = \arcsin(\alpha\sqrt{1 - \beta^2} + \beta\sqrt{1 - \alpha^2})$.

(*b*) $\arcsin \alpha + \arcsin \beta = \arccos(\sqrt{1 - \alpha^2}\sqrt{1 - \beta^2} - \alpha\beta)$.

(*c*) $\arctan \alpha + \arctan \beta = \arctan \dfrac{\alpha + \beta}{1 - \alpha\beta}$.

Differentiate the expressions in Exercises 3–16 and write down the corresponding integral formula.

3. $\dfrac{\sqrt[n]{x}}{1 + x}$.

4. $\sqrt{x}\cos^2 x$.

5. $\dfrac{1 + \sqrt{x}}{1 - \sqrt{x}}$

6. $\dfrac{\sqrt[3]{x}}{1 - \tan x}$.

7. $(\text{arc sin } x)(\text{arc cos } x)$.

8. $\dfrac{1 + \text{arc tan } x}{1 - \text{arc tan } x}$.

9. $\dfrac{\text{arc sin } x}{\text{arc tan } x}$.

10. $5 \text{ arc cot } x + \dfrac{1}{\text{arc cos } x}$.

11. $x(\log x - 1)$.

12. $\log(x^3 e^{7x})$.

13. $e^x \log x$.

14. $e^x(a \cos x + b \sin x)$.

15. $x^n e^x$.

16. $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

17. Find the *n*th derivative of:

(*a*) $x^3 e^{ax}$,

(*b*) $(\log x)^2$,

(*c*) $e^x \cos 2x$,

(*d*) $(1 + x)^6 e^x$.

PROBLEMS

SECTION 3.2, page 206

1. Let $y = e^x(a \sin x + b \cos x)$. Show that y'' can be expressed as a linear combination of y and y', that is,

$$y'' = py' + qy,$$

where p and q are constants. Express all higher derivatives as linear combinations of y' and y.

***2.** Find the *n*th derivative of arc sin x at $x = 0$, and then of $(\text{arc sin } x)^2$ at $x = 0$.

Answers to Exercises

SECTION 3.2, page 206

1. $\dfrac{dy}{dx} = 4 = \dfrac{1}{dx/dy}$.

2. Verify identity of the derivatives with respect to β and equality when $\beta = 0$.

3. $\dfrac{\sqrt[n]{x}\{1 - (n - 1)x\}}{nx(1 + x)^2}$.

4. $\dfrac{\cos^2 x}{2\sqrt{x}} - 2\sqrt{x} \sin x \cos x$.

5. $\dfrac{1}{\sqrt{x}\,(1 - \sqrt{x})^2}$.

6. $\dfrac{(1 - \tan x) + 3x(1 + \tan^2 x)}{3x^{2/3}(1 - \tan x)^2}$.

7. $(\arccos x - \arcsin x)/\sqrt{1 - x^2}$.

8. $\dfrac{2}{(1 + x^2)(1 - \arctan x)^2}$.

9. $\dfrac{1}{\sqrt{1 - x^2}\arctan x} - \dfrac{\arcsin x}{(1 + x^2)(\arctan x)^2}$.

10. $-\dfrac{5}{1 + x^2} + \dfrac{1}{\sqrt{1 - x^2}(\arccos x)^2}$.

11. $\log x$.

12. $\dfrac{3}{x} + 7$.

13. $e^x\left(\dfrac{1}{x} + \log x\right)$.

14. $e^x[(a + b)\cos x - (a - b)\sin x]$.

15. $x^{n-1}e^x(x + n)$.

16. $\dfrac{4}{(e^x + e^{-x})^2}$.

17. (*a*) $e^{ax}[a^n x^3 + 3na^{n-1}x^2 + 3n(n - 1)a^{n-2}x + n(n - 1)(n - 2)a^{n-3}]$.

(*b*) $\dfrac{2(-1)^n(n - 1)!}{x^n}\left(\sum_{\nu=1}^{n-1}\dfrac{1}{\nu} - \log x\right)$.

(*c*) $$e^x\left[\left(\sum_{l=0}^{[n/2]}(-1)^l\binom{n}{2l}2^{2l}\right)\cos 2x + \left(\sum_{l=0}^{[n-1/2]}(-1)^{l+1}\binom{n}{2l + 1}2^{2l+1}\right)\sin 2x\right] = 5^{n/2}e^x\cos(2x + n\alpha)$$

where $\tan\alpha = 2$.

(*d*) $e^x \cdot \sum_{\nu=0}^{6}\nu!\binom{6}{\nu}\binom{n}{\nu}(1 + x)^{6-\nu}$.

Solutions and Hints to Problems

SECTION 3.2, page 206

1. $p = -q = 2$, $y^{(4n)} = (-4)^n y$, $y^{(4n+1)} = (-4)^n y'$, $y^{(4n+2)} = 2(-4^n) \times (y' - y)$, $y^{(4n+3)} = 2(-4)^n(y' - 2y)$.

2. Let $y = \text{arc}\sin x$. Then

$$\frac{d^n y}{dx^n} = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{\sqrt{(1-x^2)}}\right) = \frac{d^{n-2}}{dx^{n-2}}\left[\frac{x}{(1-x^2)^{3/2}}\right].$$

Apply Leibnitz's rule to this last expression.

$$\left.\frac{d^n y}{dx^n}\right|_{x=0} = (n-2)\frac{d^{n-3}}{dx^{n-3}}\left(\frac{1}{(1-x^2)^{3/2}}\right)_{x=0}$$
$$= 3\cdot(n-2)\frac{d^{n-4}}{dx^{n-4}}\left[\frac{x}{(1-x^2)^{5/2}}\right]_{x=0},$$

and continue the process

$$\left.\frac{d^n y}{dx^n}\right|_{x=0} = 1\cdot 3\cdot 5\cdots(2\nu-1)\cdot(n-2)(n-4)\cdots(n-2\nu+2)\frac{d^{n-2\nu}}{dx^{n-2\nu}}$$
$$\times\left.\left[\frac{x}{(1-x^2)^{(2\nu+1)/2}}\right]\right|_{x=0}$$

For n even,

$$\frac{d^n y}{dx^n} = 0;$$

for n odd, $n = 2m+1$,

$$\frac{d^n y}{dx^n} = 1^2\cdot 3^2\cdot 5^2\cdots(2m-1)^2.$$

Apply Leibnitz's rule to obtain

$$\left.\frac{d^{2m}}{dx^{2m}}(\text{arc}\sin x)^2\right|_{x=0}$$
$$= \sum_{k=0}^{m-1}[1^2\cdot 3^2\cdot 5^2\cdots(2k-1)^2]\cdot[1^2\cdot 3^2\cdot 5^2\cdots(2m-2k-3)^2]\binom{2m}{2k+1}$$

and

$$\left.\frac{d^{2m+1}}{dx^{2m+1}}(\text{arc}\sin x)^2\right|_{x=0} = 0.$$

EXERCISES

SECTION 3.3, page 217

Differentiate the following functions.

1. $(x+1)^3$. **2.** $(3x+5)^2$.

3. $(x^9 - 3x^6 - x^3)^5$. **4.** $\dfrac{1}{1+x}$.

5. $\dfrac{1}{1 - x^2}$.

6. $(ax + b)^n$ (n an integer).

7. $\dfrac{1}{x + \sqrt{(x^2 - 1)}}$.

8. $\sqrt{\dfrac{ax^2 + bx + c}{lx^2 + mx + n}}$.

9. $(\sqrt{(1 - x)^{2/3}})^5$.

10. $\sin^2 x$

11. $\sin (x^2)$.

12. $\sqrt{1 + \sin^2 x}$.

13. $x^2 \sin \dfrac{1}{x^2}$.

14. $\tan \dfrac{1 + x}{1 - x}$.

15. $\sin (x^2 + 3x + 2)$.

16. $\arcsin (3 + x^3)$.

17. $\arcsin (\cos x)$.

18. $\sin (\arccos \sqrt{1 - x^2})$.

19. $x^{\sqrt{2}} - x^{-\sqrt{2}}$.

20. $[\sin (x + 7)]^{\sqrt[3]{5}}$.

21. $[\arcsin (a \cos x + b)]^{\alpha}$.

22. $e^{\tan^2 x + \log \sin x}$.

23. $(x + 2)^4(1 - x^2)^{1/3}(x^2 + 1)^{5/7}$.

24. (*a*) $x^{\sin x}$, (*b*) $(\cos x)^{\tan x}$.

PROBLEMS

SECTION 3.3, page 217

1. Find the second derivative of $f[g\{h(x)\}]$.

2. Differentiate the following function: $\log_{v(x)} u(x)$, [that is, the logarithm of $u(x)$ to the base $v(x)$; $v(x) > 0$].

3. What conditions must the coefficients α, β, a, b, c satisfy in order that

$$\frac{\alpha x + \beta}{\sqrt{(ax^2 + 2bx + c)}}$$

shall everywhere have a finite derivative that is never zero?

4. Show that $[d^n(e^{x^2/2})]/dx^n = u_n(x)e^{x^2/2}$, where $u_n(x)$ is a polynomial of degree n. Establish the recurrence relation

$$u_{n+1} = xu_n + u_n'.$$

***5.** By applying Leibnitz's rule to

$$\frac{d}{dx}(e^{x^2/2}) = xe^{x^2/2},$$

obtain the recurrence relation

$$u_{n+1} = xu_n + nu_{n-1}.$$

***6.** By combining the recurrence relations of Problems 4 and 5, obtain the differential equation

$$u_n'' + xu_n' - nu_n = 0$$

satisfied by $u_n(x)$.

7. Find the polynomial solution

$$u_n(x) = x^n + a_1x^{n-1} + \cdots + a_n$$

of the differential equation $u_n'' + xu_n' - nu_n = 0$.

***8.** If $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2 - 1)^n$, prove the relations

(*a*) $P'_{n+1} = \dfrac{x^2 - 1}{2(n + 1)} P_n'' + \dfrac{(n + 2)x}{n + 1} P_n' + \dfrac{n + 2}{2} P_n.$

(*b*) $P'_{n+1} = xP_n' + (n + 1)P_n.$

(*c*) $\dfrac{d}{dx}[(x^2 - 1)P_n'] - n(n + 1)P_n = 0.$

9. Find the polynomial solution

$$P_n = \frac{(2n)!}{2^n(n!)^2}(x^n + a_1x^{n-1} + \cdots + a_n)$$

of the differential equation

$$\frac{d}{dx}[(x^2 - 1)P_n'] - n(n + 1)P_n = 0.$$

10. Determine the polynomial $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2 - 1)^n$ by using the binomial theorem.

***11.** Let $\lambda_{n,p}(x) = \dbinom{p}{n} x^n(1 - x)^{p-n}$, $n = 0, 1, 2, \ldots, p$. Show that

$$1 = \sum_{n=0}^{p} \lambda_{n,p}(x).$$

$$x^k = \sum_{n=k}^{p} \frac{\dbinom{n}{k}}{\dbinom{p}{k}} \lambda_{n,p}(x).$$

. . . .

$$x^p = \lambda_{p,p}(x).$$

Answers to Exercises

SECTION 3.3, page 217

1. $3(x+1)^2$. **2.** $6(3x+5)$. **3.** $15x^{14}(3x^6-6x^3-1)(x^6-3x^3-1)^4$.

4. $\dfrac{-1}{(1+x)^2}$ **5.** $\dfrac{2x}{(1-x^2)^2}$. **6.** $an(ax+b)^{n-1}$.

7. $-\dfrac{1}{\sqrt{x^2-1}(x+\sqrt{x^2-1})}$.

8. $\dfrac{x^2(am-bl)+2x(an-cl)+(bn-cm)}{2\sqrt{\{(ax^2+bx+c)(lx^2+mx+n)^3\}}}$.

9. $-\frac{5}{3}(1-x)^{2/3}$. **10.** $\sin 2x$. **11.** $2x\cos(x^2)$.

12. $\dfrac{\sin x\cos x}{\sqrt{(1+\sin^2 x)}}$. **13.** $2\left(x\sin\dfrac{1}{x^2}-\dfrac{1}{x}\cos\dfrac{1}{x^2}\right)$.

14. $\dfrac{2}{(1-x)^2\cos^2\left(\dfrac{1+x}{1-x}\right)}$. **15.** $(2x+3)\cos(x^2+3x+2)$.

16. $\dfrac{3x^2}{\sqrt{\{1-(3+x^3)^2\}}}$. **17.** -1. **18.** 1.

19. $\dfrac{\sqrt{2}}{x}(x^{\sqrt{2}}+x^{-\sqrt{2}})$. **20.** $\sqrt[3]{5}\cos(x+7)\{\sin(x+7)\}^{\sqrt[3]{5}-1}$.

21. $\dfrac{a\alpha x}{\sqrt{\{1-(ax+b)^2\}}}\cdot\{\text{arc}\sin(a\cos x+b)\}^{\alpha-1}$

22. $\left(\dfrac{2\tan x}{\cos^2 x}+\cot x\right)e^{\tan^2 x+\log\sin x}$.

23. $4(x+2)^3(x^2+1)^{5/7}\sqrt[3]{(1-x^2)}-\dfrac{2x}{3\sqrt[3]{(1-x^2)^2}}(x+2)^4(x^2+1)^{5/7}$

$+\frac{10}{7}x(x^2+1)^{-2/7}(x+2)^4\sqrt[3]{(1-x^2)}$.

24. (*a*) $x^{\sin x}\left(\dfrac{\sin x}{x}+\log x\cdot\cos x\right)$.

(*b*) $(\cos x)^{\tan x}\left(-\tan^2 x+\dfrac{\log\cos x}{\cos^2 x}\right)$.

Solutions and Hints to Problems

SECTION 3.3, page 217

1. $f''[g\{h(x)\}]g'^2\{h(x)\}h'^2(x) + f'[g\{h(x)\}]g''\{h(x)\}h'^2(x)$
$+f'[g\{h(x)\}]g'\{h(x)\}h''(x)$.

2. Use $\log_v u = \dfrac{\log u}{\log v} : \dfrac{u'(x)}{u(x)\log v(x)} - \dfrac{v'(x)\log u(x)}{v(x)\{\log v(x)\}^2}$.

3. Neither the numerator nor the denominator of the derivative may vanish for any value of x. The derivative is

$$\frac{(\alpha b - \beta a)x + \alpha c - b\beta}{(ax^2 + 2bx + c)^{3/2}}.$$

The conditions are either $\alpha b - \beta a = 0$, $\alpha c \neq b\beta$, $a > 0$ and $ac - b^2 > 0$ or $a = 0$, $b = 0$, $c > 0$ $\alpha \neq 0$.

4. Verify directly that $u_0 = 1$, $u_1 = x$ and $u_2 = (x^2 + 1)$. If $u_k(x)$ is a polynomial of degree k, then

$$\begin{aligned}\frac{d^{k+1}(e^{(1/2)x^2})}{dx^{k+1}} &= \frac{d}{dx}[u_k(x)e^{(1/2)x^2}]\\ &= [xu_k(x) + u_k'(x)]e^{(1/2)x^2}\\ &= u_{k+1}(x)e^{(1/2)x^2}.\end{aligned}$$

Thus u_{k+1} is a polynomial of degree $k + 1$ and satisfies the stated recurrence relation.

5. By Leibnitz's rule

$$\frac{d^{n+1}}{dx^{n+1}}(e^{x^2/2}) = \frac{d^n}{dx^n}(xe^{x^2/2}) = x\frac{d^n}{dx^n}(e^{x^2/2}) + n\frac{d^{n-1}}{dx^{n-1}}(e^{x^2/2}).$$

6. From the two recurrence relations we have $u_n' = nu_{n-1}$. Differentiate either recurrence relation and employ this relation to obtain the differential equation.

7. For the coefficient a_ν of $x^{n-\nu}$ in u_n, the equation $u_n'' + xu_n' - nu_n = 0$ yields

$$(n - \nu)(n - \nu - 1)a_\nu - (\nu + 2)a_{\nu+2} = 0.$$

Thus

$$u_n = x^n + \frac{n(n-1)}{2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8}x^{n-4} + \cdots.$$

8. Apply Leibnitz's rule to

(*a*) $\dfrac{d^{n+2}}{dx^{n+2}}(x^2 - 1)^{n+1} = \dfrac{d^{n+2}}{dx^{n+2}}[(x^2 - 1)\cdot(x^2 - 1)^n]$,

(*b*) $\dfrac{d^{n+2}}{dx^{n+2}}(x^2 - 1)^{n+1} = \dfrac{d^{n+1}}{dx^{n+1}}[2(n + 1)x\cdot(x^2 - 1)^n]$.

(*c*) Equate the two expressions for P'_{n+1} in (*a*) and (*b*).

9. For the coefficient a_ν of $x^{n-\nu}$ the differential equation yields

$$a_{\nu+2} = -\frac{(n-\nu)(n-\nu-1)}{(\nu+2)(2n-\nu-1)}a_\nu$$

whence

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4}\cdots\right]$$

$$= \frac{1}{2^n n!}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\frac{(2n-2k)!}{(n-2k)!}$$

10. Same as solution of Problem 9.

11. By the binomial theorem, $\sum_{n=0}^{p}\lambda_{n,p}(x) = [x + (1 - x)]^p = 1$. Also, differentiating

$$(a + x)^p = \sum_{n=0}^{p}\binom{p}{n}a^{p-n}x^n$$

k times, we have

$$\binom{p}{k}(a + x)^{p-k} = \sum_{n=k}^{p}\binom{p}{n}\binom{n}{k}a^{p-n}x^{n-k}.$$

Multiplying by x^k and putting $a = 1 - x$, we have

$$\binom{p}{k}x^k = \sum_{n=k}^{p}\binom{n}{k}\binom{p}{n}(1 - x)^{p-n}x^n = \sum_{n=k}^{p}\binom{n}{k}\lambda_{n,p}(x).$$

EXERCISES

SECTION 3.4, page 223

1. A quantity of radium weighs 1 gm at time $t = 0$. At time $t = 10$ (years) it has diminished to 0.997 gm. After what time will it have diminished to 0.5 gm?

2. Under favorable conditions, bacterial cultures grow at a rate in proportion to their size. Assuming the size of the culture increases by one-third in an hour, find the instantaneous rate of change of the culture. Estimate the time it would take to increase the size of the culture one-thousandfold.

3. Solve the following differential equations:

(*a*) $y' = \alpha(y - \beta)$, (*b*) $y' - \alpha y = \beta$,

(*c*) $y' - \alpha y = \beta e^{\alpha x}$, (*d*) $y' - \alpha y = \beta e^{\gamma x}$, $\gamma \neq \alpha$.

PROBLEMS

SECTION 3.4, page 223

1. The function $f(x)$ satisfies the equation

$$f(x + y) = f(x)f(y).$$

(*a*) If $f(x)$ is differentiable, either $f(x) \equiv 0$ or $f(x) = e^{ax}$.

*(*b*) If $f(x)$ is continuous, either $f(x) \equiv 0$ or $f(x) = e^{ax}$.

2. If a differentiable function $f(x)$ satisfies the equation

$$f(xy) = f(x) + f(y),$$

then $f(x) = \alpha \log x$.

3. Prove that if $f(x)$ is continuous and

$$f(x) = \int_0^x f(t)\, dt,$$

then $f(x)$ is identically zero.

Answers to Exercises

SECTION 3.4, page 223

1. $-10 \log 2/\log 0.997 = 2.31 \times 10^3$ years.

2. For a culture of size ν

$$\frac{d\nu}{dt} = ae^{at}, \qquad a = \log\frac{4}{3}.$$

Twenty-seven hours.

3. (*a*) $y = \beta + ce^{\alpha x}$; (*b*) $y = -\dfrac{\beta}{\alpha} + ce^{\alpha x}$, $\alpha \neq 0$; $y = \beta x + c$, $\alpha = 0$;

(*c*) $y = \beta x e^{\alpha x} + ce^{\alpha x}$; (*d*) $y = \dfrac{\beta}{\gamma - \alpha} e^{\gamma x} + ce^{\alpha x}$

Solutions and Hints to Problems

SECTION 3.4, page 223

1. (*a*) We have

$$\frac{d}{dy}f(x+y)\bigg|_{y=0} = f'(x) = f(x)f'(0).$$

Consequently,

$$f(x) = Ce^{\alpha x},$$

where $C = f(0)$. In order to satisfy the given functional relation for $x = 0$, $y = 0$, we have $C = C^2$. Consequently, either $C = 0$ or $C = 1$.

(*b*) From $f(x) = f(x/2)^2$, it follows that $f(x) \geq 0$ for all x. If $f(x_0) = 0$ for any x_0, then $f(x) = f(x - x_0)f(x_0) = 0$ implies that $f(x)$ is identically zero. Otherwise $f(x)$ is everywhere positive as we assume henceforth.

Since $f(0) = f(0)^2$, we have $f(0) = 1$. Consequently, from $f(x - x) = f(0)$ we have

$$f(-x) = \frac{1}{f(x)}\,.$$

For any natural number n we have

$$f(nx) = f[(n-1)x]f(x),$$

from which it follows that

$$f(xn) = f(x)^n$$

and combining this with the results above we see that this relation holds for all integers n, no matter what sign. In particular, taking $x = 1$, we have

$$f(n) = a^n,$$

where $a = f(1)$. For a rational number p/q we have

$$f\left(\frac{p}{q}\right) = f\left(\frac{1}{q}\right)^p$$

and, using $f\left(\frac{1}{q}\right)^q = f(1) = a$,

$$f\left(\frac{p}{q}\right) = a^{p/q}.$$

Finally, since f is continuous and a^x is the continuous extension of a^r from the domain of rational values r to all real values x,

$$f(x) = a^x.$$

2. Take the derivative with respect to y at $y = 1$.

3. Since $f'(x) = f(x)$, we have $f(x) = Ce^x$; but $f(0) = 0$. (Compare the solution of Problem 2.3, No. 2.)

EXERCISES

SECTION 3.5, page 228

1. Given the value of one of the following, $\sinh x$, $\cosh x$, $\tanh x$, determine the possible values of the others.

(*a*) $\cosh x = \frac{5}{3}$.

(*b*) $\sinh x = -\frac{5}{12}$.

(*c*) $\tanh x = \frac{12}{13}$.

2. Derive the following hyperbolic identities.

(*a*) $\sinh 2x = 2 \sinh x \cosh x$.

(*b*) $\cosh 2x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$.

(*c*) $\tanh \dfrac{x}{2} = \dfrac{\cosh x - 1}{\sinh x}$.

(*d*) $\tanh (a + b) = \dfrac{\tanh a + \tanh b}{1 + \tanh a \tanh b}$.

(*e*) $\cosh x = \dfrac{1 + \tanh^2 \frac{1}{2}x}{1 - \tanh^2 \frac{1}{2}x}$

(*f*) $\sinh x = \dfrac{2 \tanh \frac{1}{2}x}{1 - \tanh^2 \frac{1}{2}x}$.

3. Prove $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.

PROBLEMS

SECTION 3.5, page 228

1. Prove the formula

$$\sinh a + \sinh b = 2 \sinh \left(\frac{a + b}{2}\right) \cosh \left(\frac{a - b}{2}\right).$$

Obtain similar formulas for $\sinh a - \sinh b$, $\cosh a + \cosh b$, $\cosh a - \cosh b$.

2. Express $\tanh (a \pm b)$ in terms of $\tanh a$ and $\tanh b$.

Express $\coth (a \pm b)$ in terms of $\coth a$ and $\coth b$.

Express $\sinh \frac{1}{2}a$ and $\cosh \frac{1}{2}a$ in terms of $\cosh a$.

3. Differentiate

(*a*) $\cosh x + \sinh x$, (*b*) $e^{\tanh x + \coth x}$,
(*c*) $\log \sinh (x + \cosh^2 x)$, (*d*) $\operatorname{ar} \cosh x + \operatorname{ar} \sinh x$, (*e*) $\operatorname{ar} \sinh (\alpha \cosh x)$,
(*f*) $\operatorname{ar} \tanh (2x/(1 + x^2))$.

4. Calculate the area bounded by the catenary $y = \cosh x$, the ordinates $x = a$ and $x = b$, and the x-axis.

Answers to Exercises

SECTION 3.5, page 228

1. (*a*) $\sinh x = \pm\frac{4}{3}$, etc.
(*b*) $\cosh x = \frac{13}{12}$, etc.
(*c*) $\cosh x = \frac{13}{5}$, $\sinh x = \frac{12}{5}$, etc.

2. (*a*)
$$2 \sinh x \cosh x = \tfrac{1}{2}(e^x - e^{-x})(e^x + e^{-x}) = \tfrac{1}{2}(e^{2x} - e^{-2x}) = \sinh 2x.$$

The hyperbolic identities, like the corresponding trigonometric identities, may also be derived from the addition theorems.

3. Proceed by induction. True for $n = 1$. If true for $n = k$ then

$$\begin{aligned}(\cosh x + \sinh x)^{k+1} &= (\cosh x + \sinh x)(\cosh kx + \sinh kx)\\ &= (\cosh x \cosh kx + \sinh x \sinh kx)\\ &\quad + (\cosh x \sinh kx + \sinh x \cosh kx)\\ &= \cosh (k + 1)x + \sinh (k + 1)x.\end{aligned}$$

Solutions and Hints to Problems

SECTION 3.5, page 228

1. Apply the addition theorems to the right side.

$$\sinh a - \sinh b = 2 \cosh \left(\frac{a + b}{2}\right) \sinh \left(\frac{a - b}{2}\right).$$

$$\cosh a + \cosh b = 2 \cosh \left(\frac{a + b}{2}\right) \cosh \left(\frac{a - b}{2}\right).$$

$$\cosh a - \cosh b = 2 \sinh \left(\frac{a + b}{2}\right) \sinh \left(\frac{a - b}{2}\right).$$

2. $\tanh(a \pm b) = \dfrac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b}$

$\coth(a \pm b) = \dfrac{1 \pm \coth a \coth b}{\coth a \pm \coth b}$.

$\sinh \frac{1}{2}a = \sqrt{\left(\dfrac{\cosh a - 1}{2}\right)}$; $\cosh \frac{1}{2}a = \sqrt{\left(\dfrac{\cosh a + 1}{2}\right)}$.

3. (*a*) $\sinh x + \cosh x$, (*b*) $-8\,\dfrac{e^{\tanh x + \coth x}}{\cosh 4x - 1}$,

(*c*) $(1 + \sinh 2x)\coth(x + \cosh^2 x)$, (*d*) $\dfrac{1}{\sqrt{x^2 - 1}} + \dfrac{1}{\sqrt{x^2 + 1}}$

(*e*) $\dfrac{\alpha \sinh x}{\sqrt{\alpha^2 \cosh^2 x + 1}}$, (*f*) $\dfrac{2}{(1 - x^2)}$.

4. $\sinh b - \sinh a$.

EXERCISES

SECTION 3.6, page 236

1. Find the maxima, minima, and points of inflection of the following functions. Graph them and determine the regions of increase and decrease, and of convexity and concavity:

(*a*) $x^3 - 6x + 2$, (*b*) $x^{2/3}(1 - x)$,

(*c*) $\dfrac{2x}{1 + x^2}$, (*d*) $\dfrac{x^3}{x^4 + 1}$,

(*e*) $\sin^2 x$.

2. Which point of the hyperbola $y^2 - \frac{1}{2}x^2 = 1$ is nearest to the point $x = 0, y = 3$?

3. Let P be a fixed point with coordinates x_0, y_0 in the first quadrant of a rectangular coordinate system. Find the equation of the line through P such that the length intercepted between the axes is a minimum.

4. A statue 12 ft high stands on a pillar 15 ft high. At what distance must a man whose eyes are at an elevation of 6 feet be in order that the statue may subtend the greatest possible angle at his eye?

5. Two sources of light, of intensities a and b, are at a distance d apart. At which point of the line joining them is the illumination least? (Assume that the illumination is proportional to the intensity and inversely proportional to the square of the distance.)

6. Of all rectangles with a given area, find
(*a*) The one with the smallest perimeter.
(*b*) The one with the shortest diagonal.

7. In the ellipse $x^2/a^2 + x^2/b^2 = 1$ enscribe the rectangle of greatest area.

8. Two sides of a triangle are a and b. Determine the third side so that the area is a maximum.

9. A circle of radius r is divided into two segments by a line g at a distance h from the center. In the smaller of these segments inscribe the rectangle of greatest possible area.

10. Of all circular cylinders with a given volume, find the one with the least area.

11. A dairy farmer wishes to enclose a rectangular pasture using 3000 ft of available fencing for three sides and an already existent 1000 ft stone wall for the fourth side. How can this be done so as to maximize the area?

12. A swimmer wishes to reach a particular point along the shore in the least possible time. Assuming the shore to be straight and that he swims at constant speed, then runs along shore at a faster constant speed, find the path of least time (discuss all possible cases).

13. For what bases a can the inequality $x < \log_a x$ be satisfied?

PROBLEMS

SECTION 3.6, page 236

1. Determine the maxima, minima, and points of inflection of $x^3 + 3px + q$. Discuss the nature of the roots of $x^3 + 3px + q = 0$.

2. Given the parabola $y^2 = 2px$, $p > 0$, and a point $P(x = \xi,\ y = \eta)$ within it ($\eta^2 < 2p\xi$), find the shortest path (consisting of two line segments) leading from P to a point Q on the parabola and then to the focus $F(x = \frac{1}{2}p,\ y = 0)$ of the parabola. Show that the angle FQP is bisected by the normal to the parabola, and that QP is parallel to the axis of the parabola (principle of the parabolic mirror).

3. Among all triangles with given base and given vertex angle, the isosceles triangle has the maximum area.

4. Among all triangles with given base and given area, the isosceles triangle has the maximum vertex angle.

***5.** Among all triangles with given area, the equilateral triangle has the least perimeter.

***6.** Among all triangles with given perimeter the equilateral triangle has the maximum area.

***7.** Among all triangles inscribed in a circle the equilateral triangle has the maximum area.

8. Prove that if $p > 1$ and $x > 0$, $x^p - 1 \geq p(x - 1)$.

9. Prove the inequality $1 > (\sin x)/x \geq 2/\pi$, $0 \leq x \leq \pi/2$.

10. Prove that (*a*) $\tan x \geq x$, $0 \leq x < \pi/2$.

(*b*) $\cos x \geq 1 - x^2/2$.

***11.** Given $a_1 > 0, a_2 > 0, \ldots, a_n > 0$, determine the minimum of

$$\frac{\dfrac{a_1 + \cdots + a_{n-1} + x}{n}}{\sqrt[n]{a_1 a_2 \cdots a_{n-1} x}}$$

for $x > 0$. Use the result to prove by mathematical induction that (cf. Problem 13, page 109)

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

12. (*a*) Given n fixed numbers $a_1, \ldots, a_n$, determine x so that $\sum_{i=1}^{n} (a_i - x)^2$ is a minimum.

*(*b*) Minimize $\sum_{i=1}^{n} |a_i - x|$.

*(*c*) Minimize $\sum_{i=1}^{n} \lambda_i |a_i - x|$, where $\lambda_i > 0$.

13. Sketch the graph of the function

$$y = (x^2)^x, \; y(0) = 1.$$

Show that the function is continuous at $x = 0$. Has the function maxima, minima, or points of inflection?

***14.** Find the least value α such that

$$\left(1 + \frac{1}{x}\right)^{x+\alpha} > e$$

for all positive x. (*Hint:* It is known that $[1 + (1/x)]^{x+1}$ decreases monotonically and $[1 + (1/x)]^x$ increases monotonically to the limit e at infinity.)

***15.** (*a*) Find the point such that the sum of the distances to the three sides of a triangle is a minimum.

(*b*) Find the point for which the sum of the distances to the vertices is a minimum.

16. Prove the following inequalities:

(*a*) $e^x > 1/(1 + x)$, $x > 0$.

(*b*) $e^x > 1 + \log(1 + x)$, $x > 0$.

(*c*) $e^x > 1 + (1 + x)\log(1 + x)$, $x > 0$.

17. Suppose $f''(x) < 0$ on (a, b). Prove:

(*a*) Every arc of the graph within the interval lies above the chord joining its end points.

(*b*) The graph lies below the tangent at any point within (a, b).

***18.** Let f be a function possessing a second derivative on (a, b).

(*a*) Show that either condition *a* or *b* of Problem 17 is sufficient for $f''(x) \leq 0$.

(*b*) Show that the condition

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$$

for all x and y in (a, b) is sufficient for $f''(x) \leq 0$.

***19.** Let a, b be two positive numbers, p and q any nonzero numbers $p < q$. Prove that

$$\frac{[\theta a^p + (1-\theta)b^p]^{1/p}}{[\theta a^q + (1-\theta)b^q]^{1/q}} \leq 1$$

for all values of θ in the interval $0 < \theta < 1$.

(This is Jensen's inequality, which states that the pth power mean $[\theta a^p + (1-\theta)b^p]^{1/p}$ of two positive qualities a, b is an increasing function of p.)

20. Show that the equality sign in the above inequality holds if, and only if, $a = b$.

21. Prove that $\lim\limits_{p\to 0} [\theta a^p + (1-\theta)b^p]^{1/p} = a^\theta b^{1-\theta}$.

22. Defining the zeroth power mean of a, b as $a^\theta b^{1-\theta}$, show that Jensen's inequality applies to this case, and becomes $(a \neq b)$,

$$a^\theta b^{1-\theta} \gtrless [\theta a^q + (1-\theta)b^q]^{1/q} \quad \text{according to whether} \quad q \lessgtr 0.$$

For $q = 1$,
$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b.$$

23. Prove the inequality

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b,$$

$a, b > 0$, $0 < \theta < 1$, without reference to Jensen's inequality, and show that equality holds only if $a = b$. (This inequality states that the θ, $1 - \theta$ geometric mean is less than the corresponding arithmetic mean.)

***24.** Let f be continuous and positive on $[a, b]$ and let M denote its maximum value. Prove

$$M = \lim_{n\to\infty} \sqrt[n]{\int_a^b [f(x)]^n\,dx}.$$

Answers to Exercises

SECTION 3.6, page 236

1. (*a*) Maximum for $x = -\sqrt{2}$, minimum for $x = \sqrt{2}$, inflection for $x = 0$.
(*b*) Maximum for $x = \frac{2}{5}$, minimum for $x = 0$, inflection for $x = -\frac{1}{5}$.
(*c*) Maximum for $x = 1$, minimum for $x = -1$, inflection for $x = 0$, $\pm\sqrt{3}$.
(*d*) Maximum for $x = \sqrt[4]{3}$, minimum for $x = -\sqrt[4]{3}$, inflection for $x = 0$, $\pm\sqrt[4]{6 \pm \sqrt{33}}$.
(*e*) Maximum for $x = (n + \frac{1}{2})\pi$, minimum for $x = n\pi$, inflection for $x = \dfrac{2n + 1}{4}\pi$.

2. The point (0, 1).

3. Equation of line is $(y - y_0)/(x - x_0) = -\sqrt[3]{(y_0/x_0)}$.

4. $\sqrt{189}$ ft.

5. The point dividing the line *ab* in the ratio $\sqrt[3]{a}$: $\sqrt[3]{b}$.

6. The square.

7. The rectangle with corners $x = \pm a/\sqrt{2}$, $y = \pm b/\sqrt{2}$.

8. The right-angled triangle, that is $c^2 = a^2 + b^2$.

9. The side of rectangle opposite to g must be at the distance $\frac{1}{4}\{\sqrt{(8r^2 + h^2)} + h\}$ from the center.

10. The cylinder whose height is equal to the diameter of its base.

11. Take each side equal to 1000 ft.

12. Suppose a is the swimming speed and b the running speed where $a < b$. Let P be the ultimate destination of the swimmer and let 0 be the nearest point on shore. If OP subtends an angle ϕ at his eye and $\sin \phi \leq a/b$, least time is consumed by the straight path to P. If $\sin \varphi > a/b$, the path of least time is a broken path whose first segment extends to the point Q along the shore for which OQ subtends the angle arc sin a/b, at the swimmer's eye, thence along the shore to P.

13. If $a < 1$, the inequality can clearly be satisfied by taking $x = a$. If $a > 0$, we must satisfy the condition that the minimum of $x - \log_a x < 0$. The minimum occurs at $x = 1/\log a$. Take $a < e^{1/e}$.

Solutions and Hints to Problems

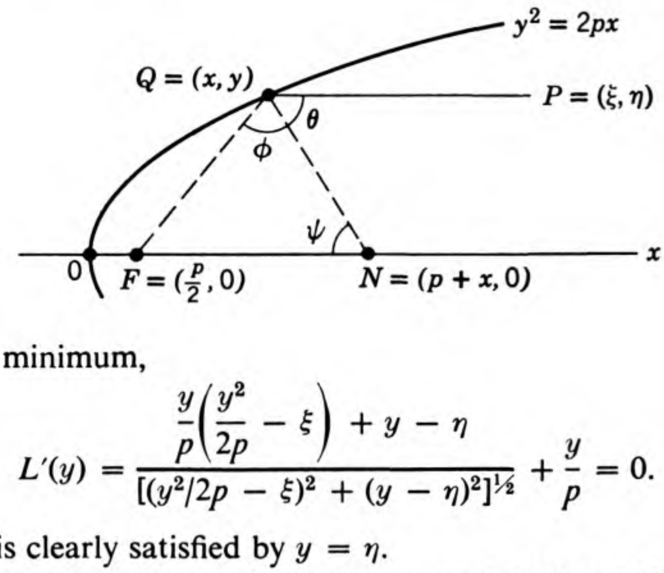
SECTION 3.6, page 236

1. An inflection at $x = 0$. When $p \geq 0$, no maxima or minima. When $p < 0$, a maximum at $x = -\sqrt{-p}$ and a minimum at $x = \sqrt{-p}$. All roots real for $q^2 \leq -4p^3$; one real root for $q^2 > -4p^3$.

2. We have for the total distance

$$L(y) = \left[\left(\frac{y^2}{2p} - \xi\right)^2 + (y - \eta)^2\right]^{1/2} + \left[\left(\frac{y^2}{2p} - \frac{p}{2}\right)^2 + y^2\right]^{1/2}$$
$$= \left[\left(\frac{y^2}{2p} - \xi\right)^2 + (y - \eta)^2\right]^{1/2} + \left[\frac{y^2}{2p} + \frac{p}{2}\right].$$

(See figure.)



For L to be a minimum,

$$L'(y) = \frac{\dfrac{y}{p}\left(\dfrac{y^2}{2p} - \xi\right) + y - \eta}{[(y^2/2p - \xi)^2 + (y - \eta)^2]^{1/2}} + \frac{y}{p} = 0.$$

The equation is clearly satisfied by $y = \eta$.

Let θ and ϕ be the angles made by the segments PQ and FQ, respectively, with the normal QN to the parabola. Note that

$$L'(y) = \frac{\sqrt{y^2 + p^2}}{p}(-\sin\theta + \sin\phi).$$

Thus, at a minimum, $\theta = \phi$.

Furthermore, it is easily verified that triangle QFN is isosceles, $|QF| = |FN| = x + p/2$ so that $\psi = \theta$ and PQ is again seen to be parallel to the x-axis.

3. Let a denote the base, x, y the legs of the triangle and θ, ξ, η the corresponding opposite angles. From the law of sines we have

$$x = \frac{a \sin\xi}{\sin\theta}, \qquad y = \frac{a\sin\eta}{\sin\theta}.$$

Consequently, the area A is given by

$$A = \frac{1}{2}xy \sin \theta = \frac{1}{2}a^2 \frac{\sin \xi \sin \eta}{\sin \theta}$$

$$= \frac{1}{2}\frac{a^2 \sin \xi \sin (\theta + \xi)}{\sin \theta},$$

where we have employed $\xi + \eta + \theta = \pi$.
The derivative with respect to ξ is $A' = a^2 \sin (2\xi + \theta)/2 \sin \theta$, from which it follows that the minimum occurs at $\xi = \frac{1}{2}(\pi - \theta)$ and hence that the triangle is isosceles.

4. If the base and area are fixed, then the altitude also is fixed. Let $(a, 0)$ and $(-a, 0)$ be the end points of the base and (x, h) the vertex. If θ is the vertex angle then

$$\cos \theta = \frac{h^2 + x^2 - a^2}{\sqrt{(a^2 - x^2)^2 + 2h^2(a^2 + x^2) + h^4}}$$

and $\cos \theta$ can then be shown to have a minimum (hence θ, a maximum) at $x = 0$.

5. The proof employs the result of Example 1 in the text. Let T be the triangle of given area and least perimeter, and let b be any side of it. Then, keeping b fixed, T must be the triangle of given base b and given area having the least perimeter. Hence T must be isosceles, and the two sides of T other than b are equal to one another. But b is any side, and T is therefore equilateral.

6. First show as claimed in the text that of all triangles with a given perimeter and a given base the isosceles triangle has the greatest area. For this purpose, in the notation of Problem 4, let u be the length of the side joining the vertices $(-a, 0)$ and (x, h), and v the length of the side joining $(a, 0)$ and (x, h). The length $u + v = 2b$ is fixed and

$$A = ah = a\sqrt{v^2 - (a - x)^2} = a\sqrt{u^2 - (a + x)^2}.$$

Eliminate u and v to obtain

$$A = a\sqrt{(b^2 - a^2)(1 - x^2/b^2)},$$

from which it is easily verified that A has a maximum at $x = 0$.

Finally, if T is a triangle of maximum area for a fixed parimeter, then T must be equilateral, for if it has base b, then by the preceding argument the other two sides must be equal. Since b may be any side, the triangle is equilateral.

7. Given a fixed base, the triangles inscribed in a circle have a constant vertical angle. Apply the result of Problem 3.

8. The function $f(x) = (x^p - 1) - p(x - 1)$ has a unique minimum at $x = 1$.

9. Find the minima of $x - \sin x$ and $\sin x - (2/\pi)x$ in the interval $0 \le x \le \pi/2$. Or show that $(\sin x)/x$ is monotonic in that interval.

10. (*a*) In the interval $0 \le x \le \pi/2$ for $f(x) = \tan x - x$ we have $f'(x) = \tan^2 x \ge 0$. Consequently, f is monotonically increasing and $f(x) \ge f(0)$, which yields the result.

(*b*) Set $f(x) = \cos x + x^2/2 - 1$. From Problem 9, $f'(x) = x - \sin x \ge 0$ for $0 \le x \le \pi/2$, and the inequality is obviously valid for $x > \pi/2 > 1$. Consequently, $f(x) \ge f(0)$ for all x.

11. The minimum is

$$\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^{(n-1)/n} \frac{1}{\sqrt[n]{(a_1a_2\cdots a_{n-1})}}$$

Set $A_\nu = \frac{1}{\nu}\sum_{k=1}^{\nu} a_k$, $G_\nu = \sqrt[\nu]{a_1a_2\cdots a_\nu}$. For $\nu = 1$, $G_1 \le A_1$. Furthermore, if $A_{\nu-1} \ge G_{\nu-1}$, then by the result just obtained

$$\frac{A_\nu}{G_\nu} \ge \left(\frac{A_{\nu-1}}{G_{\nu-1}}\right)^{1-1/\nu} \ge \left(\frac{A_{\nu-1}}{G_{\nu-1}}\right)^0 = 1.$$

12. (*a*)

$$x = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

(*b*) Take $a_1 \le a_2 \le \cdots \le a_n$. Note that

$$|a_n - x| + |a_1 - x| \ge a_n - a_1$$

and that the value $a_n - a_1$ is taken when $a_1 \le x \le a_n$. If n is even, $n = 2m$ the minimum is taken when $a_m \le x \le a_{m+1}$ and has the value

$$a_{m+1} + a_{m+2} + \cdots + a_{2m} - a_1 - a_2 - \cdots - a_m,$$

and if n is odd, $n = 2m - 1$, then the minimum is taken when $x = a_m$ and has the value

$$a_{m+1} + a_{m+2} + \cdots + a_{2m-1} - a_1 - a_2 - \cdots - a_{m-1}.$$

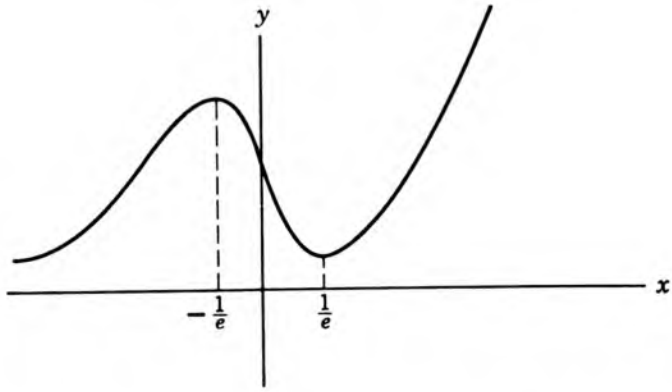
The value of x which yields the minimum is the statistic called the *median* of the values a_i.

(*c*) Without loss of generality we may assume $a_1 < a_2 < \cdots < a_n$. The function $f(x) = \sum_{i=1}^{n} \lambda_i |a_i - x|$ is piecewise linear and has the piecewise constant derivative given by

$$f'(x) = \tau_\nu = \lambda_\nu + \lambda_{\nu+1} + \cdots + \lambda_n - \lambda_1 - \lambda_2 - \cdots - \lambda_{\nu-1}$$

when $a_{\nu-1} < x < a_\nu$. There exists a value ν such that $\tau_\nu \ge 0$ and $\tau_{\nu+1} < 0$. If $\tau_\nu = 0$, then $a_{\nu-1} \le x \le a_\nu$ yields the minimum. If $\tau_\nu > 0$, then $x = a_\nu$ yields the minimum.

13. Show that $\lim_{x\to 0} x \log x^2 = 0$; hence that $\lim_{x\to 0} (x^2)^x = \lim_{x\to 0} e^{x \log x^2} = 1$. Maximum at $x = -1/e$, minimum at $x = 1/e$, inflections for $x = 0$, and for the value of x satisfying $(2 + \log x^2)^2 + 2/x = 0$.



14. We have $\lim_{x\to\infty} (1 + 1/x)^{x+\alpha} = e$. For $f(x) = (1 + 1/x)^{x+\alpha}$ we have

$$f'(x) = f(x)g(x),$$

where $g(x) = \log(1 + 1/x) - (x + \alpha)/(x^2 + x)$. At infinity, $g(x) = 0$. Since

$$g'(x) = \frac{(2\alpha - 1)x + \alpha}{(x^2 + x)^2},$$

we see that $g'(x) > 0$ and $g(x)$ increases to 0 at infinity if $\alpha \geq \frac{1}{2}$, and decreases to 0 if $\alpha < \frac{1}{2}$. It follows that $f(x)$ decreases to e for $\alpha \geq \frac{1}{2}$, and if $\alpha < \frac{1}{2}$ that $f(x)$ has values less than e for large x.

15. (*a*) Let the vertices of the triangle be located at $P = (a, b)$, $Q = (-c, 0)$, and $R = (c, 0)$, where $b > 0$ and $c > 0$. We also assume $a \geq 0$ (if $a < 0$ we replace the triangle by its reflection in the y-axis). For any point (x, y) set $p = y$,

$$q = \frac{bx - (a - c)y - bc}{\sqrt{b^2 + (a - c)^2}}, \qquad r = \frac{bx - (a + c)y + bc}{\sqrt{b^2 + (a + c)^2}},$$

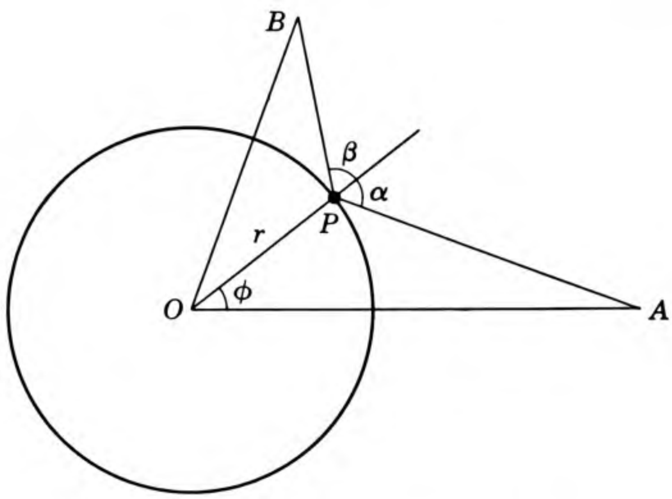
where p, q, and r denote the directed distances from the sides (extended if necessary) QR, RP, and PQ, with signs positive for (x, y) above, to the left and to the right of the respective sides. Restrict (x, y) to a line parallel to the base, $y =$ constant. On this line minimize $f(x) = |p| + |q| + |r|$. Observe that on the line $y = b$ the minimum occurs at P and if $y > b$, the value of the minimum must be greater. It is therefore necessary to consider only $y \leq b$. We have

$$f'(x) = \frac{b}{\sqrt{b^2 + (a + c)^2}} \operatorname{sgn} r + \frac{b}{\sqrt{b^2 + (a - c)^2}} \operatorname{sgn} q.$$

It follows that the minimum on the line y = constant occurs on the line RP, that is, $|q| = 0$. The value of the minimum is

$$g(y) = y + 2c\frac{b - y}{\sqrt{b^2 + (a + c)^2}}.$$

If the side RP is greater than the base QR, the over-all minimum must then occur at the vertex R. If the side RP is less than the base, the minimum occurs at the vertex P. Thus the minimum falls on the vertex opposite the longest side of the triangle. If the two longer sides are equal in length, then the minimum is reached at any point of the shortest side. If the triangle is equilateral, the minimum is reached at any point within or on the triangle.



(b) Let the vertices be located at the points O, $A = (a, 0)$, $B = (b\cos\theta, b\sin\theta)$. We minimize the sum of the distances from a point P on a circle of radius r about O. The sum of the distances of P from the vertices is given as a function of the angle ϕ of inclination of OP:

$$f(\phi) = r + \sqrt{(r\cos\phi - a)^2 + (r\sin\phi)^2} + \sqrt{(r\cos\phi - b\cos\theta)^2 + (r\sin\phi - b\sin\theta)^2};$$

hence

$$f'(\phi) = \frac{ar\sin\phi}{|PA|} - \frac{br\sin(\theta - \phi)}{|PB|}.$$

If P is inside the triangle, then by the law of sines (see figure)

$$f'(\phi) = r(\sin\alpha - \sin\beta)$$

and the condition for a minimum on the circle is $\sin\alpha = \sin\beta$. Thus if P yields the over-all minimum, the angle subtended by two of the vertices at P is bisected by the line joining P to the third vertex. It follows that the angle subtended by the three sides at P are all equal and equal to $2\pi/3$.

If any of the vertex angles A, B, or C is equal to or greater than $2\pi/3$, then there is no interior minimum. Thus suppose $C \geq 2\pi/3$. The minimum point P cannot be interior to the triangle. Furthermore, it cannot lie on the side of AB opposite to C since a lesser value would be given by its image in AC. If $\rho = |PC| > 0$, the line PC must then lie outside the angle subtended by AB at P. The only possibility left is that $\rho = 0$ or the minimum point is C itself.

16. (*a*) The derivative of $f(x) = (1 + x)e^x$ is positive for $x \geq 0$; consequently, $f(x) > f(0) = 1$.

(*b*) Differentiate and use (*a*).

(*c*) Differentiate and use (*b*).

17. Essentially the same as Miscellaneous Problems, Chapter 2, Nos. 4 and 5.

(*a*) The statement that the arc lies above the chord is expressed by

$$f[\theta a + (1 - \theta)\beta] > \theta f(\alpha) + (1 - \theta)f(\beta),$$

where $0 < \theta < 1$.

Set $u = \theta\alpha + (1 - \theta)\beta$. Take $\alpha < \beta$; then $\alpha < u < \beta$. By the mean value theorem

$$f(u) - f(\alpha) = f'(\xi)(u - \alpha),$$

$$f(\beta) - f(u) = f'(\eta)(\beta - u),$$

where

$$\alpha < \xi < u < \eta < \beta.$$

Since $f'(x)$ is a decreasing function, $f'(\eta) < f'(\xi)$. Add to obtain

$$f(\beta) - f(\alpha) < f'(\xi)(\beta - \alpha),$$

whence

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} < \frac{f(u) - f(\alpha)}{u - \alpha},$$

and therefore

$$f(\alpha) + (u - \alpha)\frac{f(\beta) - f(\alpha)}{\beta - \alpha} < f(u).$$

Enter the expression for u in this inequality to obtain the desired result.

(*b*) The equation of the tangent at $x = \xi$ is $y = f(\xi) + f'(\xi)(x - \xi)$. For

$$g(x) = f(\xi) - f(x) + f'(\xi)(x - \xi)$$

we have

$$g'(x) = f'(\xi) - f'(x).$$

Thus $g'(x) < 0$ for $x < \xi$ and $g'(x) > 0$ for $x > \xi$. Consequently, $g(x)$ has a strict minimum at $x = \xi$, or $g(x) > g(\xi) = 0$.

18. (*a*) Suppose $\alpha < u < \beta$; then if all arcs lie above the corresponding chords,

$$f(\alpha) + (\xi - \alpha)\,\frac{f(u) - f(\alpha)}{u - \alpha} < f(\xi) \qquad \text{for} \quad \alpha < \xi < u.$$

and

$$f(\beta) + (\eta - \beta)\frac{f(u) - f(\beta)}{u - \beta} < f(\eta) \qquad \text{for} \quad u < \eta < \beta,$$

whence

$$\frac{f(\xi) - f(\alpha)}{\xi - \alpha} > \frac{f(u) - f(\alpha)}{u - \alpha},$$

$$\frac{f(\beta) - f(\eta)}{\beta - \eta} < \frac{f(\beta) - f(u)}{\beta - u}.$$

Also taking $u = \beta$ in the first of these equations and $u = \alpha$ in the second and replacing ξ and η by u in both,

$$\frac{f(\xi) - f(\alpha)}{\xi - \alpha} > \frac{f(u) - f(\alpha)}{u - \alpha}$$

$$> \frac{f(\beta) - f(u)}{\beta - u} > \frac{f(\beta) - f(\eta)}{\beta - \eta}.$$

Taking the limits as $\xi \to \alpha$ and $\eta \to \beta$ we obtain for $\alpha < \beta$,

$$f'(\alpha) > f'(\beta).$$

It follows that $f''(x) \leq 0$.

Suppose that the arc lies below all its tangents; that is, for all ξ

$$f(x) \leq f(\xi) + f'(\xi)(x - \xi).$$

By the mean value theorem

$$f(x) - f(\xi) = f'(\eta)(x - \xi),$$

for some η between x and ξ. Then $f'(\eta) \geq f'(\xi)$ for $x < \xi$ and $f'(\eta) \leq f'(\xi)$ for $x > \xi$. Taking $\lim_{x\to\xi} \dfrac{f'(\eta) - f'(\xi)}{\eta - \xi}$, we obtain $f''(\xi) \leq 0$.

(*b*) Induction on ν for $n = 2^\nu$ shows that

$$\frac{f(a_1 + a_2 + \cdots + a_n)}{n} \geq \frac{1}{n}[f(a_1) + f(a_2) + \cdots + f(a_n)].$$

For natural numbers p, q with $p + q = 2^\nu$, then

$$f\left(\frac{px + qy}{p + q}\right) \geq \frac{pf(x) + qf(y)}{p + q}.$$

Now let θ be any real number satisfying $0 < \theta < 1$. Take p as the integer part of $\theta 2^\nu$, $p = [\theta 2^\nu]$, and $q = 2^\nu - p$, and employ the continuity of f to derive the condition a of Problem 17.

19. To show that

$$\phi(x) = [\theta a^x + (1 - \theta)b^x]^{1/x} = u^{1/x}$$

is monotonically increasing, note on applying the result of 17*a* to the convex function $u \log u$ that

$$\phi'(x) = [\theta a^x \log a^x + (1 - \theta)b^x \log b^x - u \log u\,] \frac{u^{1-1/x}}{x^2}$$

is nonnegative.

20. For $g(u) = u \log u$ we have strict inequality, $g''(u) = u^{-1} > 0$, for positive u. It follows in the solution of Problem 19 that $\phi'(x) = 0$ only if $a = b$.

21. Observe for the inequality proved in Problem 19 that the proof imposes no restriction on the sign of x other than $x \neq 0$. Consider

$$\begin{aligned}\lim_{x\to 0+} \log \phi(x) &= \lim_{x \to 0+} \frac{\log [\theta a^x + (1 - \theta)b^x]}{x} \\ &= \frac{d}{dx} \log [\theta a^x + (1 - \theta)b^x]_{x=0+} \\ &= \theta \log a + (1 - \theta) \log b.\end{aligned}$$

22. Clear from the remarks above.

23. Since $\log x$ is a concave function, the arc lies above its chord. Consequently,

$$\begin{aligned}\log a^\theta b^{1-\theta} &= \theta \log a + (1 - \theta) \log b \\ &\leq \log [\theta a + (1 - \theta)b].\end{aligned}$$

24. In some subinterval $[\alpha, \beta]$ of $[a, b]$ we have $f(x) > M - \epsilon$. It follows that

$$(M - \epsilon)^n(\beta - \alpha) < \int_a^b f(x)^n \, dx \leq M^n(b - a).$$

Take the *n*th root and pass to the limit.

EXERCISES

SECTION 3.7, page 248

1. Compare the following functions with powers of x as regards their order of magnitude as $x \to \infty$:

(*a*) $e^{x^\beta} - 1$, $(\beta > 0)$

(*b*) $(\log x)^\beta$, $(\beta > 0)$

(*c*) $\sin x$,

(*d*) $\sinh x$,

(*e*) $x^{1/2} \sin x \cdot \arctan x$,

(*f*) $x^{1/2} \sin x + \dfrac{x^2 \cos^3 x}{x^2 + 1}$,

(*g*) $\dfrac{e^{-1/x}}{1 - e^{-1/x}}$,

(*h*) $x^x - 1$,

(*i*) $\log (x \log x)$.

2. Compare the functions of Exercise 1 with e^{ax}, e^{x^a}, and $(\log x)^a$ for $a > 0$.

3. Compare the functions of Exercise 1 with powers of x as $x \to 0$.

4. (*a*) What are the limits, as $x \to \infty$, of $e^{(-e^x)}$ and $e^{(e^{-x})}$?

(*b*) Does the limit $\lim_{x \to \infty} e^{x^n} e^{(-e^x)}$ exist?

5. If $\phi(x) \to \infty$ as $x \to \infty$, show that $\log \phi(x)$ is of a lower order and $e^{\phi(x)}$ of a higher order of magnitude than $\phi(x)$.

PROBLEMS

SECTION 3.7, page 248

1. Let $f(x)$ be a continuous function vanishing, together with its first derivative, for $x = 0$. Show that $f(x)$ vanishes to a higher order than x as $x \to 0$

2. Show that $f(x) = \dfrac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{b_0 x^m + b_1 x^{m-1} + \cdots + b_m}$, when a_0, $b_0 \neq 0$, is of the same order of magnitude as x^{n-m}, when $x \to \infty$.

***3.** Prove that e^x is not a rational function.

***4.** Prove that e^x cannot satisfy an algebraic equation with polynomials in x as coefficients.

5. If the order of magnitude of the positive function $f(x)$ as $x \to \infty$ is higher, the same, or lower than that of x^m, prove that $\int_a^x f(\xi)\, d\xi$ has the corresponding order of magnitude relative to x^{m+1}.

6. Compare the order of magnitude as $x \to \infty$ of $\int_a^x f(\xi)\, d\xi$ relative to $f(x)$ for the following functions $f(x)$:

(*a*) $\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$, (*b*) e^x,

(*c*) xe^{x^2}, (*d*) $\log x$

Answers to Exercises

SECTION 3.7, page 248

1. (*a*) Higher than x^N, (*b*) lower than x^ϵ, (*c*) $0(x^\epsilon)$, (*d*) higher than x^N, (*e*), (*f*) not comparable, since the functions do not tend to infinity or to definite limits. (*g*) same as x, (*h*) higher than x^N, (*j*) lower than x^ϵ.

2. Higher than e^{ax}, $(\log x)^a$, lower than e^{ax}, higher, same, or lower than e^{x^a} corresponding to positive, zero, or negative sign of $\beta - \alpha$, (*b*) lower than e^{ax}, e^{x^a}, (*c*) bounded, (*d*) same as e^x, lower than e^{x^a}, higher than $(\log x)^a$, (*e*), (*f*), (*g*) lower than e^{ax}, e^{x^a}, higher than $(\log x)^\alpha$, (*h*) higher than $e^{x^{1-\epsilon}}$, lower than $e^{x^{1+\epsilon}}$, higher than e^{ax}, $(\log x)^a$, (*j*) same as $\log x$, lower than e^{ax}, e^{x^a}.

3. (*a*) Same as x^β, (*b*) lower than $\left(\dfrac{1}{x}\right)^\epsilon$, (*c*) same as x, (*d*) same as x, (*e*) same as $x^{5/2}$, (*f*) same as $x^{3/2}$, (*g*) higher than x^N, (*h*) higher than $x^{1-\epsilon}$, lower than x, (*j*) lower than $\left(\dfrac{1}{x}\right)^\epsilon$.

4. (*a*) 0; 1.
(*b*) Yes; 0.

Solutions and Hints to Problems

SECTION 3.7, page 248

1. $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = f'(0) = 0.$

2. $\dfrac{f(x)}{x^{n-m}} = \dfrac{a_0 + a_1/x + \cdots + a_n/x^n}{b_0 + b_1/x + \cdots + b_m/x^m}$;

hence

$$\lim_{x\to\infty} \frac{f(x)}{x^{n-m}} = \frac{a_0}{b_0}.$$

3. e^x has higher order of magnitude at ∞ than any power, hence the result of Problem 2 could not be satisfied.

4. If $\sum\limits_{k=0}^{n} p_k(x)e^{kx} = 0$, then

$$p_n(x) = -\sum_{k=0}^{n-1} p_k(x)e^{-(n-k)x}.$$

Consequently, $\lim\limits_{x\to\infty} p_n(x) = 0$. It follows that the polynomial $p_n(x)$ is identically zero, and sequentially that $p_{n-1}(x), p_{n-2}(x), \ldots, p_0(x)$ are all identically zero.

5. If $f(x)$ has the same order of magnitude as does x^m, then

$$Bx^m < f(x) < Cx^m.$$

It follows that

$$F(x) = \int_a^x f(x)\,dx < \frac{C}{m+1}(x^{m+1} - a^{m+1})$$
$$< \frac{C}{m+1}x^{m+1}\left[1 + \left(\frac{|a|}{x}\right)^{m+1}\right]$$
$$< \frac{2C}{m+1}x^{m+1},$$

for $x > |a|$.
Similarly

$$F(x) > \frac{B}{m+1}x^{m+1}\left[1 - \left(\frac{|a|}{x}\right)^{m+1}\right] > \frac{B}{2(m+1)}x^{m+1}$$

for $x > 2\,|a|$.

Comparisons of the same type establish the other two cases. For example, if $f(x)$ has lower order than x^m, then given any positive ϵ, there exists a $b(\epsilon)$ such that $f(x) < \epsilon x^m$ whenever $x > b(\epsilon)$. We have

$$F(x) = F(b) + \int_b^x f(x)\,dx$$
$$< F(b) + \int_b^x \epsilon x^m\,dx;$$

hence

$$F(x) < x^{m+1}\left[\frac{\epsilon}{m+1} + \frac{F(b)}{x^{m+1}} - \frac{\epsilon b^{m+1}}{x^{m+1}}\right].$$

6. Let $F(x) = \int_a^x f(\xi)\,d\xi$.

(*a*) $F(x) = 0(e^{\sqrt{x}})$, higher.
(*b*) $F(x) = 0(e^{x})$, same.
(*c*) $F(x) = 0(e^{x^2})$, lower.
(*d*) $F(x) = 0(x \log x)$, higher.

PROBLEMS

SECTION 3.8, page 264

1. Find the limit as $n \to \infty$ of $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$.

***2.** Find the limit of

$$b_n = \frac{1}{\sqrt{n^2 - 0}} + \frac{1}{\sqrt{n^2 - 1}} + \frac{1}{\sqrt{n^2 - 4}} + \cdots + \frac{1}{\sqrt{n^2 - (n-1)^2}}.$$

***3.** If α is any real number greater than -1, evaluate

$$\lim_{n\to\infty} \frac{1^\alpha + 2^\alpha + 3^\alpha + \cdots + n^\alpha}{n^{\alpha+1}}.$$

Solutions and Hints to Problems

SECTION 3.8, page 264

1. The number a_n is the lower sum for $1/x$ in the interval $[1, 2]$ for a subdivision into n equal parts (cf. Problems S.2, No. 1). $a_n \to \log 2$.

2. Compare with $\dfrac{1}{\sqrt{1 - x^2}}$ at $x = 0, \dfrac{1}{n}, \dfrac{2}{n}, \dots, \dfrac{n-1}{n} \cdot \dfrac{\pi}{2}$.

3. $\dfrac{1}{1 + \alpha}$.

EXERCISES

SECTION 3.9, page 264

Evaluate the following integrals and verify the results by differentiation.

1. $\int xe^{x^2}\, dx.$

2. $\int x^3 e^{-x^4}\, dx.$

3. $\int x^2 \sqrt{1 + x^3}\, dx.$

4. $\int \frac{\log x}{x}\, dx.$

5. $\int \frac{dx}{x(\log x)^n}.$

6. $\int x\sqrt{x + 1}\, dx.$

7. $\int x\sqrt{ax + b}\, dx.$

8. $\int x^2(\sqrt[n]{x + a})\, dx.$

9. $\int \frac{6x}{2 + 3x}\, dx.$

10. $\int \frac{x^4}{1 - x}\, dx.$

11. $\int x^3(\sqrt{1 - x^2})^5\, dx.$

12. $\int_0^1 \frac{\arctan x}{1 + x^2}\, dx.$

13. $\int_0^\pi \cos^n x \sin x\, dx.$

14. $\int_0^4 \frac{x\, dx}{\sqrt{1 + 3x^2}}.$

15. $\int_a^b \frac{x}{(1 + x^2)^2}\, dx.$

16. $\int_a^b \frac{x^3}{1 - x}\, dx \quad (1 < a < b).$

17. $\int_0^{\pi/2} x \sin 2x^2\, dx.$

18. Evaluate $\int_0^1 (1 - x)^n\, dx$ (where n is a positive integer) by substitution.

Perform the integrations in Exercises 19–20.

19. $\int \frac{e^{2x}}{\sqrt[4]{e^x + 1}}\, dx.$

20. $\int \frac{dx}{x\sqrt{x^{2n} - 1}}.$

21. Given $F(x) = \int f(x)\, dx$, find $\int f(ax + b)\, dx$.

Answers to Exercises

SECTION 3.9, page 264

1. $\frac{1}{2}e^{x^2}$.

2. $-\frac{1}{4}e^{-x^4}$.

3. $\frac{2}{9}(1 + x^3)^{3/2}$.

4. $\frac{1}{2}(\log x)^2$.

5. $-\frac{1}{n - 1}\left(\frac{1}{\log x}\right)^{n-1}.$

6. Set $x = v^2 - 1$; $\frac{2(x + 1)^{5/2}}{5} - \frac{2(x + 1)^{3/2}}{3}.$

7. $\frac{2}{a^2}\left[\frac{(ax + b)^{5/2}}{5} - \frac{(ax + b)^{3/2}}{3}\right].$

8. $n(x + a)^{1/n}\left[\frac{(x + a)^3}{3n + 1} - 2a\frac{(x + a)^2}{2n + 1} + \frac{a^2(x + a)}{n + 1}\right].$

9. $2x - \frac{4}{3}\log|2 + 3x|$.

10. $\frac{-x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} - x - \log|x - 1|.$

11. $\frac{1}{9}(1 - x^2)^{9/2} - \frac{1}{7}(1 - x^2)^{7/2}$.

12. $\frac{\pi^2}{8}.$

13. $\frac{1 + (-1)^n}{n + 1}$

14. 2.

15. $\frac{1}{2(1 + a^2)} - \frac{1}{2(1 + b^2)}$

16. $\frac{1}{3}(a^3 - b^3) + \frac{1}{2}(a^2 - b^2) + (a - b) + \log \frac{a-1}{b-1}$.

17. $\frac{1}{4}\left(1 - \cos \frac{\pi^2}{2}\right)$

18. $\frac{1}{n+1}$.

19. $\frac{4}{7}(1 + e^x)^{7/4} - \frac{4}{3}(1 + e^x)^{3/4}$.

20. $\frac{1}{n} \arccos \frac{1}{x^n}$.

21. $\frac{F(ax + b)}{a}$.

EXERCISES

SECTION 3.10, page 271

Evaluate the following integrals and verify the results by differentiation:

1. $\int \frac{3\,dx}{9x^2 - 6x + 2}$.

2. $\int \frac{dx}{\sqrt{x^2 - 2x + 5}}$.

3. $\int \frac{x + 1}{\sqrt{1 - x^2}}\,dx$.

4. $\int \frac{dx}{\sqrt{5 + 2x + x^2}}$.

5. $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$.

6. $\int \frac{x\,dx}{x^2 - x + 1}$.

7. $\int \frac{x\,dx}{\sqrt{x^2 - 4x + 1}}$.

8. $\int \frac{(x + 1)\,dx}{\sqrt{2 + 2x - 3x^2}}$.

9. $\int \frac{dx}{x^2 + x + 1}$.

10. $\int \frac{dx}{x^2 - x + 1}$.

11. $\int \frac{dx}{x^2 + 2ax + b}$.

12. $\int \sin^3 x \cos^4 x\,dx$.

13. $\int \sin^2 x \cos^5 x\,dx$.

14. $\int \frac{x^2}{\sqrt{1 - x^2}}\,dx$.

15. $\int \frac{x + 1}{\sqrt{1 + x^2}}\,dx$.

16. $\int \frac{x^2}{\sqrt{1 + x^2}}\,dx$.

17. $\displaystyle\int_0^a \frac{dx}{\sqrt{x^2 + a^2}}.$

18. $\displaystyle\int_0^a \frac{dx}{x^2 + a^2}.$

19. $\displaystyle\int_0^1 x^2 \sqrt{1 - x^2}\, dx.$

20. $\displaystyle\int^1 x^2(1 - x^2)^{3/2}\, dx.$

Answers to Exercises

SECTION 3.10, page 271

1. $\arctan (3x - 1).$

2. $\log \left[\frac{x - 1}{2} + \sqrt{1 + \left(\frac{x - 1}{2}\right)^2}\right]$

or

$$\operatorname{ar\,sinh} \frac{x - 1}{2}.$$

3. $\arcsin x - \sqrt{1 - x^2}.$

4. $\log \left[\frac{x + 1}{2} + \sqrt{1 + \left(\frac{x + 1}{2}\right)^2}\right]$

or

$$\operatorname{ar\,sinh} \frac{x + 1}{2}.$$

5. $\arcsin \frac{x + 1}{2}.$

6. $\frac{1}{2} \log (x^2 - x + 1) + \frac{1}{\sqrt{3}} \arctan \frac{2x - 1}{\sqrt{3}}.$

7. $2 \operatorname{ar\,cosh} \left(\frac{x - 2}{\sqrt{3}}\right) + \sqrt{x^2 - 4x + 1}.$

8. $-\frac{1}{3} \sqrt{2 + 2x - 3x^2} + \frac{4}{3\sqrt{3}} \arcsin \frac{3x - 1}{\sqrt{7}}.$

9. $\frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}}.$

10. $\frac{2}{\sqrt{3}} \arctan \frac{2x - 1}{\sqrt{3}}.$

11. $\frac{1}{\sqrt{b - a^2}} \arctan \frac{x + a}{\sqrt{b - a^2}}$, if $b - a^2 > 0$; $-\frac{1}{x + a}$, if $b - a^2 = 0$;

$$-\frac{1}{\sqrt{a^2 - b}} \operatorname{ar\,tanh} \frac{x + a}{\sqrt{a^2 - b}}, \text{ if } b - a^2 < 0.$$

12. $-\dfrac{\cos^5 x}{5} + \dfrac{\cos^7 x}{7}$.

13. $\dfrac{\sin^3 x}{3} - 2\,\dfrac{\sin^5 x}{5} + \dfrac{\sin^7 x}{7}$.

14. $\frac{1}{2} \arcsin x - \frac{1}{2}x\sqrt{1 - x^2}$.

15. $\sqrt{1 + x^2} + \operatorname{ar\,sinh} x$.

16. $\frac{1}{2}x\sqrt{1 + x^2} - \frac{1}{2} \operatorname{ar\,sinh} x$.

17. $\operatorname{ar\,sinh} 1 = \log(1 + \sqrt{2})$.

18. $\dfrac{1}{a} \arctan \dfrac{x}{a}$.

19. $\dfrac{\pi}{2}$.

20. $\dfrac{\pi}{32}$.

EXERCISES

SECTION 3.11, page 274

Evaluate the integrals in Exercises 1–14.

1. $\displaystyle\int \frac{x \cos x}{\sin^2 x}\, dx$.

2. $\displaystyle\int \frac{x^7}{(1 - x^4)^2}\, dx$.

3. $\displaystyle\int x^2 \cos x\, dx$.

4. $\displaystyle\int x^3 e^{-x^2}\, dx$.

5. $\displaystyle\int_{-\pi}^{\pi} x^2 \cos nx\, dx$ (n a positive integer).

6. $\displaystyle\int_{-\pi}^{\pi} x^2 \sin nx\, dx$ (n a positive integer).

7. $\displaystyle\int x^3 \cos x^2\, dx$.

8. $\displaystyle\int \sin^4 x\, dx$.

9. $\displaystyle\int \cos^6 x\, dx$.

10. $\displaystyle\int x^4 \sqrt{1 - x^2}\, dx$.

11. $\displaystyle\int x^2 e^x\, dx$.

12. $\displaystyle\int \frac{\log x}{x^n}\, dx$ ($n \neq 1$).

13. $\displaystyle\int x^m \log x\, dx$ ($m \neq 1$).

14. $\displaystyle\int x^2 (\log x)^2\, dx$.

15. Prove the formula

$$\int e^x p(x)\,dx = e^x[p(x) - p'(x) + p''(x) - + \cdots],$$

where $p(x)$ is any polynomial.

Evaluate the integrals in Exercises 16–19.

16. $\displaystyle\int_0^{\pi/2} \cos^n x\,dx.$

17. $\displaystyle\int_0^{\pi/6} \cos^7 3\theta \sin^4 6\theta\,d\theta.$

18. $\displaystyle\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}}.$

19. $\displaystyle\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}}.$

Obtain recurrence formulas for the integrals of Exercises 20–22.

20. $\displaystyle\int x^a(\log x)^m\,dx.$

21. $\displaystyle\int x^n e^{ax} \sin bx\,dx.$

22. $\displaystyle\int x^n e^{ax} \cos bx\,dx.$

Evaluate the integrals of Exercises 23 and 24.

23. $\displaystyle\int e^{ax} \sinh bx\,dx.$

24. $\displaystyle\int e^{ax} \cosh bx\,dx.$

25. Evaluate $\displaystyle\int_{-1}^1 x^3 e^{-x^4} \cos 2x\,dx.$

PROBLEMS

SECTION 3.11, page 274

1. Show that for all odd positive values of n the integral $\int e^{-x^2} x^n\,dx$ can be evaluated in terms of elementary functions.

2. Show that if n is even, the integral $\int e^{-x^2} x^n\,dx$ can be evaluated in terms of elementary functions and the integral $\int e^{-x^2}\,dx$ (for which tables have been constructed).

3. Prove that

$$\int_0^x \left[\int_0^u f(t)\,dt\right] du = \int_0^x f(u)(x-u)\,du.$$

***4.** Problem 3 gives a formula for the second iterated integral. Prove that the nth iterated integral of $f(x)$ is given by

$$\frac{1}{(n-1)!}\int_0^x f(u)(x-u)^{n-1}\,du.$$

5. Prove for the binomial coefficient $\binom{n}{k}$ that

$$\binom{n}{k} = \left[(n+1)\int_0^1 x^k(1-x)^{n-k}\,dx\right]^{-1}.$$

6. Obtain a recursive formula for

$$\int x^p(ax^n + b)^q\,dx$$

and use this relation to integrate

$$\int x^3(x^7 + 1)^4\,dx.$$

***7.** (*a*) Let $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2 - 1)^n$. Show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0, \qquad \text{if} \quad m \neq n.$$

(*b*) Prove that $\displaystyle\int_{-1}^{1} P_n{}^2(x)\,dx = \frac{2}{2n+1}$.

(*c*) Prove that $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = 0$, if $m < n$.

(*d*) Evaluate $\displaystyle\int_{-1}^{1} x^n P_n(x)\,dx$.

Answers to Exercises

SECTION 3.11, page 274

1. $-\dfrac{x}{\sin x} + \log\left|\tan\dfrac{x}{2}\right|$.

2. $\dfrac{1}{4}\left[\dfrac{1}{1-x^4} + \log|1 - x^4|\right]$.

3. $(x^2 - 2)\sin x + 2x\cos x$.

4. $-\frac{1}{2}(x^2 + 1)e^{-x^2}$. **5.** $\dfrac{4\pi(-1)^n}{n^2}$. **6.** 0 (integral of an odd function).

7. $\frac{1}{2}(x^2\sin x^2 + \cos x^2)$. **8.** $\frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{8}x$.

9. $\frac{1}{192}\sin 6x + \frac{3}{64}\sin 4x + \frac{15}{64}\sin 2x + \frac{5}{16}x$.

10. Put $x = \cos\theta$: $x\sqrt{1-x^2}(-\frac{1}{16} - \frac{1}{24}x^2 + \frac{1}{6}x^4) + \frac{1}{16}\arcsin x$.

11. $e^x(x^2 - 2x + 2)$. **12.** $-\dfrac{1}{(n-1)x^{n-1}}\log x - \dfrac{1}{(n-1)^2x^{n-1}}$.

13. $\dfrac{x^{m+1}}{m+1}\log x - \dfrac{x^{m+1}}{(m+1)^2}$. **14.** $\frac{1}{3}x^3\{(\log x)^2 - \frac{2}{3}\log x + \frac{2}{9}\}$.

16. $\dfrac{(n-1)(n-3)\cdots 1}{n(n-2)\cdots 2}\cdot\dfrac{\pi}{2}$ if n is even; $\dfrac{(n-1)(n-3)\cdots 2}{n(n-2)\cdots 3}$ if n is odd.

17. $\dfrac{2^{12}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}$.

18. $\dfrac{(2n!)}{2^{2n}(n!)^2}\cdot\dfrac{\pi}{2}$.

19. $\dfrac{2^{2n}(n!)^2}{(2n+1)!}$.

20. $\displaystyle\int x^a(\log x)^m\,dx = \frac{x^{a+1}(\log x)^m}{a+1} - \frac{m}{a+1}\int x^a(\log x)^{m-1}\,dx.$

21. $$\int x^n e^{ax}\sin bx\,dx = \frac{x^n e^{ax}}{a^2+b^2}(a\sin bx - b\sin bx) - \frac{an}{a^2+b^2}\int x^{n-1}e^{ax}\sin bx\,dx + \frac{bn}{a^2+b^2}\int x^{n-1}e^{ax}\cos bx\,dx.$$

22. $$\int x^n e^{ax}\cos bx\,dx = \frac{x^n e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) - \frac{an}{a^2+b^2}\int x^{n-1}e^{ax}\cos bx\,dx - \frac{bn}{a^2+b^2}\int x^{n-1}e^{ax}\sin bx\,dx.$$

23. $\displaystyle\int e^{ax}\sinh bx\,dx = \frac{e^{ax}}{b^2-a^2}(b\cosh bx - a\sinh bx).$

24. $\displaystyle\int e^{ax}\cosh bx\,dx = \frac{e^{ax}}{b^2-a^2}(b\sinh bx - a\cosh bx).$

25. 0 (integral of an odd function).

Solutions and Hints to Problems

SECTION 3.11, page 274

1. Set $n = 2m - 1$ and put $t = x^2$. We have

$$\int e^{-x^2}x^n\,dx = \frac{1}{2}\int e^{-t}t^{m-1}\,dt.$$

Integrate by parts.

2. Set $n = 2m$. Integrate by parts:

$$\int (e^{-x^2}x)x^{2m-1}\,dx = -\tfrac{1}{2}e^{-x^2}x^{2m-1} + \frac{2m-1}{2}\int e^{-x^2}x^{2m-2}\,dx.$$

We have a recursive formula which permits the expression of the integral in the stated form.

3. Integrate by parts or differentiate with respect to x.

4. Proof by induction: If the nth iterated integral $f_n(x)$ of $f(x)$ is given by the formula, then

$$f_{n+1}(x) = \frac{1}{(n-1)!}\int_0^x f_1(u)(x-u)^{n-1}\,du$$
$$= \frac{1}{(n-1)!}\int_0^x \int_0^u f(t)\,dt(x-u)^{n-1}\,du.$$

Integrate by parts to obtain

$$f_{n+1}(x) = \frac{1}{n!}\int_0 f(u)(x-u)^n\,du.$$

5. Integrate by parts to obtain

$$I = \int_0^1 x^k(1-x)^{n-k}\,dx = \frac{n-k}{k+1}\int_0^1 x^{k+1}(1-x)^{n-k-1}\,dx.$$

Repeatedly apply this formula to obtain

$$I = \frac{(n-k)(n-k-1)\cdots 1}{(k+1)(k+2)\cdots n}\int_0^1 x^n\,dx$$
$$= \frac{(n-k)!\,k!}{n!\,(n+1)}.$$

Compare this solution with that of Miscellaneous Problems, Chapter 2, No. 2.

6. $$\frac{x^{p-n+1}(ax^n+b)^{q+1}}{a(p+nq+1)} - \frac{b(p-n+1)}{a(p+nq+1)}\int x^{p-n}(ax^n+b)^q\,dx$$

or

$$\frac{x^{p+1}(ax^n+b)^q}{p+nq+1} + \frac{bnq}{p+nq+1}\int x^p(ax^n+b)^{q-1}\,dx.$$

Use the second of the recursive formulas above to obtain

$$\int x^3(x^7+1)^4\,dx.$$

7. (*a*) Integrate by parts:

$$\int_{-1}^1 P_n(x)P_m(x)\,dx = \frac{(-1)^k}{2^{n+m}m!\,n!}\int_{-1}^1\left[\frac{d^{m+k}}{dx^{m+k}}(x^2-1)^m\right]\left[\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n\right]dx,$$

where we employ

$$\frac{d^\nu}{dx^\nu}(x^2 - 1)^\mu\Big|_{x=\pm 1} = 0$$

for $\nu < \mu$. For $m < n$ take $k = n$ above to obtain the result.

Alternatively, from Problems 3.3, Nos. 8 and 9,

$$\int_{-1}^{1} P_m\left\{\frac{d}{dx}[(x^2 - 1)P_n'] - n(n + 1)P_n\right\} dx = 0.$$

Integrate the first term of the integrand by parts to obtain

$$\int_{-1}^{1} P_m\frac{d}{dx}[(x^2 - 1)P_n']\, dx = \int_{-1}^{1} P_n \frac{d}{dx}[(x^2 - 1)P_m']\, dx$$

whence

$$n(n + 1)\int_{-1}^{1} P_n P_m\, dx = m(m + 1)\int_{-1}^{1} P_n P_m\, dx.$$

(*b*) Apply the first formula in the solution of part *a*:

$$\int_{-1}^{1} P_n(x)^2\, dx = \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(2n)!(x^2 - 1)^n\, dx.$$

Now integrate by parts:

$$\begin{aligned}\int_{-1}^{1}(x^2 - 1)^n\, dx &= \frac{-2n}{2n + 1}\int_{-1}^{1}(x^2 - 1)^{n-1}\, dx \\ &= (-1)^n\frac{2^{n+1}n(n - 1)\cdots 1}{(2n + 1)(2n - 1)\cdots 1} \\ &= \frac{(-1)^n 2^{2n+1}(n!)^2}{(2n + 1)!}\end{aligned}$$

from which the result follows.

(*c*) Integrate by parts:

$$\begin{aligned}\int_{-1}^{1} x^m P_n(x)\, dx &= -\frac{m}{2^n n!}\int_{-1}^{1} x^{m-1}D^{n-1}(x^2 - 1)^n\, dx \\ &= \frac{(-1)^k m!}{2^n n!(m - k)!}\int_{-1}^{1} x^{m-k}D^{n-k}(x^2 - 1)^n\, dx;\end{aligned}$$

for $n > m$ take $k = m$ and integrate.

(*d*) Apply the formula in the solution of part *c*:

$$\int_{-1}^{1} x^n P_n(x)\, dx = \frac{(-1)^n}{2^n}\int_{-1}^{1}(x^2 - 1)^n\, dx.$$

The value of the last integral is given in the solution to part *b*. *Answer:*

$$\frac{2^{n+1}(n!)^2}{(2n + 1)!}.$$

EXERCISES

SECTION 3.12, page 282

Integrate the following.

1. $\displaystyle\int \frac{dx}{2x - 3x^2}$.

2. $\displaystyle\int \frac{dx}{x^2 - x}$.

3. $\displaystyle\int \frac{3\,dx}{x(x + 1)^3}$.

4. $\displaystyle\int \frac{x^2 + x + 1}{3x^2 - 2x - 5}\,dx$.

5. $\displaystyle\int \frac{dx}{(x - 1)^2(x^2 + 1)}$.

6. $\displaystyle\int \frac{x^2\,dx}{(x - 1)^2(x^2 + 1)}$.

7. $\displaystyle\int \frac{dx}{1 - x^3}$.

8. $\displaystyle\int \frac{dx}{1 + x^3}$.

9. $\displaystyle\int \frac{(x - 4)}{(x^2 + 1)(x - 2)}\,dx$.

10. $\displaystyle\int \frac{x + 4}{(x^2 - 1)(x + 2)}\,dx$.

11. $\displaystyle\int \frac{x^6}{1 - x^4}\,dx$.

12. $\displaystyle\int \frac{x^2}{x^4 + x^2 - 2}\,dx$.

13. $\displaystyle\int \frac{dx}{x^2(x^2 + 1)^2}$.

14. $\displaystyle\int \frac{1 + \sqrt[3]{x}}{1 + \sqrt[4]{x}}\,dx$.

15. $\displaystyle\int \frac{x\,dx}{\sqrt[3]{1 + x} - \sqrt{1 + x}}$.

16. $\displaystyle\int \frac{x^2 - 1}{x^4 + x^2 + 1}\,dx$.

17. $\displaystyle\int \frac{dx}{x(x + 1)\cdots(x + n)}$.

PROBLEMS

SECTION 3.12, page 282

***1.** Integrate

$$\int \frac{dx}{x^6 + 1}.$$

2. Use the partial fraction expansion to prove Newton's formulas

$$\frac{\alpha_1{}^k}{g'(\alpha_1)} + \frac{\alpha_2{}^k}{g'(\alpha_2)} + \cdots + \frac{\alpha_n{}^k}{g'(\alpha_n)} = \begin{cases} 0 & \text{for } k = 0, 1, 2, \ldots, n - 2 \\ 1 & \text{for } k = n - 1, \end{cases}$$

where $g(x)$ is a polynomial of the form $(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$ with distinct nonzero $\alpha_1, \ldots, \alpha_n$.

Answers to Exercises

SECTION 3.12, page 282

1. $\log\sqrt{\left|\dfrac{x}{2-3x}\right|}$. **2.** $\log\left|1-\dfrac{1}{x}\right|$.

3. $\log\left|\dfrac{x}{x+1}\right|^3+\dfrac{3}{x+1}+\dfrac{3}{2(x+1)^2}$.

4. $\dfrac{x}{3}-\frac{1}{8}\log|x+1|+\frac{49}{72}\log|3x-5|$.

5. $-\dfrac{1}{2(x-1)}+\log\sqrt[4]{\dfrac{1+x^2}{(1-x)^2}}$. **6.** $\dfrac{-1}{2(x-1)}-\log\sqrt[4]{\dfrac{1+x^2}{(1-x)^2}}$.

7. $\log\dfrac{1}{\sqrt[3]{|x-1|}}+\frac{1}{6}\log(x^2+x+1)+\dfrac{1}{\sqrt{3}}\arctan\dfrac{2x+1}{\sqrt{3}}$.

8. $\log\sqrt[3]{|x+1|}-\frac{1}{6}\log|x^2-x+1|+\dfrac{1}{\sqrt{3}}\arctan\dfrac{2x-1}{\sqrt{3}}$.

9. $\log\dfrac{1}{\sqrt[5]{(x-2)^2}}+\log\sqrt[5]{1+x^2}+\frac{9}{5}\arctan x$.

10. $\frac{2}{3}\log|x+2|+\frac{5}{6}\log|x-1|-\frac{3}{2}\log|x+1|$.

11. $-\dfrac{x^3}{3}+\log\sqrt[4]{\left|\dfrac{x+1}{x-1}\right|}-\frac{1}{2}\arctan x$.

12. $\frac{1}{6}\log\dfrac{x-1}{x+1}+\dfrac{\sqrt{2}}{3}\arctan\dfrac{x}{\sqrt{2}}$.

13. $-\dfrac{3x^2+2}{2x(x^2+1)}-\dfrac{3}{2}\arctan x$.

14. $\frac{12}{13}x^{13/12}-\frac{6}{5}x^{5/6}+\frac{4}{3}x^{3/4}+\frac{12}{7}x^{7/12}-2x^{1/2}-3x^{1/3}+4x^{1/4}+12x^{1/12}$

$-2\log(1+x^{1/4})-4\log(1+x^{1/12})-4\sqrt{3}\arctan\dfrac{2}{\sqrt{3}}x^{1/12}-\frac{1}{2})$.

15. $-6\sqrt[3]{(1+x)^2}(\frac{1}{4}+\frac{1}{5}\sqrt[6]{(1+x)}+\frac{1}{6}\sqrt[3]{(1+x)}+\frac{1}{7}\sqrt{(1+x)}$
$+\frac{1}{8}\sqrt[3]{(1+x)^2}+\frac{1}{9}\sqrt[6]{(1+x)^5})$.

16. Put $x+\dfrac{1}{x}=t\cdot\dfrac{1}{2}\log\dfrac{x^2-x+1}{x^2+x+1}$.

17. $\dfrac{1}{n!}\Bigg[\log x-\dbinom{n}{1}\log(x+1)+\dbinom{n}{2}\log(x+2)-\cdots$
$+(-1)^n\dbinom{n}{n}\log(x+n)\Bigg]$.

Solutions and Hints to Problems

SECTION 3.12, page 282

1. Use $x^6 + 1 = (x^2 + 1)(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)$.

Answer:

$$\tfrac{1}{3} \text{ arc tan } x + \frac{\sqrt{3}}{12} \log \frac{x^2 + \sqrt{3}x + 1}{x^2 - \sqrt{3}x + 1} + \tfrac{1}{6} \text{ arc tan } (2x + \sqrt{3})$$

$$+ \tfrac{1}{6} \text{ arc tan } (2x - \sqrt{3}).$$

Note: $\text{arc tan } (2x + \sqrt{3}) + \text{arc tan } (2x - \sqrt{3}) = \text{arc tan } \dfrac{x}{1 - x^2}$.

2. Use the partial fraction decomposition for x^ν, $\nu < n$, to obtain

$$x = \sum_{i=1}^{n} \frac{\alpha_i{}^\nu g(x)}{g'(\alpha_i)(x - \alpha_i)}$$

$$= \sum_{i=1}^{n} \frac{\alpha_i{}^\nu}{g'(\alpha_i)} (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{i-1})(x - \alpha_{i+1}) \cdots (x - \alpha_n)$$

Take $x = 0$ above to obtain

$$0 = \left[\sum_{i=1}^{n} \frac{\alpha_i^{\nu-1}}{g'(\alpha_i)} \right] (-1)^{n-1} \alpha_1 \alpha_2 \cdots \alpha_n,$$

from which the first result follows with $k = \nu - 1$.

If $\nu = n$, then x^n must be resolved into a polynomial plus a sum of partial fractions. Thus

$$x^n - g(x) = \sum_{i=1}^{n} \frac{\alpha_i{}^n g(x)}{g'(\alpha_i)(x - \alpha_i),}.$$

from which the desired result is obtained as before.

The result may be extended to the case of a zero root. If $g(0) = 0$, then $g(x) = xh(x)$, where by assumption $h(0) \neq 0$. Take $a_n = 0$ and $\alpha_1, \alpha_2 \cdots, \alpha_{n-1}$ as the roots of $h(x)$. Use $g'(\alpha_i) = \alpha_i h'(\alpha_i)$ to obtain for $1 < \nu < n$,

$$x^{\nu-1} = \sum_{i=1}^{n} \frac{\alpha_i{}^\nu h(x)}{g'(\alpha_i)(x - \alpha_i)}$$

and proceed as before. For $\nu = n - 1$ ($k = n - 2$), however, the sum of the fractions is now 1.

EXERCISES

SECTION 3.13, page 290

Integrate the following.

1. $\displaystyle\int \frac{dx}{1 + \sin x}$.

2. $\displaystyle\int \frac{dx}{1 + \cos x}$.

3. $\displaystyle\int \frac{dx}{2 + \sin x}$.

4. $\displaystyle\int \frac{dx}{\sin^3 x}$.

5. $\displaystyle\int \frac{dx}{\cos x}$.

6. $\displaystyle\int_0^{\pi/2} \frac{dx}{3 + \cos x}$.

7. $\displaystyle\int \frac{dx}{1 + \cos^2 x}$.

8. $\displaystyle\int \frac{dx}{3 + \sin^2 x}$.

9. $\displaystyle\int \tan^3 x\, dx$.

10. $\displaystyle\int \frac{dx}{\sin x + \cos x}$.

11. $\displaystyle\int \frac{\sin^2 x + \cos^3 x}{3\cos^2 x + \sin^4 x} \sin x\, dx$.

12. $\displaystyle\int \sqrt{x^2 - 4}\, dx$.

13. $\displaystyle\int \sqrt{4 + 9x^2}\, dx$.

14. $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2 - 4x + 3}}$.

15. $\displaystyle\int x\sqrt{x^2 + 4x}\, dx$.

16. $\displaystyle\int \frac{dx}{\sqrt{x} + \sqrt{1-x}}$.

17. $\displaystyle\int \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}\, dx$.

18. $\displaystyle\int \frac{\sqrt{x-a}}{1 + \sqrt{x-a+1}}\, dx$.

19. $\displaystyle\int \frac{dx}{\sqrt{x-a} + \sqrt{x-b}}$.

Answers to Exercises

SECTION 3.13, page 290

1. $\displaystyle -\frac{2}{1 + \tan\frac{x}{2}}$. **2.** $\displaystyle \tan\frac{x}{2}$. **3.** $\displaystyle \frac{2}{\sqrt{3}} \operatorname{arc\,tan}\left(\frac{2\tan\frac{x}{2} + 1}{\sqrt{3}}\right)$.

4. $\frac{1}{8}\left(\tan^2\frac{x}{2} - \cot^2\frac{x}{2}\right) + \frac{1}{2}\log\left|\tan\frac{x}{2}\right|$.

5. $\log\left|\dfrac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}-1}\right|$.

6. $\frac{1}{\sqrt{2}}\arctan\frac{1}{2}\sqrt{2}$.

7. $\frac{1}{\sqrt{2}}\arctan\frac{\tan x}{\sqrt{2}}$.

8. $\frac{1}{2\sqrt{3}}\arctan\frac{2\tan x}{\sqrt{3}}$.

9. $\frac{1}{2\cos^2 x} + \log\cos x$.

10. $\frac{1}{\sqrt{2}}\log\left|\dfrac{\tan\frac{x}{2} - 1 + \sqrt{2}}{\tan\frac{x}{2} - 1 - \sqrt{2}}\right|$.

11. $\frac{1}{4}\log\frac{\cos^2 x - \cos x + 1}{(\cos^2 x + \cos x + 1)^3} + \frac{1}{2\sqrt{3}}\arctan\frac{2\cos x - 1}{\sqrt{3}}$

$$- \frac{1}{2\sqrt{3}}\arctan\frac{2\cos x + 1}{\sqrt{3}}.$$

12. $\frac{1}{2}x\sqrt{x^2-4} - 2\,\mathrm{ar}\cosh\frac{x}{2}$.

13. $\frac{1}{2}x\sqrt{4+9x^2} + \frac{2}{3}\,\mathrm{ar}\sinh\frac{3}{2}x$.

14. $2\arctan\sqrt{\frac{x-3}{x-1}}$.

15. $\frac{1}{3}\sqrt{(x^2+4x)^3} - (x+2)\sqrt{(x^2+4x)} + 4\,\mathrm{ar}\cosh\frac{x+2}{2}$.

16. $\sqrt{x} - \sqrt{(1-x)} + \frac{1}{2\sqrt{2}}\log\left|\dfrac{(\sqrt{x}-\sqrt{\frac{1}{2}})(\sqrt{1-x}+\sqrt{\frac{1}{2}})}{(\sqrt{x}+\sqrt{\frac{1}{2}})(\sqrt{1-x}-\sqrt{\frac{1}{2}})}\right|$.

17. $\log\left|x\cdot\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right| + \sqrt{(1-x^2)}$.

18. $\frac{1}{2}\,\mathrm{ar}\cosh(2x-2a+1) + \sqrt{(x-a)^2+(x-a)} - 2\sqrt{x-a}$.

19. $\frac{2}{3(b-a)}(\sqrt{(x-a)^3} - \sqrt{(x-b)^3})$)

PROBLEMS

SECTION 3.14, page 298

***1.** Prove that the substitution $x = (\alpha t + \beta)/(\gamma t + \delta)$ with $\alpha\delta - \gamma\beta \neq 0$ transforms the integral

$$\int \frac{dx}{\sqrt{ax^4 + bx^3 + cx^2 + dx + e}}$$

into an integral of similar type, and that if the biquadratic

$$ax^4 + bx^3 + cx^2 + dx + e$$

has no repeated factors, neither has the new biquadratic in t which takes its place. Prove that the same is true for

$$\int R\left(x, \sqrt{ax^4 + bx^3 + cx^2 + dx + e}\right) dx,$$

where R is a rational function.

2. The function

$$\phi(x) = \int_0^x \frac{du}{\sqrt{1 - k^2 \sin^2 u}}$$

is known as the *elliptic integral of the first kind.*

(*a*) Show that ϕ is continuous and increasing and hence has a continuous inverse.

(*b*) Let $am(x)$ denote the inverse of $\phi(x)$. Prove $sn(x) = \sin\,[am(x)]$, where $sn(x)$ is defined on p. 299, footnote 3.

Solutions and Hints to Problems

SECTION 3.14, page 298

1. If $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, the first integral is transformed into

$$\int \frac{dt}{\sqrt{(\gamma t + \delta)^2 f[(\alpha t + \beta)/(\gamma t + \delta)]/(\alpha\delta - \beta\gamma)^2}}.$$

2. (*a*) ϕ has a positive derivative.
(*b*) The function $am\, x$ is defined by

$$x = \int_0^{am\, x} \frac{du}{\sqrt{1 - k^2 \sin^2 u}}.$$

Substitute $u = \text{arc} \sin v$ to obtain

$$x = \int_0^{\sin am\, x} \frac{dx}{\sqrt{(1 - v^2)(1 - k^2v^2)}};$$

this relation characterizes the upper limit as $sn(x)$, hence the result.

EXERCISES

SECTION 3.15, page 301

Test the convergence of the improper integrals in Exercises 1 to 25.

1. $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}}$

2. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^6}$

3. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^3}.$

4. $\displaystyle\int_{0}^{\infty} \frac{dx}{(1 + x)\sqrt{x}}.$

5. $\displaystyle\int_{0}^{\pi} \frac{dx}{1 - \cos x}.$

6. $\displaystyle\int_{A}^{B} \frac{dx}{\sqrt{(x - a_1)(x - a_2)(x - a_3)(x - a_4)}}$, where a_1, a_2,

a_3, a_4 are all different and lie between A and B.

7. $\displaystyle\int_{0}^{\infty} \frac{\text{arc tan } x}{1 + x^2}\, dx.$

8. $\displaystyle\int_{0}^{\infty} \frac{\text{arc tan } x}{1 - x^3}\, dx.$

9. $\displaystyle\int_{1}^{\infty} \frac{x}{1 - e^x}\, dx.$

10. $\displaystyle\int_{0}^{\infty} \frac{x}{e^x - 1}\, dx.$

11. $\displaystyle\int_0^{\pi/2} \log \tan x \, dx.$

12. $\displaystyle\int_0^a \frac{dx}{\sqrt{ax - x^2}}.$

13. $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{(x^2 - 1)}}.$

14. $\displaystyle\int_0^1 \left(\log \frac{1}{x}\right)^n dx.$

15. $\displaystyle\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx.$

16. $\displaystyle\int_0^\infty e^{-x} x^m (\log x)^n \, dx.$

17. $\displaystyle\int_0^\pi \log \sin x \, dx.$

18. $\displaystyle\int_0^\pi \frac{1}{x} \log \sin x \, dx.$

19. $\displaystyle\int_0^\pi x \log \sin x \, dx.$

20. $\displaystyle\int_{-\infty}^\infty e^{-x^2} \, dx.$

21. $\displaystyle\int_0^\infty x^{2n-1} e^{-x^2} \, dx.$

22. $\displaystyle\int_0^{\pi/2} \frac{x^m \, dx}{(\sin x)^n}.$

23. $\displaystyle\int_0^\infty \frac{dx}{1 + x^4 \sin^2 x}.$

24. $\displaystyle\int_0^\infty \frac{x \, dx}{1 + x^2 \sin^2 x}.$

***25.** $\displaystyle\int_0^\infty \frac{x^\alpha \, dx}{1 + x^\beta \sin^2 x}.$

PROBLEMS

SECTION 3.15, page 301

***1.** Prove that $\displaystyle\int_0^\infty \sin^2 \left[\pi\left(x + \frac{1}{x}\right)\right] dx$ does not exist.

***2.** Prove that $\displaystyle\lim_{k \to \infty} \int_0^\infty \frac{dx}{1 + kx^{10}} = 0.$

3. For what values of s is (*a*) $\displaystyle\int_0^\infty \frac{x^{s-1}}{1 + x} dx$, (*b*) $\displaystyle\int_0^\infty \frac{\sin x}{x^s} dx$ convergent?

***4.** Does $\displaystyle\int_0^\infty \frac{\sin t}{1 + t} dt$ converge?

***5.** (*a*) If a is a fixed positive number, prove that

$$\lim_{h \to 0} \int_{-a}^a \frac{h}{h^2 + x^2} dx = \pi.$$

(*b*) If $f(x)$ is continuous in the interval $-1 \le x \le 1$, prove that

$$\lim_{h \to 0} \int_{-1}^1 \frac{h}{h^2 + x^2} f(x) \, dx = \pi f(0).$$

***6.** Prove that $\lim_{x \to \infty} e^{-x^2} \int_0^x e^{t^2}\, dt = 0.$

7. Assuming that $|\alpha| \neq |\beta|$, prove that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \sin \alpha x \sin \beta x \, dx = 0.$$

***8.** If $\int_a^\infty \frac{f(x)}{x}\, dx$ converges for any positive value of a, and if $f(x)$ tends to a limit L as $x \to 0$, show that $\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\, dx$ converges for α and β positive and has the value $L \log \frac{\beta}{\alpha}$.

9. By reference to the Problem 8, show that

(*a*) $\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\, dx = \log \frac{\beta}{\alpha}.$

(*b*) $\int_0^\infty \frac{\cos \alpha x - \cos \beta x}{x}\, dx = \log \frac{\beta}{\alpha}.$

***10.** If $\int_a^b \frac{f(x)}{x}\, dx$ converges for any positive values of a and b, and if $f(x)$ tends to a limit M as $x \to \infty$ and a limit L as $x \to 0$, show that

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\, dx = (L - M) \log \frac{\beta}{\alpha}.$$

11. Obtain the following expressions for the gamma function:

$$\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\, dx.$$

$$\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx.$$

Answers to Exercises

SECTION 3.15, page 301

1. Convergent. **2.** Convergent. **3.** Divergent.
4. Convergent. **5.** Divergent. **6.** Convergent.
7. Convergent. **8.** Divergent. **9.** Convergent.
10. Convergent. **11.** Convergent. **12.** Convergent.
13. Convergent.

14. Convergent if $n > -1$; divergent if $n \leq -1$.
15. Convergent if $n > -1$, $m > -1$; otherwise divergent.
16. Convergent if $n > 0$, $m > -1$; otherwise divergent.
17. Convergent.
18. Divergent. **19.** Convergent. **20.** Convergent.
21. Convergent if $n > 0$; divergent if $n \leq 0$.
22. Convergent if $m > n - 1$; divergent if $m \leq n - 1$.
23. Convergent. Consider

$$\int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1 + x^4 \sin^2 x} = \left(\int_{\nu\pi}^{(\nu+\epsilon)\pi} + \int_{(\nu+\epsilon)\pi}^{(\nu+1-\epsilon)\pi} + \int_{(\nu+1-\epsilon)\pi}^{(\nu+1)\pi} \right) \frac{dx}{1 + x^4 \sin^2 x}.$$

In the first and last integrals the integrand <1, and in the second integral the integrand $< \dfrac{1}{\pi^4 \nu^4 \sin^2 \epsilon\pi}$, so that

$$\int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1 + x^4 \sin^2 x} < 2\epsilon\pi + \frac{\pi}{\pi^4 \nu^4 \sin^2 \epsilon\pi}.$$

Choose $\epsilon = 1/\nu^{4/3}$; then $\sin \epsilon\pi > \frac{1}{2}\epsilon\pi$, and

$$\int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1 + x^4 \sin^2 x} < \frac{k}{\nu^{4/3}} < k \int_{\nu-1}^{\nu} \frac{dx}{x^{4/3}},$$

where k is a constant. Finally,

$$\int_A^B \frac{dx}{1 + x^4 \sin^2 x} < \int_{n\pi}^{m\pi} \frac{dx}{1 + x^4 \sin^2 x} < k \int_{(n-1)\pi}^{(m-1)\pi} \frac{dx}{x^{4/3}}$$

$$= \frac{3k}{\pi^{1/3}} \left[\frac{1}{\sqrt[3]{n-1}} - \frac{1}{\sqrt[3]{m-1}} \right] < \frac{3k}{\pi^{1/3} \sqrt[3]{n-1}} \to 0 \text{ as } n \to \infty.$$

Or $$\int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1 + x^4 \sin^2 x} < \int_{\nu\pi}^{(\nu+1)\pi} \frac{dx}{1 + (\nu\pi)^4 \sin^2 x} < \frac{2\pi}{\sqrt{1 + (\nu\pi)^4}} < \frac{k}{\nu^2}.$$

24. $\displaystyle\int_0^A \frac{x\,dx}{1 + x^2 \sin^2 x} > \int_0^A \frac{x\,dx}{1 + x^2} > \frac{1}{2} \log(1 + A^2)$; divergent.

25. Convergent if $\beta < -2$, $\beta + 1 < \alpha < -1$ or $\beta > 0$, $-1 < \alpha < \beta/2 - 1$; otherwise divergent.

Suppose that $\beta \leq 0$. Then $\displaystyle\int^{\infty} \frac{x^\alpha\,dx}{1 + x^\beta \sin^2 x}$ converges only if $\alpha < -1$; $\displaystyle\int_0 \frac{x^\alpha\,dx}{1 + x^\beta \sin^2 x}$ behaves like $\displaystyle\int_0 \frac{x^\alpha\,dx}{1 + x^{\beta+2}}$, that is, if $\beta + 2 \geq 0$, then $\alpha > -1$, contrary to the preceding; if $\beta + 2 < 0$, $\alpha - \beta - 2 > -1$.

Suppose that $\beta > 0$. Then $\displaystyle\int_0 \frac{x^\alpha\,dx}{1 + x^\beta \sin^2 x}$ converges only if $\alpha > -1$.

Furthermore,

$$\frac{\nu^{\alpha}\pi^{\alpha+1}}{\sqrt{1 + (\nu + 1)^{\beta}\pi^{\beta}}} = \int_{\nu\pi}^{(\nu+1)\pi} \frac{(\nu\pi)^{\alpha}\, dx}{1 + (\nu + 1)^{\beta}\pi^{\beta}\sin^2 x}$$

$$< \int_{\nu\pi}^{(\nu+1)\pi} \frac{x^{\alpha}\, dx}{1 + x^{\beta}\sin^2 x} < \int_{\nu\pi}^{(\nu+1)\pi} \frac{(\nu + 1)^{\alpha}\pi^{\alpha}\, dx}{1 + (\nu\pi)^{\beta}\sin^2 x} < \frac{(\nu + 1)^{\alpha}\pi^{\alpha+1}}{\sqrt{1 + (\nu\pi)^{\beta}}}$$

or

$$k_1\nu^{\alpha-\beta/2} < \int_{\nu\pi}^{(\nu+1)\pi} \frac{x^{\alpha}\, dx}{1 + x^{\beta}\sin^2 x} < k_2\nu^{\alpha-\beta/2}.$$

Hence $\displaystyle\int_{\pi}^{\infty} \frac{x^{\alpha}\, dx}{1 + x^{\beta}\sin^2 x}$ converges if, and only if, $\alpha - \frac{1}{2}\beta < -1$.

The integral may also be estimated by the method of Exercise 23.

Solutions and Hints to Problems

SECTION 3.15, page 301

1. Consider the function in the neighborhood of $x = k - \frac{1}{2}$ for integer k. We have

$$f(x) = \sin^2 \pi\left(x + \frac{1}{x}\right) = \sin^2 \pi\left[(k - \tfrac{1}{2}) + x - (k - \tfrac{1}{2}) + \frac{1}{x}\right]$$

$$\cdot \cos^2 \pi\left[(x - [k - \tfrac{1}{2}]) + \frac{1}{x}\right].$$

If $k > 7$, then $1/x < \frac{1}{6}$ and, too, if $|x - (k - \frac{1}{2})| \le \frac{1}{6}$, then

$$f(x) > \cos^2\left(\frac{\pi}{3}\right) = \frac{1}{4}$$

Consequently, for each of these disjoint intervals of length $\frac{1}{3}$, the function is greater than the positive constant $\frac{1}{4}$. Therefore

$$\int_0^k f(x)\, dx > \frac{k - 1}{12}.$$

2.

$$\int_0^{\infty} \frac{dx}{1 + kx^{10}} < \int_0^{\epsilon} dx + \int_{\epsilon}^{A} \frac{dx}{kx^{10}}$$

$$< \epsilon + \frac{1}{9\epsilon^9 k}$$

$$< 2\epsilon$$

for $k > \epsilon^{10}/9$.

3. (*a*) For $0 < s < 1$.

(*b*) For $0 < s < 2$.

4. Yes.

5. (*a*) $\int_{-a}^{a} \frac{h}{h^2 + x^2}\, dx = 2 \arc\tan \frac{a}{h}$. Thus as h approaches 0 through positive values the integral tends to π.

(*b*) Take δ so small that $|f(x) - f(0)| < \delta$ for $|x| < \epsilon$. Then

$$\left| \int_{-1}^{1} \frac{hf(x)}{h^2 + x^2}\, dx - f(0) \int_{-\delta}^{\delta} \frac{h}{h^2 + x^2}\, dx \right|$$

$$< \epsilon \int_{-\delta}^{\delta} \frac{h}{h^2 + x^2}\, dx + M \left| \int_{-1}^{-\delta} \frac{h}{h^2 + x^2}\, dx + \int_{1}^{\delta} \frac{h}{h^2 + x^2}\, dx \right|,$$

where M is an upper bound for $|f(x)|$ in $[-1, 1]$. Now pass to the limit as h approaches 0 through positive values.

6. For any positive ϵ,

$$e^{-x^2} \int_0^x e^{t^2}\, dt = \int_0^{x-\epsilon} e^{t^2 - x^2}\, dt + \int_{x-\epsilon}^{x} e^{t^2 - x^2}\, dt < [(x - \epsilon)e^{\epsilon^2}]e^{-2\epsilon x} + \epsilon.$$

Now pass to the limit as $x \to \infty$.

7. $\sin \alpha x \sin \beta x = \frac{1}{2}[\cos(\alpha - \beta)x - \cos(\alpha + \beta)x]$. Consequently,

$$\frac{1}{T} \int_0^T \sin \alpha x \sin \beta x\, dx = \frac{1}{2T} \left[\frac{\sin(\alpha - \beta)x}{\alpha - \beta} - \frac{\sin(\alpha + \beta)x}{\alpha + \beta} \right].$$

8. $$\int_a^{\infty} \frac{f(\alpha x) - f(\beta x)}{x}\, dx = \int_{a\alpha}^{\infty} \frac{f(x)}{x}\, dx - \int_{a\beta}^{\infty} \frac{f(x)}{x}\, dx$$

$$= \int_{a\alpha}^{a\beta} \frac{f(x)}{x}\, dx = \log \frac{\beta}{\alpha} + \int_{a\alpha}^{a\beta} \frac{f(x) - L}{x}\, dx.$$

Show that this last integral tends to zero as $a \to 0$.

10. Proceed as in Problem 8.

11. In the formula $\Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1}\, dt$ substitute $t = x^2$ and $t = \log \frac{1}{x}$ respectively.

PROBLEMS

SECTION 3.16, page 312

1. Obtain the addition formula for $\sin(x + y)$.

2. Without using the addition formulas prove that $\cos x$ is an even function and $\sin x$, odd.

*3. (*a*) Prove for some positive h that $\cos x < 1$ for $0 < x < h$.
(*b*) Prove if $\cos z > 0$ for $0 \leq z \leq 2^n x$ that

$$\cos(2^{n+1}x) < 2^n(\cos x - 1) + 1.$$

(*c*) Combining the results (*a*) and (*b*) prove that $\cos x$ has a zero.

4. Let a be the smallest positive zero of $\cos x$. Prove that

$$\sin(x + 4a) = \sin x,$$
$$\cos(x + 4a) = \cos x.$$

5. Fill in the steps of the following indirect proof that $\cos x$ has a zero:
(*a*) If $\cos x$ has no zeros, then $\sin x$ is monotonically increasing for $x \geq 0$.
(*b*) The functions $\sin x$ and $\cos x$ are bounded from above and below.
(*c*) The limit of $\sin x$ as x tends to infinity exists and is positive.
(*d*) The equation

$$\cos x = 1 - \int_0^x \sin t\, dt$$

stands in contradiction to (*b*).

Solutions and Hints to Problems

SECTION 3.16, page 312

1. $u = \sin(x + a)$ is a solution of the differential equation $u'' + u = 0$ satisfying at $x = 0$ the conditions $u(0) = \sin a$, $u'(0) = \cos a$. Consequently,

$$\begin{aligned}\sin(x + a) &= u(0)\cos x + u'(0)\sin x \\ &= \sin a \cos x + \cos a \sin x.\end{aligned}$$

2. $\cos(-x)$ is a solution of $u'' + u = 0$ with $u(0) = 1$ and $u'(0) = 0$. $\sin(-x)$ is a solution with $u(0) = 0$ and $u'(0) = -1$.

3 (*a*) For $f(x) = \cos x$ we have at $x = 0, f(0) = 1, f'(0) = 0, f''(0) = -1$. From the continuity of f'' there exists an interval $0 < x < h$, where $f''(x) < 0$. Then, by the mean value theorem applied twice,

$$\begin{aligned}f(h) - f(0) &= f'(x_1)h \\ &= [f'(x_1) - f'(0)]h \\ &= f''(x_2)h^2,\end{aligned}$$

where $0 < x_2 < x_1 < h$.

(*b*) Take $x > 0$ so that $0 < \cos x < 1$. Proceed by induction. For $n = 0$
$\cos 2x = 2\cos^2 x - 1$

$$< 2\cos x - 1 = 2(\cos x - 1) + 1.$$

If the inequality holds for $n = k$, then

$$\begin{aligned}\cos 2^{k+1}x &= 2\cos^2 2^k x - 1\\ &\le 2\cos 2^k x - 1\\ &< 2^{k+1}(\cos x - 1) + 1.\end{aligned}$$

(*c*) If $\cos x > 0$ for all x then from part *b*, $2^n(\cos x - 1) + 1 > 0$ for all n, or $1 - \cos x < 1/2^n$, which contradicts $1 - \cos x > 0$.

4. $$\begin{aligned}\sin(x + 4a) &= \sin x \cos 4a + \cos x \sin 4a\\ &= \sin x(2\cos^2 2a - 1)\\ &\quad + \cos x(4 \sin a \cos a \cos 2a)\\ &= \sin x[2(2\cos^2 a - 1)^2 - 1]\\ &= \sin x.\end{aligned}$$

Similarly, for $\cos(x + 4a)$.

5. (*a*) Set $u = \cos x$, $v = \sin x$. If $u(x)$ has no zeros, since $u(0) = 1$, it follows by continuity that $u(x) > 0$. From $v'(x) = u(x) > 0$, it follows that $v(x)$ is increasing for $x \ge 0$.

(*b*) Follows from $u^2(x) + v^2(x) = 1$.

(*c*) Since $v(x)$ is bounded and monotonic, it must have a limit L, and since $v(x)$ increases from zero, L is positive.

(*d*) From part *c*,

$$\cos x = 1 - \int_0^x \sin t\, dt$$

is not bounded below, in contradiction to (*b*).

MISCELLANEOUS PROBLEMS, Chapter 3

1. Prove

$$\frac{d^n}{dx^n} f(\log x) = x^{-n} \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right) \cdots \left(\frac{d}{dt} - n + 1\right) f(t)$$

when $t = \log x$. Here, we employ

$$\left(\frac{d}{dt} - k\right)\phi = \frac{d\phi}{dt} - k\phi,$$

where ϕ is any function of t and k is a constant.

2. A smooth closed curve C is said to be convex if it lies wholly to one side of each tangent. Show that for the triangle of minimum area circumscribed about C that each side is tangent to C at its midpoint.

Solutions and Hints to Problems

MISCELLANEOUS PROBLEMS, Chapter 3, page 323

1. Set

$$L_n[f(t)] = \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right)\cdots\left(\frac{d}{dt} - n + 1\right)f(t).$$

For $n = 1$,
$$\frac{d}{dx}f(\log x) = x^{-1}L_1[f(t)].$$

If for $n = k$,
$$\frac{d^k}{dx^k}f(\log x) = x^{-k}L_k[f(t)],$$

then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}f(\log x) &= \frac{d}{dx}\left\{x^{-k}L_k[f(t)]\right\}\\ &= -kx^{-k-1}L_k[f(t)]\\ &\quad + x^{-k}\frac{d}{dt}L_k[f(t)]\frac{dt}{dx}\\ &= x^{-k-1}\left(\frac{d}{dt} - k\right)L_k[f(t)]\\ &= x^{-k-1}L_{k+1}[f(t)],\end{aligned}$$

where at the last step we use the commutativity of the differential operators $(d/dt - a)$ and $(d/dt - b)$, for a and b any constants.

2. Choose a coordinate system in which the base of the circumscribed triangle has its point of tangency at the origin and vertices on the x-axis at $(-a, 0)$, $(b, 0)$, where $b \geq a > 0$. Let the equation for C in the neighborhood of the origin be given by $y = f(x)$, where, by assumption, $f(0) = 0$, $f'(0) = 0$ and the convexity condition is taken to be $f''(0) > 0$. We show that if the point of tangency is not at the midpoint of the base (if $b > a$), then a smaller circumscribed triangle may be constructed for which the base is replaced by the tangent at a point (x, y) close to the origin. Let the coordinates of the vertex be (c, d); let the ordinates of the points where the tangent at (x, y) meets the left and right sides be y_1 and y_2, respectively, and the x-intercept of the tangent, z. The area of the triangle intercepted by the new tangent is seen easily to be $A(x) = \frac{1}{2}d(b + c) - \frac{1}{2}y_1(z + a) - \frac{1}{2}y_2(b - z)$, where, for $m = f'(x)$

$$y_1 = \frac{md(z + a)}{m(c + a) - d}, \qquad y_2 = \frac{md(z - b)}{m(c - b) + d}.$$

It follows that for $b > a$,

$$A'(0) = -\tfrac{1}{2}f''(0)(b^2 - a^2) < 0$$

and the area can be lessened.

4 *Applications in Physics and Geometry*

EXERCISES

SECTION 4.1c, page 328

1. For each of the following parametric representations of a curve, find a nonparametric equation and identify the curve.

(*a*) $x = at + b$, $y = ct + d$.
(*b*) $x = a \cosh u$, $y = b \sinh u$,
(*c*) $x = |\cos z|$, $y = \sin z$.
(*d*) $x = a \cos \theta$, $y = 0$.
(*e*) $x = at + b$, $y = \alpha t^2 + \beta$.
(*f*) $x = a \sec \theta$, $y = b \tan \theta$.
(*g*) $x = |t| + t$, $y = |t| - t$.

2. Sketch the graph of the curve

$$x = a \cos 2\theta \cos \theta,$$
$$y = a \cos 2\theta \sin \theta,$$

and give its equation in nonparametric form.

3. Plot the following curves and find their equations in nonparametric form:

(*a*) $x = \dfrac{5at^2}{1 + t^5}, \qquad y = \dfrac{5at^3}{1 + t^5}.$

(*b*) $x = at + b \sin t, \qquad y = a - b \cos t.$

PROBLEMS

SECTION 4.1c, page 328

1. Sketch the hypocycloid for $a = 4c$ (the astroid) and find its nonparametric equation.

2. Prove that if c/a is rational, the general hypocycloid is closed after the moving circle has rotated an integral number of times, whereas if c/a is irrational, the curve has infinitely many points where it meets the circumference of the fixed circle and will not close.

3. Derive the parametric representation

$$x = at - b\sin t, \qquad y = a - b\cos t$$

for an ordinary trochoid, that is, for the path of a point P attached to a disc of radius a rolling along a line, P having the distance b from the center of the disc (see Fig. 4.7).

4. Find the parametric equations for the curve $x^3 + y^3 = 3axy$ (the folium of Descartes), choosing as parameter t the tangent of the angle between the x-axis and the ray from the origin to the point (x, y).

Answers to Exercises

SECTION 4.1c, page 328

1. (*a*) $ay - cx = ad - bc$. A straight line provided $a^2 + c^2 \neq 0$.

(*b*) $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$. Hyperbola, right branch.

(*c*) $x^2 + y^2 = 1$, $x \geq 0$. Semicircle.

(*d*) $y = 0$, $|x| < a$. Segment joining the points $(-a, 0)$ and $(a, 0)$.

(*e*) $a^2(y - \beta) = \alpha(x - b)^2$. Parabola.

(*f*) $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$. Hyperbola.

(*g*) $x = 0, y \geq 0$ or $y = 0, x \geq 0$.

Broken line consisting of the rays directed from the origin along the positive axes.

2. $(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$.

3. (*a*) $x^5 + y^5 = 5ax^2y^2$.

(*b*) $x = a \arccos \dfrac{a - y}{b} + \sqrt{b^2 - (a - y)^2}$.

Solutions and Hints to Problems

SECTION 4.1c, page 328

1. $\left(\frac{x}{4c}\right)^{2/3} + \left(\frac{y}{4c}\right)^{2/3} = 1$.

2. Set $\lambda = a/c$. The parametric representation of the hypocycloid can be written

$$x(t) = c(\lambda - 1)\cos t + c\cos(\lambda - 1)t$$
$$y(t) = c(\lambda - 1)\sin t - c\sin(\lambda - 1)t.$$

If λ is rational, $\lambda = p/q$ in lowest terms, then clearly $x(t + 2n\pi q) = x(t)$, $y(t + 2n\pi q) = y(t)$.

Conversely, if $x(t + \alpha) = x(t)$, $y(t + \alpha) = y(t)$ for some $\alpha > 0$, then $y(\alpha) = y(0) = 0$ and $x'(\alpha) = x(0) = 0$, or

$$c(\lambda - 1)\sin\alpha - c\sin(\lambda - 1)\alpha = 0,$$
$$-c(\lambda - 1)\sin\alpha - c\sin(\lambda - 1)\alpha = 0.$$

Since $\lambda > 1$, it follows that $\sin\alpha = \sin(\lambda - 1)\alpha = 0$ and therefore $\alpha = m\pi$ and $(\lambda - 1)\alpha = n\pi$ for some positive integers m and n. Consequently, $\lambda = n/m + 1$ is rational.

Finally, the points where the hypocycloid meets the circle satisfy $\cos \lambda t = 1$. Let $t_n = 2n\pi/\lambda$ for any integer n. If $x(t_n) = x(t_m)$ and $y(t_n) = y(t_m)$ for distinct values of n and m, then $\cos t_n = \cos t_m$ and $\sin t_n = \sin t_m$, whence

$$t_m = t_n + 2k\pi,$$

where k is some nonzero integer. It follows that $\lambda = k/(m - n)$ for λ rational. If λ is irrational, then the hypocycloid never meets the fixed circle more than once at a given point.

4. $x = \dfrac{3at}{1 + t^3}, \qquad y = \dfrac{3at^2}{1 + t^3}.$

EXERCISES

SECTION 4.1d, page 333

1. Discuss the orientation of the curves of Exercises 4.1c, No. 1.

Answers to Exercises

SECTION 4.1d, page 333

1. (*a*) If $a > 0$, the line is traversed from left to right.

(*b*) If $a > 0$ and $b > 0$, the right branch of the hyperbola is traversed upward.

(*c*) For $(2n - \frac{1}{2})\pi \le z \le (2n + \frac{1}{2})\pi$, the semicircle is traversed upward; for $(2n + \frac{1}{2})\pi \le z \le (2n + \frac{3}{2})\pi$, downward.

(*e*) For $a > 0$ the parabola is traversed from left to right.

(*f*) If $a > 0$ and $b > 0$ the right branch of the hyperbola is traversed upward for $(2n - \frac{1}{2})\pi \le \theta \le (2n + \frac{1}{2}\pi)$; the left branch upward for $(2n + \frac{1}{2})\pi \le \theta \le (2n + \frac{3}{2})\pi$.

(*g*) Along the x-axis from $+\infty$ to 0, thence along the y-axis from 0 to $+\infty$.

EXERCISES

SECTION 4.1e, page 343

1. For each of the following curves obtain the equations of the tangent and normal lines at the indicated points.

(*a*) $x = a \cos \theta$, $y = a \sin \theta$, $\left(\theta = \dfrac{\pi}{6}\right)$.

(*b*) $x = \sin 2\alpha$, $y = \sin \alpha$, $\left(\alpha = \dfrac{\pi}{4}\right)$.

(*c*) $x = a\theta \cos \theta$, $y = a\theta \sin \theta$, $(\theta = 2n\pi)$.

(*d*) $x = a\dfrac{1 - t^2}{1 + t^2}$, $y = \dfrac{2bt}{1 + t^2}$, $(t = 0, 1)$.

(*e*) $x = \dfrac{3a \cos^2 \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}$, $y = \dfrac{3a \cos \theta \sin^2 \theta}{\cos^3 \theta + \sin^3 \theta}$, $\left(\theta = 0, \dfrac{\pi}{2}\right)$.

(Why should the answers be different at the same point of the curve?)

2. Prove for the curve

$$x = f(t),\ y = g(t)$$

that the equation of the tangent line at $t = \tau$ is given in parametric form by

$$x = f(\tau) + uf'(\tau),$$
$$y = g(\tau) + ug'(\tau).$$

Obtain a similar representation for the normal line.

3. A constant length l is measured off along the normal to the parabola $y^2 = 4px$. Find the curve described by the extremity of this segment.

PROBLEMS

SECTION 4.1e, page 343

1. The angle α between two curves at a point of intersection is defined to be the angle between their tangents at the point. Find a formula for $\cos \alpha$ in terms of the parametric representations of the curves.

2. Let $x = f(t)$ and $y = g(t)$. Derive formulas for d^2y/dx^2 and d^3y/dx^3 in terms of derivatives with respect to the parameter t.

3. Find the formula for the angle α between two curves $r = f(\theta)$ and $r = g(\theta)$ in polar coordinates.

4. Find the equations of the curves which everywhere intersect the straight lines through the origin at the same angle α.

5. Prove: if $x = f(t)$ and $y = g(t)$ are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) with $x'^2 + y'^2 > 0$, then there is at least one point on the open arc

$$x = f(t), \qquad y = g(t), \qquad (a < t < b),$$

where the tangent is parallel to the chord joining the end points.

6. Let P be the point of a circle which traces out a cycloid as the circle rolls on a given line. Let Q be the point of contact of the circle with the line. Prove that at any instant, the normal to the cycloid at P passes through Q. What similar property holds for the tangent at P?

7. Prove that the length of the segment of the tangent to the astroid,

$$x = 4c\cos^3\theta, \qquad y = 4c\sin^3\theta,$$

cut off by the coordinate axes is constant.

***8.** Show that the two families of ellipses and hyperbolas, $0 < a < b$

$$\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1, \quad \text{for} \quad 0 < \lambda < a,$$

$$\frac{x^2}{a^2 - \tau^2} + \frac{y^2}{b^2 - \tau^2} = 1 \quad \text{for} \quad a < \tau < b,$$

are confocal (that is, have the same foci) and intersect at right angles.

9. (*a*) Show for the ellipse that the angle between the two rays from the foci to a point on the curve is bisected by the normal at the point.

(*b*) Show for the hyperbola that the angle is bisected by the tangent.

Answers to Exercises

SECTION 4.1e, page 343

1. (*a*) Tangent: $y = -\sqrt{3}x + 2a$; normal: $y = \dfrac{x}{\sqrt{3}}$.

(*b*) Tangent: $x = 1$; normal: $y = \dfrac{1}{\sqrt{2}}$.

(*c*) Tangent: $y = 2n\pi x - 4n^2\pi^2(-1)^n a$;

normal: $y = -\dfrac{x}{2n\pi} + (-1)^n a$.

(*d*) For $t = 0$; tangent: $x = a$;
normal: $y = 0$. For $t = 1$; tangent: $y = b$;
normal $x = 0$.

(*e*) For $\theta = 0$; tangent: $y = 0$; normal: $x = 0$.

For $\theta = \frac{\pi}{2}$; tangent: $x = 0$; normal: $y = 0$.

For $0 \leq \theta \leq \frac{\pi}{2}$ the curve describes a closed loop with a corner at the origin.

3. $x = t - \dfrac{l\sqrt{p}}{\sqrt{t+p}}, y^2 = 4pt\left(1 + \dfrac{l}{2\sqrt{p}\sqrt{t+p}}\right)^2.$

Solutions and Hints to Problems

SECTION 4.1e, page 343

1. Let the two curves be given by parametric representations $[x_1(u), y_1(u)]$, $[x_2(v), y_2(v)]$.

$$\cos\alpha = \frac{\dfrac{dx_1}{du}\dfrac{dx_2}{dv} + \dfrac{dy_1}{du}\dfrac{dy_2}{dv}}{\sqrt{\left(\dfrac{dx_1}{du}\right)^2 + \left(\dfrac{dy_1}{du}\right)^2}\sqrt{\left(\dfrac{dx_2}{dv}\right)^2 + \left(\dfrac{dy_2}{dv}\right)^2}}.$$

2. $$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2},$$

$$\frac{d^3y}{dx^3} = \frac{(\dot{x}\dddot{y} - \dot{y}\dddot{x})\dot{x} - 2(\dot{x}\ddot{y} - \dot{y}\ddot{x})\ddot{x}}{\dot{x}^3}$$

where the dot denotes differentiation with respect to t.

3. At a point of intersection $r = f(\theta) = g(\phi)$, where $\phi = \theta + 2k\pi$, and for the angle α between the two curves

$$\cos\alpha = \frac{f'(\theta)g'(\phi) + r^2}{\sqrt{f'(\theta)^2 + r^2}\quad\sqrt{g'(\phi)^2 + r^2}}.$$

4. Let θ be the angle of inclination of the straight line and let the line be represented in terms of the parameter t, $x_1 = (\cos\theta)t$, $y_1 = (\sin\theta)t$. Let the curve be given in terms of the parameter θ by $x_2 = r(\theta)\cos\theta$, $y_2 = r(\theta)\sin\theta$. Inserting these expressions in the solution to Problem 1 we obtain

$$\cos\alpha = \frac{r'}{\sqrt{r'^2 + r^2}}$$

whence $r'/r = (d/d\theta)\log r = \pm\tan\alpha$, and $r = Ce^{\pm\theta\tan\alpha}$. For unknown reasons this exponential spiral is usually referred to as a "logarithmic" spiral.

5. The distance of a point $(f(t), g(t))$ on the arc from the chord joining the points $(f(a), g(a))$ and $(f(b), g(b))$ is proportional to

$$\Phi(t) = f(t)[g(b) - g(a)] - g(t)[f(b) - f(a)] + f(b)g(a) - f(a)g(b).$$

Since $\Phi(a) = \Phi(b) = 0$, it follows by Rolle's theorem that for some point τ in the open interval (a, b)

$$\Phi'(\tau) = f'(\tau)[g(b) - g(a)] - g'(\tau)[f(b) - f(a)] = 0$$

which proves the theorem.

6. We have

$$P = (x, y) = \{a[t - \sin t], a[1 - \cos t]\}$$

and $Q = (at, 0)$. Consequently, the slope of the line PQ is $-(1 - \cos t)/a \sin t$. The slope of the tangent at P is $\dot{y}/\dot{x} = (\sin t)/(1 - \cos t)$ which is the negative reciprocal of the slope of PQ.

The tangent to the cycloid passes through the highest point on the circle, the point $(at, 2a)$.

7. The equation of the tangent line is

$$\frac{x}{4c \cos \theta} + \frac{y}{4c \sin \theta} = 1.$$

8. Foci at $(0, \pm\sqrt{b^2 - a^2})$. At a point of intersection (x, y)

$$x^2 = \frac{(a^2 - \lambda^2)(\tau^2 - a^2)(\tau^2 - \lambda^2)}{(a^2 - \lambda^2)(b^2 - \tau^2) + (b^2 - \lambda^2)(\tau^2 - a^2)},$$

$$y^2 = \frac{(b^2 - \lambda^2)(b^2 - \tau^2)(\tau^2 - \lambda^2)}{(a^2 - \lambda^2)(b^2 - \tau^2) + (b^2 - \lambda^2)(\tau^2 - a^2)};$$

hence, for the product of the slopes,

$$-\frac{x^2}{y^2}\frac{(b^2 - \lambda^2)(b^2 - \tau^2)}{(a^2 - \lambda^2)(a^2 - \tau^2)} = -1.$$

9. The angle of inclination γ of the angle bisector is $\frac{1}{2}(\alpha + \beta)$, where α and β are the angles of inclination of the rays from the foci. Let the ellipse be represented in parametric form by

$$x = a \cos \theta, \qquad y = b \sin \theta,$$

where $a > b$ and the foci by $(\pm c, 0)$, where $c = \sqrt{a^2 - b^2}$. From

$$\tan \alpha = \frac{b \sin \theta}{a(\cos \theta) - c},$$

$$\tan \beta = \frac{b \sin \theta}{a(\cos \theta) + c},$$

it follows easily that

$$\tan (\alpha + \beta) = \frac{2ab \sin \theta \cos \theta}{b^2 \cos^2 \theta - a^2 \sin^2 \theta} = \tan 2\gamma.$$

PROBLEMS

SECTION 4.1f, page 348

1. Prove that the curve defined by

$$y = \begin{cases} x^2 \sin \dfrac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

has finite length, but that the continuous curve defined by

$$y = \begin{cases} x \sin \dfrac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

is not rectifiable.

2. Prove that if the function f is defined and monotone on the closed interval $[a, b]$, then the arc defined by

$$y = f(x), \qquad (a \leq x \leq b)$$

is rectifiable.

Solutions and Hints to Problems

SECTION 4.1f, page 348

1. (*a*) Verify that $\int_\epsilon^1 \sqrt{1 + y'^2}\, dx$ is bounded and monotonic in ϵ for $0 < \epsilon < 1$.

(*b*) Consider the chords joining successive points of the sequence $x_m = \dfrac{1}{(m + \frac{1}{2})\pi}$, $y_m = (-1)^m x_m$ for $m = 0, 1, \ldots$. The length of the inscribed polygon between x_0 and x_m is

$$\sum_{k=1}^{m} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} > \sum_{k=1}^{n} |y_k - y_{k-1}|$$

$$\geq \frac{2}{\pi} \sum_{k=1}^{n} \frac{k}{k^2 - \frac{1}{4}} > \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k}.$$

However, $\sum_{k=1}^{m} \dfrac{1}{k}$ is the upper sum for the integral $\int_1^{m+1} \dfrac{1}{x}\, dx$, when the interval is subdivided into m equal parts. Consequently,

$$\sum_{k=1}^{m} \frac{1}{k} \geq \log (m + 1)$$

and the length of the inscribed polygon becomes infinite in the limit, $m \to \infty$ or $x_m \to 0$.

It follows also from Miscellaneous Problems to Chapter 1, No. 1, that $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge.

2. Consider any subdivision $a = x_0 < x_1 < x_2 < \cdots < x_n = k$ of the interval. Set $y_n = f(x_n)$. The inscribed polygon with the successive vertices (x_n, y_n) has length

$$L = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$
$$\leq \sum_{k=1}^{n} [|x_k - x_{k-1}| + |y_k - y_{k-1}|].$$

Since $f(x)$ is monotone, $|y_k - y_{k-1}|$ has the same sign for all k. Consequently,

$$L \leq b - a + |f(b) - f(a)|.$$

Thus the inscribed polygons have an upper bound and therefore the arclength (or least upper bound of the lengths of inscribed polygons) exists.

EXERCISES

SECTION 4.1g, page 352

1. Give expressions for arclength of the following curves:

(*a*) The catenary, $y = \cosh x$.
(*b*) The semicubical parabola, $y = x^{3/2}$.
(*c*) The astroid, $x = \cos^3 \theta$, $y = \sin^3 \theta$ for $0 \leq \theta \leq 2\pi$.
(*d*) The cardioid, $x = 2a \cos \theta - a \cos 2\theta$, $y = 2a \sin \theta - a \sin 2\theta$.
(*e*) The Archimedean spiral, $r = a\theta (a > 0)$.
(*f*) The logarithmic spiral, $r = e^{m\theta}$.
(*g*) The curve, $r = a(\theta^2 - 1)$.

2. (*a*) Find the parametric representation of the cardioid when the length of arc is used as parameter.

(*b*) Do the same for the cycloid.

3. Find the length of arc of the epicycloid

$$x = (a + b) \cos t - b \cos \frac{a + b}{b} t,$$

$$y = (a + b) \sin t - b \sin \frac{a + b}{b} t,$$

reckoned from the initial point $t = 0$.

***4.** Obtain the formula for the length of a curve in polar coordinates.

PROBLEMS

SECTION 4.1g, page 352

1. An *elliptic integral of the second kind* has the form

$$\int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

(*a*) Show that the arc length of the ellipse $x = a \cos \theta$, $y = b \sin \theta$ can be expressed in terms of an elliptic integral of the second kind.

(*b*) Do the same for the trochoid

$$x = at - b \sin t, \qquad y = a - b \cos t.$$

*(*c*) Show that the arc length of the hyperbola can be expressed in terms of elliptic integrals of the first and second kinds.

Answers to Exercises

SECTION 4.1g, page 352

1. (*a*) $\sinh x$, (*b*) $(x + \frac{4}{9})^{3/2}$, (*c*) 6.

(*d*) $8a(1 - \cos 2\theta)$, (*e*) $\frac{1}{2}a(\operatorname{ar\,sinh} \theta + \theta\sqrt{1 + \theta^2})$. (*f*) $\dfrac{\sqrt{1 + m^2}}{m}\, e^{m\theta}$.

(*g*) $a\left(\dfrac{\theta^3}{3} + \theta\right)$.

2. (*a*) $x = R + s\left(1 - \dfrac{s}{2R} + \dfrac{s^2}{32R^2}\right)\left(1 - \dfrac{s}{16R}\right)$,

$$y = R\left(\frac{s}{R} - \frac{s^2}{16R^2}\right)^{3/2}\left(1 - \frac{s}{8R}\right), \qquad \text{for } 0 \le s \le 16R.$$

(*b*) $x = 2a \arccos\left(1 - \dfrac{s}{4a}\right) - \left(1 - \dfrac{s}{4a}\right)\sqrt{s\left(1 - \dfrac{s}{8a}\right)/2a}$,

$$y = s - \frac{s^2}{8a} \qquad \text{for } 0 \le s \le 8a.$$

3. $\dfrac{4b(a + b)}{a}\left(1 - \cos \dfrac{a}{2b}\, t\right)$.

4. $\displaystyle\int \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$

Solutions and Hints to Problems

SECTION 4.1g, page 352

1. Elliptic integrals of the first and second kind are denoted by

$$F(k, \phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

and

$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

(*a*) The arclength for $\phi_1 \leq \theta \leq \phi_2$ for $b > a$, $k = 1 - a^2/b^2$ is

$$b[E(k, \phi_2) - E(k, \phi_1)].$$

(*b*) The arclength for $\phi_1 \leq t \leq \phi_2$ is

$$2(a + b)\{E[k, \tfrac{1}{2}(\phi_2 + \pi)] - E[k, \tfrac{1}{2}(\phi_1 + \pi)]\}$$

where $k^2 = 4ab/(a + b)^2$.

(*c*) For the hyperbola,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

set $v = x/a\alpha$, where $\alpha^2 = a^2/(a^2 + b^2)$. The arclength is then given by

$$I = a\alpha \int_{v_1}^{v_2} \frac{\sqrt{1 + v^2}}{\sqrt{1 + \alpha^2 v^2}}\, dv.$$

Set $v = \tan \phi$:

$$I = a\alpha \int_{\phi_1}^{\phi_2} \frac{d\phi}{\cos^2 \phi \sqrt{1 - \beta^2 \sin^2 \phi}}.$$

$$= \frac{a}{\alpha}\left[\frac{\sqrt{1 - \beta^2 \sin^2 \phi}\tan \phi}{\alpha^2} + \alpha^2 F(\beta, \phi) - E(\beta, \phi)\right]_{\phi_1}^{\phi_2},$$

where $\beta^2 = 1 - \alpha^2$ as may be verified by differentiation.

EXERCISES

SECTION 4.1h, page 354

1. Find the equation of the osculating circle to the parabola $y = x^2$ when $x = 1$.

2. (*a*) Determine the curvature of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the ends of the two axes.

(*b*) Determine the radius of curvature of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at a vertex. Do the same for the parabola

$$y = ax^2.$$

(*c*) Write the equation of the osculating circle in each of the above cases.

3. Give formulas for the curvature for each of the curves of Exercises 4.1g, No. 1.

4. Find the radius of curvature of

(*a*) the parabola $y = x^2$.

(*b*) the ellipse $x = a\cos\phi$. $y = b\sin\phi$ as a function of x and of ϕ respectively.

Find the maxima and minima of the radius of curvature and the points at which these maxima and minima occur.

5. Sketch the curve

$$x = \int_0^t \frac{\cos u}{\sqrt{u}}\,du, \qquad y = \int_0^t \frac{\sin u}{\sqrt{u}}\,du$$

and determine its radius of curvature ρ.

6. Sketch the curve defined by the equations

$$x = \int_0^t \cos\left(\frac{1}{2}\pi t^2\right) dt, \qquad y = \int_0^t \sin\left(\frac{1}{2}\pi t^2\right) dt.$$

What is the behavior of the curve as t runs from $-\infty$ to $+\infty$? Calculate the curvature k as a function of the length of arc.

PROBLEMS

SECTION 4.1h, page 354

1. Let P be a point of the rolling circle which generates a cycloid and let Q be the lowest point of the circle at any given instant. Show that Q bisects the segment joining P to the center of the osculating circle of the cycloid at P.

2. Find the center of curvature for $y = x^2$ when $x = 0$. Determine the point of intersection of the normal lines to the curve when $x = 0$ and when $x = \epsilon$. Calculate the distance of the intersection from the center of curvature. Suggest an alternative definition for the center of curvature. Prove that this definition is equivalent to the definition given in the text.

3. Consider the question of whether the osculating circle crosses the curve at the point of contact.

***4.** Prove that the circle of curvature at a point P of the curve C is the limit of the circles through three points P, P_1, P_2 as P_1 and P_2 tend to P.

5. Let $r = f(\theta)$ be the equation of a curve in polar coordinates. Prove that the curvature is given by the formula

$$k = \frac{2r'^2 - rr'' + r^2}{(r'^2 + r^2)^{3/2}},$$

where

$$r' = \frac{df}{d\theta}, \qquad r'' = \frac{d^2f}{d\theta^2}.$$

6. The curve for which the length of the tangent intercepted between the point of contact and the y-axis is always equal to 1 is called the tractrix. Find its equation. Show that the radius of curvature at each point of the curve is inversely proportional to the length of the normal intercepted between the point on the curve and the y-axis. Calculate the length of arc of the tractrix and find the parametric equations in terms of the length of arc.

7. Let $x = x(t), y = y(t)$ be a closed curve. A constant length p is measured off along the normal to the curve. The extremity of this segment describes a curve which is called a parallel curve to the original curve. Find the area, the length of arc, and the radius of curvature of the parallel curve.

8. Show that the only curves whose curvature is a fixed constant k are circles of radius $1/k$.

***9.** If the curvature of a curve in the xy-plane is a monotonic function of the length of arc, prove that the curve is not closed and that it has no double points.

Answers to Exercises

SECTION 4.1h, page 354

1. $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$.

2. (*a*) At $(\pm a, 0)$, $\kappa = \dfrac{a}{b^2}$.

(*b*) For the hyperbola, $\dfrac{a}{b^2}$, for the parabola, $2a$.

4. (*a*) $\frac{1}{2}(1 + 4x^2)^{3/2}$; min $\frac{1}{2}$ at $x = 0$.

(*b*) $\dfrac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}{ab}$; if $a > b$, min $\dfrac{b^2}{a}$ at $\phi = 0, \pi$; max $\dfrac{a^2}{b}$ at $\phi = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

5. $\rho = \dfrac{1}{\sqrt{t}}$

6. $\kappa = \pi s$.

Solutions and Hints to Problems

SECTION 4.1h, page 354

1. The point Q lies in the direction of the normal from P at the distance $a\sqrt{2(1 - \cos t)}$ equal to half the radius of curvature.

2. Center of curvature at $(0, \frac{1}{2})$. Intersection of normals at $(0, \frac{1}{2} + \epsilon^2)$ at distance ϵ^2 from the center of curvature. The center of curvature to the curve at a point P is the limiting position of the intersection of the normals at P and Q as Q approaches P.

Choose coordinates so that P falls at the origin and the tangent at P is the x-axis. Let $y = f(x)$ be the equation of the curve in the neighborhood of P; then $f(0) = f'(0) = 0$. The center of curvature then lies at $[0, 1/f''(0)]$. The equation of the normal at $Q = [\epsilon, f(\epsilon)]$ is

$$y - f(\epsilon) = -\frac{x - \epsilon}{f'(\epsilon)}$$

and this line meets the normal at the origin (the y-axis) at

$$y = f(\epsilon) + \epsilon/f'(\epsilon) = (\epsilon - 0)/(f'(\epsilon) - f'(0)) + f(\epsilon).$$

Take the limit as ϵ approaches 0.

3. Employ the coordinates of Problem 2. The osculating circle has the equation (for center of curvature at $(0, a)$, where $a = 1/f''(0)$),

$$g(x) = a - \sqrt{a^2 - x^2}.$$

For $\phi(x) = f(x) - g(x)$ we have $\phi(0) = \phi'(0) = \phi''(0)$, but $\phi'''(0) = f'''(0)$. If $f'''(x) \neq 0$, employing the mean value theorem repeatedly, we have

$$\phi(x) = \phi'(x_1)x = \phi''(x_2)xx_1 = \phi'''(x_3)xx_1x_2$$

where x_{i+1} lies between x_i and 0. We see then, if $f'''(0) \neq 0$ and the third derivative is continuous [hence $\phi'''(x_3)$ has the same sign as $f'''(0)$ if x is sufficiently close to 0] that $\phi(x)$ changes sign at the origin and the two curves cross. If $f'''(0) = 0$ it is necessary to consider higher derivatives.

4. Again, adopt the convention of the preceding exercises, taking P at the origin. Set $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, where $y_i = f(x_i)$. The center of the circle through O, P_1, and P_2 lies at (α, β), where

$$\alpha = \frac{y_2(x_1^2 + y_1^2) - y_1(x_2^2 + y_2^2)}{2(y_2x_1 - x_2y_1)}$$

$$\beta = \frac{x_1(x_2^2 + y_2^2) - x_2(x_1^2 + y_1^2)}{2(y_2x_1 - x_2y_1)}.$$

We assume x_1, x_2, 0 distinct and $f''(x) \neq 0$ in the neighborhood of $x = 0$. Assume $f''(x)$ continuous. Then for $|x| < \delta$

$$f''(0) - \epsilon \leq f''(x) \leq f''(0) + \epsilon.$$

Integrating twice from 0 to x we obtain

$$|f(x) - \tfrac{1}{2}f''(0)x^2| < \tfrac{1}{2}\epsilon x^2$$

or

$$f(x) = \tfrac{1}{2}[f''(0) + e(x)]x^2,$$

where $\lim_{x\to 0} e(x) = 0$. Entering this above, we obtain

$$\lim_{\substack{x_1\to 0\\ x_2\to 0}} \alpha = 0, \qquad \lim_{\substack{x_1\to 0\\ x_2\to 0}} \beta = \frac{1}{f''(0)}.$$

For a general treatment of the kind of estimate used above, see Problems 5.3*b*, No. 2.

5. Use the parametric representation of the curve in the form $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$.

6. $y = -\text{ar}\cosh\dfrac{1}{x} + \sqrt{(1 - x^2)} + \text{constant}$; $s = \log\left(\dfrac{x}{x_0}\right)$;

$$x = e^s,\ y = -\text{ar}\cosh e^{-s} + \sqrt{(1 - e^{2s})} + \text{constant}.$$

7. Let ds, ds' be the lengths of arc, l, l' the total lengths, A, A' the areas, and k, k' the curvatures of the curve and the parallel curve respectively. Then

$$ds' = (1 + pk)\,ds; \qquad k' = \frac{k}{1 + pk};$$

$$A' = A + lp + \pi p^2; \qquad l' = l + 2\pi p.$$

8. Choose coordinates for the curve such that for a given point of the curve the origin is at the center of curvature on the positive y-axis and the x-axis is tangent at the point. Set $y' = p$. The condition of constant curvature is

$$\frac{p'}{(1 + p^2)^{3/2}} = \frac{1}{k}.$$

Integrate with respect to x and solve for p to obtain

$$y' = -\frac{x}{\sqrt{r^2 - x^2}}$$

and integrate again to obtain

$$y = \sqrt{r^2 - x^2}.$$

9. If the curve is closed and has arclength σ, then for the parametric representation $x = f(s)$, $y = g(s)$ we have the periodicity condition $f(s + \sigma) = f(s)$, $g(s + \sigma) = g(s)$. Since the curvature must then also be periodic, $\kappa(s + \sigma) = \kappa(s)$, it cannot be monotone.

Let κ be an increasing function of s, P be a point of the curve. As s increases the osculating circles shrink within the osculating circle at P, and the curve passes from the exterior to the interior of its osculating circles. It follows that once a curve enters an osculating circle it can never emerge; it can never return to P. For the proof show for a nearby point Q that the line joining the centers of the osculating circles at P and Q crosses the curve between P and Q. The arc PQ is interior to one circle, exterior to the other; whence show the circles do not meet. Since the radius of curvature increases with s, the osculating circle contracts within itself as s increases.

PROBLEMS

SECTION 4.1i, page 360

1. Show that the expression for the curvature of a curve $x = x(t)$, $y = y(t)$ is unaltered by rotation of axes and also by change of parameter given by $t = \phi(\tau)$, where $\phi'(\tau) > 0$.

EXERCISES

SECTION 4.1k, page 365

1. Calculate the area bounded by the semicubical parabola $y = x^{3/2}$, the x-axis, and the lines $x = a$ and $x = b$.

2. Calculate the area of the region bounded by the line $y = x$ and the lower half of the loop of the folium of Descartes.

$$x^3 + y^3 - 3axy = 0.$$

3. Calculate the area of a sector of the Archimedean spiral $r = a\theta$ $(a > 0)$.

4. Calculate the area of the cardioid, using polar coordinates (see Exercises 4.1g, No. 1*d*).

5. Calculate the area of the astroid (see Exercises 4.1g, No. 1*c*).

6. Find the area of the hypocycloid of n cusps generated when the radius of the fixed circle is n times the radius of the rolling circle.

Answers to Exercises

SECTION 4.1k, page 365

1. $\frac{2}{5}(b^{5/2} - a^{5/2})$. **2.** $\dfrac{3a^2}{4}$. **3.** $\frac{1}{6}a^2(\theta_2{}^3 - \theta_1{}^3)$. **4.** $6\pi R^2$.

5. $6\pi r^2$.

6. If the radius of the inner circle is a, the area is $\pi a^2(n - 1)(n - 2)$.

EXERCISES

SECTION 4.1*l*, page 373

1. Find the center of mass of an arbitrary arc
(*a*) of a circle of radius *r*.
(*b*) of a catenary.

Answers to Exercises

SECTION 4.1*l*, page 373

1. (*a*) $$\xi = \frac{r(\sin\phi_2 - \sin\phi_1)}{\varphi_2 - \varphi_1},$$

$$\eta = \frac{-r(\cos\phi_2 - \cos\phi_1)}{\varphi_2 - \varphi_1},$$

where ϕ_1, ϕ_2 are the θ-coordinates of the extremities of the arc.

(*b*) $$\xi = \frac{(x_2 \sinh x_1 - x_2 \sinh x_1 - \cosh x_2 + \cosh x_1)}{\sinh x_2 - \sinh x_1}$$

$$\eta = \frac{\{2(x_2 - x_1) + \sinh 2x_2 - \sinh 2x_1\}}{4(\sinh x_2 - \sinh x_1)},$$

where (x_1, y_1), (x_2, y_2) are the extremities of the arc.

EXERCISES

SECTION 4.1m, page 374

1. Using Guldin's rule, calculate the surface area and volume of (*a*) a sphere, (*b*) truncated cone, (*c*) sector of the paraboloid obtained by the revolution of an arc of a parabola about the axis.

2. Find the volume and surface area of the torus obtained by rotating the circle

$$x^2 + (y - a)^2 = b^2, \qquad (0 < b \leq a)$$

about the x-axis.

3. Obtain the volume of the ellipsoid of revolution obtained by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

4. Find the volume and surface area of a zone of a sphere of radius r, that is, of the portion of the sphere cut off by two parallel planes distant h_2, h_1 respectively from the center.

5. Find the area of the catenoid, the surface obtained by rotating an arc of the catenary $y = \cosh x$ about the x-axis.

Answers to Exercises

SECTION 4.1m, page 374

1. (*a*) Sphere of radius r: $4\pi r^2$; $\frac{4}{3}\pi r^3$.

(*b*) Truncated cone, base radii a and b; height h: $\dfrac{\pi h}{3}(a^2 + ab + b^2)$; $\pi(a + b)\sqrt{h^2 + (a - b)^2}$.

(*c*) Parabola $y^2 = 4px$, arc between (x_1, y_1) and (x_2, y_2): $2\pi p(x_2{}^2 - x_1{}^2)$; $\dfrac{8\pi}{3}\sqrt{p}[(x_2 + p)^{3/2} - (x_1 + p)^{3/2}]$

2. $2\pi^2a^2b$; $4\pi^2ab$.

3. $\frac{4}{3}\pi ab^2$.

4. Volume $\pi r^2(h_2 - h_1) - \frac{1}{3}\pi(h_2{}^3 - h_1{}^3)$. Surface $2\pi(h_2 - h_1)r$.

5. $\pi(x_1 - x_0) + \dfrac{\pi}{2}(\sinh 2x_1 - \sinh 2x_0)$.

EXERCISES

SECTION 4.1n, page 375

1. Calculate the moment of inertia about the x-axis of the boundary of the rectangle $a \leq x \leq b$, $\alpha \leq y \leq \beta$.

2. Given a triangle with sides of length a, b, c, calculate the moment of inertia about the side of length a.

3. Calculate the moment of inertia of an arc of the catenary $y = \cosh x$ (*a*) about the x-axis, (*b*) about the y-axis.

4. The equation, $y = f(x) + \alpha$, $a \leq x \leq b$, represents a family of curves, one for each value of the parameter α. Prove that in this family the curve with the least moment of inertia about the x-axis is that which has its center of mass on the x-axis.

5. Obtain an expression for the moment of inertia of a planar curve about an axis perpendicular to the plane. Prove that this moment is a minimum when the axis passes through the center of mass of the curve.

Answers to Exercises

SECTION 4.1n, page 375

1. $(\alpha^2 + \beta^2)(b - a) + \frac{2}{3}(\beta^3 - \alpha^3)$.

2. $\dfrac{4}{3}\dfrac{b + c}{a^2}\,[(a^2 + b^2 + c^2)^2 - 4b^2c^2]$

3. (*a*) $\sinh x_2 - \sinh x_1 + \frac{1}{3}(\sinh^3 x_2 - \sinh^3 x_1)$,
(*b*) $(x_2{}^2 + 2)\sinh x_2 - (x_1{}^2 + 2)\sinh x_1 - 2x_2\cosh x_2 + 2x_1\cosh x_1$, if $0 \leq x_1 \leq x_2$.

4. Expand $T(\alpha) = \int_a^b (z + \alpha)^2 \sqrt{(1 + z')^2}\, dx$, where $z = f(x)$:

$$T = \int_a^b z^2 \sqrt{(1 + z')^2}\, dx + 2\alpha \int_a^b z\sqrt{(1 + z')^2}\, dx$$

$$+ \alpha^2 \int_a^b \sqrt{1 + z'^2}\, dx.$$

For a minimum $T' = 0$,

$$\alpha = -\frac{\int_a^b z\sqrt{1 + z'^2}\, dx}{\int_a^b \sqrt{1 + z'^2}\, dx},$$

from which the result follows easily.

5. Let the axis pass through the point (a, b) of the plane. The moment of inertia is

$$T = \int [(x - a)^2 + (y - b)^2]\, ds.$$

For a fixed b the minimum of T will occur when $a = \int x\, ds$ independently of the value of b; similarly, for a fixed a the minimum will occur when $b = \int y\, ds$. Consequently, the over-all minimum occurs when the axis passes through the center of mass.

EXERCISES

SECTION 4.3b, page 384

1. Show if $\mathbf{R} = \lambda Q$ that $|\mathbf{R}| = |\lambda| \cdot |\mathbf{Q}|$.

2. (*a*) Find the unit tangent vector to the parabola $y = x^2$ at $(1, 1)$.

(*b*) Determine the cosine of the angle between the parabola and the line $y = x$ at $(1, 1)$.

3. (*a*) Obtain a vector parametric equation for the line $\overline{P_0P_1}$ from the parallelism of the segment P_0P_1 to any segment P_0P joining P_0 to another point P on the line.

(*b*) Obtain the vector normal form of the equation of the line in terms of the vector $\overrightarrow{OP}$, where P is the foot of the perpendicular from 0 onto the line.

4. Let $\mathbf{R}_i = \overrightarrow{OP_i}$.

(*a*) Prove that $\frac{1}{2}(\mathbf{R}_1 + \mathbf{R}_2)$ is the position vector of the midpoint of the segment P_1P_2.

(*b*) Prove that $\mathbf{R} = t\mathbf{R}_1 + (1 - t)\mathbf{R}_2$ for $0 \le t \le 1$ is the parametric equation for the segment joining $\mathbf{R}_1$ and $\mathbf{R}_2$. What is the geometric significance of the parameter t?

(*c*) Prove that $\frac{1}{3}(\mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3)$ is the position vector of the intersection of the medians of the triangle $P_1P_2P_3$.

5. Prove that

$$|\mathbf{R}_1| - |\mathbf{R}_2| \le |\mathbf{R}_1 + \mathbf{R}_2| \le |\mathbf{R}_1| + |\mathbf{R}_2|.$$

6. Represent $\mathbf{R}_1$ as the sum of a vector parallel to $\mathbf{R}_2$ (assuming $\mathbf{R}_2 \neq 0$), and a vector perpendicular to $\mathbf{R}_2$.

7. Prove that the vector which bisects the angle between $\mathbf{R}_1$ and $\mathbf{R}_2$ is

$$\mathbf{R} = \left(\frac{|\mathbf{R}_2|}{|\mathbf{R}_1| + |\mathbf{R}_2|}\right)\mathbf{R}_1 + \left(\frac{|\mathbf{R}_1|}{|\mathbf{R}_1| + |\mathbf{R}_2|}\right)\mathbf{R}_2.$$

Answers to Exercises

SECTION 4.3b, page 384

2. (*a*) $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$; (*b*) $\frac{3}{\sqrt{10}}$.

3. (*a*) Let R_0 be the position vector of P_0 and set $\mathbf{T} = \overline{P_0P_1}$. Then for an arbitrary point $\mathbf{R}$ on the line,

$$\mathbf{R} = \mathbf{R}_0 + \lambda \mathbf{T}.$$

(*b*) Let $\mathbf{R}_0$ be the position vector of P, $\mathbf{N} = \overline{OP}$. Then

$$(\mathbf{R} - \mathbf{R}_0) \cdot \mathbf{N} = 0.$$

4. (*a*) Set $\mathbf{M} = \frac{1}{2}(\mathbf{R}_1 + \mathbf{R}_2)$; then $\mathbf{R}_1 - \mathbf{M} = \frac{1}{2}(\mathbf{R}_1 - \mathbf{R}_2) = \mathbf{M} - \mathbf{R}_2$ so that $\mathbf{R}_1$ and $\mathbf{R}_2$ lie in opposite directions from $\mathbf{M}$ at equal distances.

(*b*) $\mathbf{R}_1 - \mathbf{R} = (1 - t)(\mathbf{R}_1 - \mathbf{R}_2)$,
$\mathbf{R}_2 - \mathbf{R} = t(\mathbf{R}_2 - \mathbf{R}_1)$;

thus $\mathbf{R}_1$ and $\mathbf{R}_2$ lie in opposite directions from $\mathbf{R}$ and $\mathbf{R}$ must lie on the segment joining the two. Furthermore, the distance of $\mathbf{R}$ from $\mathbf{R}_2$ is $t\,|\mathbf{R}_2 - \mathbf{R}_1|$ and varies continuously from 0 to $|\mathbf{R}_2 - \mathbf{R}_1|$ as t goes from 0 to 1; thus every point of the segment can be described in this form.

(*c*) Set $\mathbf{I} = \frac{1}{3}(\mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3)$. The midpoint of P_2P_3 is given by $\mathbf{M}_1 = \frac{1}{2}(\mathbf{R}_2 + \mathbf{R}_3)$. Now $\mathbf{I} = \frac{1}{3}(\mathbf{R}_1 + 2\mathbf{M}_1)$, and from part *b* the position vector of a point on the median from P_1, likewise on the medians from P_2 and P_3.

5. Apply Cauchy's inequality

$$\mathbf{R}_1 \cdot \mathbf{R}_2 \leq |\mathbf{R}_1|\,|\mathbf{R}_2|$$

as follows:

$$\begin{aligned} |\mathbf{R}_1 + \mathbf{R}_2|^2 &= |\mathbf{R}_1|^2 + 2\mathbf{R}_1 \cdot \mathbf{R}_2 + |\mathbf{R}_2|^2 \\ &\leq |\mathbf{R}_1|^2 + 2\,|\mathbf{R}_1|\,|\mathbf{R}_2| + |\mathbf{R}_2|^2 \\ &\leq (|\mathbf{R}_1| + |\mathbf{R}_2|)^2. \end{aligned}$$

Thus

$$|\mathbf{R}_1 + \mathbf{R}_2| \leq |\mathbf{R}_1| + |\mathbf{R}_2|.$$

Next, in the preceding inequality replace $\mathbf{R}_1$ by $\mathbf{R}_1 + \mathbf{R}_2$ and $\mathbf{R}_2$ by $-\mathbf{R}_2$:

$$|\mathbf{R}_1 + \mathbf{R}_2 - \mathbf{R}_2| \leq |\mathbf{R}_1 + \mathbf{R}_2| + |\mathbf{R}_2|,$$

whence

$$|\mathbf{R}_1| - |\mathbf{R}_2| \leq |\mathbf{R}_1 + \mathbf{R}_2|.$$

6. $\mathbf{R}_1 = \dfrac{(\mathbf{R}_1 \cdot \mathbf{R}_2)\mathbf{R}_2}{|\mathbf{R}_2|^2} + \left[\mathbf{R}_1 - \dfrac{(\mathbf{R}_1 \cdot \mathbf{R}_2)\mathbf{R}_2}{|\mathbf{R}_2|^2}\right]$

7. Note that

$$\frac{\mathbf{R} \times \mathbf{R}_1}{|\mathbf{R}|\,|\mathbf{R}_1|} = -\frac{\mathbf{R} \times \mathbf{R}_2}{|\mathbf{R}|\,|\mathbf{R}_2|}.$$

EXERCISES

SECTION 4.3d, page 394

1. At what point of the motion

$$x = a \cos \omega t, \qquad y = b$$

is the absolute acceleration a maximum? A minimum?

2. Determine the speed at which a particle may move along a catenary $y = \cosh x$ so that the normal component of its acceleration is a constant.

PROBLEMS

SECTION 4.3d, page 394

1. Prove if the acceleration is always perpendicular to velocity that the speed is constant.

2. The velocity vector, considered as a position vector, traces out a curve known as the hodograph. Show whether or not a particle moving on a closed curve may have a straight line as its hodograph.

3. Assuming the rolling circle moves at constant speed, find the velocity and acceleration of the point P which generates the cycloid.

4. Let A be a fixed point of the plane and suppose that the acceleration vector for a moving point P is always directed toward A and proportional to $1/|AP|^2$. Prove that the hodograph (cf. Problem 2) is a circle.

5. Let A be a fixed point on a circle. Let P be a point of the circle moving so that the acceleration vector points to A. Prove that the acceleration is proportional to $|AP|^{-5}$.

Answers to Exercises

SECTION 4.3d, page 394

1. Max: $t = \dfrac{n\pi}{\omega}$; min: $t = \dfrac{(n + \frac{1}{2})\pi}{\omega}$.

2. If a_n is the magnitude of the normal acceleration,

$$v = \frac{a_n}{1 - \tanh x}.$$

Solutions and Hints to Problems

SECTION 4.3d, page 394

1. $0 = \ddot{\mathbf{R}} \cdot \dot{\mathbf{R}} = \dfrac{1}{2}\dfrac{d}{dt}(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}}) = \dfrac{1}{2}\dfrac{d}{dt}(v^2)$; hence v^2 = constant.

2. A closed curve cannot have a straight hodograph. A smooth closed curve has tangents in all directions. However, if the hodograph were straight, a tangent in the direction of the hodograph would be precluded. (The argument can easily be extended to a curve with corners if it is observed that at a corner it is necessary to impose the condition $v = 0$; otherwise the acceleration is infinite. Then the hodograph has a corner at the origin corresponding to the corner in the curve.)

3. Take $x = vt - a \sin vt/a$, $y = a - a \cos vt/a$. Then

$$\dot{\mathbf{R}} = \left(v - v \cos \frac{vt}{a}, v \sin \frac{vt}{a}\right),$$

$$\ddot{\mathbf{R}} = \left(\frac{v^2}{a} \sin \frac{vt}{a}, \frac{v^2}{a} \cos \frac{vt}{a}\right).$$

Thus the acceleration is directed toward the center of the rolling circle and is constant.

4. Show the curvature of the hodograph is constant. (See Problems 4.1h, No. 8.) Set $\mathbf{R} = \rho\mathbf{x}$, where $\mathbf{x} = (\cos\theta, \sin\theta)$ and obtain the condition that $\ddot{\mathbf{R}}$ is parallel to $\mathbf{x}$:

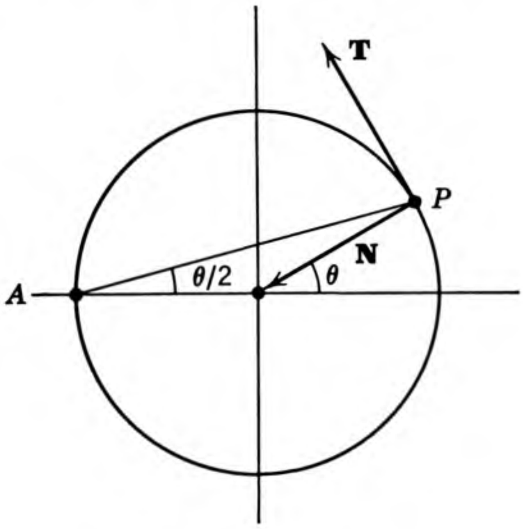
$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

or

$$\frac{d}{dt} \log \frac{r^2}{\dot{\theta}} = 0.$$

Thus $r^2\dot{\theta} = \mu$, where μ is constant. Furthermore, since $\ddot{\mathbf{R}} = \dfrac{\lambda}{r^2}\mathbf{x}$, where λ is constant, the curvature of the hodograph is $\lambda r^2\dot{\theta} = \lambda\mu$.

5. Let the radius of the circle be 1. For the unit tangent **T** and unit normal **N** take $\mathbf{T} = (-\sin\theta, \cos\theta)$ and $\mathbf{N} = (-\cos\theta, -\sin\theta)$. (See figure.) For



$\mathbf{R} = \overrightarrow{AP}$, $\mathbf{R} = 2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2}, \sin\dfrac{\theta}{2}\right)$, the tangential and normal components are

$$\mathbf{R}\cdot\mathbf{T} = -2\cos\frac{\theta}{2}\sin\frac{\theta}{2},$$

$$\mathbf{R}\cdot\mathbf{N} = -2\cos^2\left(\frac{\theta}{2}\right);$$

hence the tangential and normal components of the acceleration are $\ddot{\theta}$ and $\dot{\theta}^2$, respectively. The equation $\ddot{\mathbf{R}} = -\lambda^2\mathbf{R}$ yields

$$\dot{\theta}^2 = \lambda^2\cos^2\left(\frac{\theta}{2}\right), \qquad \ddot{\theta} = \lambda^2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right).$$

Consequently,

$$\frac{d\lambda}{d\theta} = \frac{1}{\dot{\theta}}\frac{d}{d\theta}\left(\frac{\dot{\theta}}{\cos\frac{1}{2}\theta}\right) = \frac{\ddot{\theta}\cos\frac{1}{2}\theta + \frac{1}{2}\dot{\theta}^2\sin\frac{1}{2}\theta}{\dot{\theta}\cos^2(\frac{1}{2}\theta)}$$
$$= \frac{3\lambda\sin\frac{1}{2}\theta}{2\cos\frac{1}{2}\theta}.$$

Integrate $(1/\lambda)(d\lambda/d\theta)$ with respect to θ to obtain

$$\lambda = C_1(\cos\tfrac{1}{2}\theta)^{-3} = C_2 r^{-3}.$$

But now $a^2 = \dot{\theta}^4 + \ddot{\theta}^2 = \lambda^4\cos^2(\frac{1}{2}\theta) = C_3 r^{-10}$. It follows that a is proportional to r^{-5}.

EXERCISES

SECTION 4.4, page 397

In the following exercises take the acceleration of gravity in round numbers as 32 ft/sec^2.

1. Find the height from which a body must be dropped in order to reach the ground at a speed of 60 miles per hour (88 ft/sec).

2. A particle starts from the origin with velocity 4, and under the influence of gravity slides down a straight wire until it reaches the vertical line $x = 2$. What must the slope of the path be so that the point may reach the vertical line in the shortest time?

3. Find the equations of motion for a particle sliding without resistance on a plane inclined at angle θ to the horizontal.

4. Let v_0 be the initial speed at which a projectile is thrown. Assuming the initial points and end points of the motion are on a level plane, what angle of inclination of the initial velocity vector maximizes the horizontal distance between the points? Obtain the nonparametric equation for this trajectory.

5. A point A moves with constant velocity 1 on a circle with radius r and center the origin. The point A is connected to to a point B by a line of constant length $l(>r)$; B is constrained to move on the x-axis (cf. the crank, connecting rod, and piston of a steam engine). Calculate the velocity and acceleration of B as functions of the time.

Answers to Exercises

EXERCISE 4.4, page 397

1. 121 ft.

2. 0.

3. Takes the x-axis as a horizontal line in the plane, the y-axis inclined at angle θ; then

$$\ddot{x} = 0, \qquad \ddot{y} = -g \sin \theta.$$

4. Inclination $\frac{\pi}{4}$. $y = x - \dfrac{x^2}{{v_0}^2 g}$.

5. $x = r\cos\dfrac{t}{r} + \sqrt{l^2 - r^2\sin^2 t/r}$;

$$\frac{dx}{dt} = -\frac{r\sin(2t/r)}{2\sqrt{[l^2 - r^2\sin^2(t/r)]}} - \sin\frac{t}{r};$$

$$\frac{d^2x}{dt^2} = -\frac{l^2\cos(2t/r) + r^2\sin^4(t/r)}{\sqrt{[l^2 - r^2\sin^2(t/r)]^3}} - \frac{1}{r}\cos\frac{t}{r}.$$

PROBLEMS

SECTION 4.5, page 402

1. A particle moves in a straight line subject to a resistance producing the retardation ku^3, where u is the velocity and k a constant. Find expressions for the velocity u and the time t in terms of s, the distance from the initial position, and v_0, the initial velocity.

2. A particle of unit mass moves along the x-axis and is acted upon by a force $f(x) = -\sin x$.

(*a*) Determine the motion of the point if at time $t = 0$ it is at the point $x = 0$ and has velocity $v_0 = 2$. Show that as $t \to \infty$ the particle approaches a limiting position, and find this limiting position.

(*b*) If the conditions are the same, except that v_0 may have any value, show that if $v_0 > 2$ the point moves to an infinite distance as $t \to \infty$, and that if $v_0 < 2$ the point oscillates about the origin.

3. Choose axes with their origin at the center of the earth, whose radius we shall denote by R. According to Newton's law of gravitation, a particle of unit mass lying on the y-axis is attracted by the earth with a force $-\mu M/y^2$, where μ is the "gravitational constant" and M is the mass of the earth.

(*a*) Calculate the motion of the particle after it is released at the point $y_0\,(> R)$; that is, if at time $t = 0$ it is at the point $y = y_0$ and has the velocity $v_0 = 0$.

(*b*) Find the velocity with which the particle in (*a*) strikes the earth.

(*c*) Using the result of (*b*), calculate the velocity of a particle falling to the earth from infinity.[1]

***4.** A particle perturbed slightly from rest on top of a circle slides downward under the force of gravity. At what point does it fly unconstrained off the circle?

***5.** A particle of mass m moves along the ellipse $r = k/(1 - e\cos\theta)$. The force on the particle is cm/r^2 directed toward the origin. Describe the motion of the particle, find its period, and show that the radius vector to the particle sweeps out equal areas in equal times.

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[1] This is the same as the least velocity with which a projectile would have to be fired in order that it should leave the earth and never return.

Solutions and Hints to Problems

SECTION 4.5, page 402

1. $u = \dfrac{v_0}{1 + ksv_0}, \quad t = \dfrac{s}{v_0} + \tfrac{1}{2}ks^2.$

2. (*a*) $x = 4 \text{ arc tan } e^t - \pi; \ x = \pi.$

3. (*a*) $t = \dfrac{1}{\sqrt{2\mu M}}\left\{y_0^{1/2}\sqrt{y(y_0 - y)} - y_0^{3/2} \text{ arc sin}\sqrt{\dfrac{y}{y_0}} + \tfrac{1}{2}\pi y_0^{3/2}\right\}.$

(*b*) $-\sqrt{2\mu M\left(\dfrac{1}{R} - \dfrac{1}{y_0}\right)}.$

(*c*) $-\sqrt{\dfrac{2\mu M}{R}}.$

4. At the point where its normal acceleration equals the centripetal component of gravitational acceleration. If r is the radius and θ is the central angle measured from the vertical,

$$\cos \theta = \frac{r}{1 + r}.$$

5. $\theta = at, r = \dfrac{k}{1 - e \cos at}, \quad$ where $a = \dfrac{(1 - e)^2}{k^2}\sqrt{ck};$

$$\text{period} = \frac{2\pi}{a} = \frac{2\pi}{(1 - e)^2 c^{1/2}} \cdot k^{3/2}.$$

EXERCISES

SECTION 4.6, page 404

1. (*a*) A spring stretches in proportion to the applied force. At what frequency will the spring oscillate when supporting a load of 100 gm. If the stationary load stretches the spring 4 cm?

(*b*) If the amplitude of the oscillation is 2 cm, find the maximum velocity of motion. (Take 980 cm per second per second as the acceleration of gravity.)

Answers to Exercises

SECTION 4.6, page 404

1. (*a*) $\sqrt{245} \approx 15.8$ cycles per second,
(*b*) $2\sqrt{245}$ cm/sec.

EXERCISES

SECTION 4.9, page 418

1. On a frictionless wire in the shape of the parabola $y = x^2$, a bead of mass m is released at the point (a, a^2). Assuming gravity to be in the direction of the negative y-axis, find the speed at which the bead reaches the vertex and the force which it exerts on the wire at that point.

Answers to Exercises

SECTION 4.9, page 418

1. Speed: $a\sqrt{2g} = 8a$ ft/sec. Force: $mg(1 + 4a^2)$.

EXERCISES

SECTION 4.A.1, page 424

1. Find the evolutes of each of the following curves:

(*a*) The parabola, $y = x^2/4p$.
(*b*) The hyperbola, $xy = 1$.
(*c*) The catenary, $y = c \cosh x/c$.

2. Find the involutes for (*a*) and (*c*) in Exercise 1.

PROBLEMS

SECTION 4.A.1, page 424

1. Show that the evolute of an epicycloid (Example, p. 329) is another epicycloid similar to the first, which can be obtained from the first by rotation and contraction.

2. Show that the evolute of a hypocycloid (Example, p. 331) is another hypocycloid, which can be obtained from the first by rotation and expansion.

Answers to Exercises

SECTION 4.A.1, page 424

1. (*a*) $x = -\dfrac{t^3}{4p^2}, \; y = 2p + \dfrac{3t^2}{4p}$

(*b*) $x = \dfrac{3t}{2} + \dfrac{1}{2t^3},$

$y = \dfrac{t^2}{2} + \dfrac{1}{t} + \dfrac{1}{2t^2}.$

(*c*) $x = t - c \cosh \dfrac{t}{c} \sinh \dfrac{t}{c}.$

$y = 2c \cosh \dfrac{t}{c}.$

2. For the parabola $y = x^2/4p$.

5 *Taylor's Expansion*

EXERCISES

SECTION 5.4b, page 450

1. Prove that

$$x - \frac{x^2}{2} + \frac{x^3}{3}(1 + x)^{-1} < \log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3},$$

for $x > 0$. Using this result find $\log \frac{4}{3}$ to two places.

2. Calculate $\log \frac{6}{5}$ to three places, using the series

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots.$$

Hint: From the estimate for the error of the series, $|R_n| \leq |x|^{n+1}/(1 + x)$, find n so large that $|R_n| < \frac{1}{1000}$ when $1 + x = \frac{6}{5}$, and then calculate the nth order polynomial approximation for this value of x.

3. How many terms of the series for $\log(1 + x)$ should be used in order to obtain $\log(1 + x)$ to within 10% if $30 \leq x \leq 31$. (*Hint:* Use the methods of Problem 2.)

4. Verify for the coefficients a_ν in the series for $\log(1 + x)$ and arc tan x, that $\nu!\, a_\nu = f^{(\nu)}(0)$, $\nu = 0, 1, 2, \ldots$.

5. How many terms of the series $\log(1 + x)$ are needed to calculate $\log 2$ to three decimal place accuracy if we set $x = 1$? Be a little more ingenious in your use of the series and achieve the desired accuracy in a few terms.

6. Find the value of θ in Lagrange's form of the remainder R_n for the expansions of $1/(1 - x)$ and $1/(1 + x)$ in powers of x.

PROBLEMS

SECTION 5.4b, page 450

1. Give the complete formal derivation of the remainder formula (27), p. 452 using mathematical induction.

2. (*A Variant of Proof of Taylor's Theorem*)

(*a*) If $g(h)$ has continuous derivatives through the $(n + 1)$th order for $0 \leq h \leq A$, and if $g(0) = g'(0) = \cdots = g^{(n)}(0) = 0$, while $|g^{(n+1)}(h)| \leq M$ on $[0, A]$, for M a constant, show that $|g^{(n)}(h)| \leq Mh$, $|g^{(n-1)}(h)| \leq Mh^2/2!, \ldots,$ $|g^{(n-i)}(h)| \leq Mh^i/i!, \ldots, |g(h)| \leq Mh^n/n!$, for all h in the interval.

(*b*) Let $f(x)$ be a sufficiently differentiable function on $a \leq x \leq b$ and $T_n(h)$ be the Taylor polynomial for $f(x)$ at $x = a$. Apply the result of (*a*) to the function $g(h) = R_n = f(a + h) - T_n(h)$ to obtain Taylor's formula with a rough estimate for the remainder.

3. Let $f(x)$ have a continuous derivative in the interval $a \leq x \leq b$, and let $f''(x) \geq 0$ for every value of x. Then if ξ is any point in the interval, the curve nowhere falls below its tangent at the point $x = \xi$, $y = f(\xi)$.

(Use the Taylor expansion to two terms.)

4. Deduce the integral formula for the remainder R_n by applying integration by parts to

$$f(x + h) - f(x) = \int_0^h f'(x + \tau)\, d\tau.$$

5. Integrate by parts the formula

$$R_n = \frac{1}{n!}\int_0^h (h - \tau)^n f^{(n+1)}(x + \tau)\, d\tau,$$

and so obtain

$$R_n = f(x + h) - f(x) - hf'(x) - \cdots - \frac{h^n}{n!} f^{(n)}(x).$$

***6.** Suppose that in some way a series for the function $f(x)$ has been obtained, namely

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + R_n(x),$$

where $a_0, a_1, \ldots, a_n$ are constants, $R_n(x)$ is n times continuously differentiable, and $R_n(x)/x^n \to 0$ as $x \to 0$. Show that $a_k = (f^k(0)/k!)$ $(k = 0, \ldots, n)$, that is, that the series is a Taylor series.

Answers to Exercises

SECTION 5.4b, page 450

1. For $0 < x < 1$,

$$R_2(x) = \int_0^x \frac{t^2}{1 + t}\, dt > \int_0^x \frac{t^2}{1 + x}\, dt$$

$$\geq \frac{t^3}{3(1 + x)}.$$

The upper estimate is obtained as in the text.
To two places, $\log \frac{4}{3} = 0.29$.

2. Take 4 terms; $\log \frac{6}{5} = 0.1823$.

3. Impossible; series not valid.

5. Use $\log (1 - \frac{1}{2}) = -\log 2$.

6.
$$\frac{1}{1 - x}: \qquad \theta = \frac{1 - (1 - x)^{1/(n+1)}}{x}.$$
$$\frac{1}{1 + x}: \qquad \theta = \frac{(1 + x)^{1/(n+2)} - 1}{x}.$$

Solutions and Hints to Problems

SECTION 5.4b, page 450

1. Let the polynomials ϕ_n be defined recursively by $\phi_0(t) = 1$, $\phi'_{n+1}(t) = \phi_n(t)$, and $\phi_n(b) = 0$. Then
$$\phi_n(t) = \frac{(t - b)^n}{n!}.$$
This proposition holds for $n = 1$. If it holds for n, then from $\phi'_{n+1}(x) = \phi_n(x)$, and
$$\phi_{n+1}(t) = \int_b^t \phi_n(x)\, dx = \frac{(t - b)^{n+1}}{(n + 1)!},$$
it holds as well for $n + 1$.

Now, for $n = 0$,
$$f(b) - f(a) = \int_a^b \phi_0 f'(t)\, dt.$$
If for $n = k$,
$$f(b) = \sum_{\nu=1}^{k} \frac{f^{(\nu)}(a)(b - a)^\nu}{\nu!} + (-1)^k \int_a^b f^{(k+1)}(t)\phi_k(t)\, dt,$$
then from integration by parts
$$\begin{aligned}\int_a f^{(k+1)}(t)\phi_k(t)\, dt &= \int_a^b f^{(k+1)}(t)\phi'_{k+1}(t)\, dt \\ &= \phi_{k+1}(t) f^{(k+1)}(t)\Big|_a^b - \int_a^b f^{(k+2)}(t)\phi_{k+1}(t)\, dt \\ &= f^{(k+1)}(a)\frac{(a - b)^{k+1}}{(k + 1)!} - \int_a^b f^{(k+2)}(t)\phi_{k+1}(t)\, dt.\end{aligned}$$
Consequently,
$$f(b) = \sum_{\nu=1}^{k+1} \frac{f^{(\nu)}(a)(b - a)^\nu}{\nu!} + (-1)^{k+1} \int_a^b f^{(k+2)}(t)\phi_{k+1}(t)\, dt.$$

2. (*a*) Let $\phi(h)$ satisfy $\phi(0) = 0$ and $|\phi'(h)| \le M|h|^\nu$ for $|h| \le A$; then, for $h > 0$

$$-\frac{Mh^{\nu+1}}{\nu+1} \le \int_0^h \phi'(x)\,dx \le \frac{Mh^{\nu+1}}{\nu+1},$$

and, for $h < 0$

$$\frac{(-1)^\nu Mh^{\nu+1}}{\nu+1} \le \int_0^h \phi'(x)\,dx \le \frac{(-1)^{\nu+1}Mh^{\nu+1}}{\nu+1}.$$

In either case, $|\phi(h)| \le \dfrac{M|h|^{\nu+1}}{\nu+1}$.

Now we apply this result recursively to functions $g^{(n+1)}(h), g^{(n)}(h), \ldots$. For $\phi(h) = g^{(n+1)}(h)$, $\nu = 1$ and

$$|g^{(n)}(h)| \le M\,|h|.$$

If

$$|g^{(n-k)}(h)| \le \frac{M\,|h|^{k+1}}{(k+1)!}, \qquad (k \le n)$$

then for $\phi(h) = g^{(k)}(h)$, $\nu = n - k + 1$ and

$$|g^{(n-k+1)}(h) \le \frac{M\,|h|^{k+2}}{(k+2)!}.$$

(*b*) $g(h) = f(a+h) - \sum_{k=0}^{n} \dfrac{h^k f^{(k)}(a)}{k!}$. Clearly, $g^k(0) = 0$ for $k \le n$. If $|f^{(n+1)}(a)| \le M$ in $[a - A, a + A]$, then g satisfies the conditions of part *a*, and $|g(h)| = |R_{n+1}(h)| \le \dfrac{M\,|h|^{n+1}}{n+1}$ for $|h| < A$.

3. The equation of the tangent is

$$g(x) = f(\xi) + f'(\xi)(x - \xi).$$

However, for some number η between ξ and x,

$$f(x) - [f(\xi) + f'(\xi)(x - \xi)] = \tfrac{1}{2}f''(\eta)(x-\xi)^2 \ge 0.$$

4 and **5.** Integrate by parts taking $\tau + h$ instead of τ as the integral of $d\tau$:

$$\int_0^h f'(x+\tau)\,d\tau = f'(x+\tau)(\tau - h)\Big|_0^h - \int_0^h f''(x+\tau)(\tau-h)\,d\tau,$$

whence

$$f(x+h) - f(x) - f'(x)h = \int_0^h f''(x+\tau)(h-\tau)\,d\tau.$$

Suppose

$$f(x+h) - \sum_{\nu=0}^{k} \frac{f^{(\nu)}(x)h^\nu}{\nu!} = \int_0^h f^{(k+1)}(x+\tau)\frac{(h-\tau)^k}{k!}\,d\tau.$$

Since

$$\int_0^h f^{(k+1)}(x+\tau)\frac{(h-\tau)^k}{k!}\,d\tau = -f^{(k+1)}(x+\tau)\frac{(h-\tau)^{k+1}}{(k+1)!}\Bigg|_0^h + \int_0^h f^{(k+2)}(x+\tau)\frac{(h-\tau)^{k+1}}{(k+1)!}\,d\tau,$$

the formula holds when k is replaced by $k + 1$.

6. Apply Leibnitz's rule to $R_n(x) = Q(x)x^n$:

$$R_n^{(k)}(x) = \sum_{\nu=0}^{k} n!\binom{k}{\nu} Q^{(\nu)}(x)x^{n-k+\nu}/(n-k+\nu)!$$
$$= n!\,x^{n-k}\left[\frac{Q(x)}{(n-k)!} + \frac{kQ'(x)x}{(n-k+1)!} + \cdots\right].$$

Thus $\lim_{x\to 0}\dfrac{R_n^{(k)}(x)}{x^{n-k}} = 0$, for $k \le n$. Consequently, $R_n^{(k)}(0) = 0$. Now differentiate and set $x = 0$, n times.

EXERCISES

SECTION 5.5, page 453

1. Expand $(1 + x)^{1/2}$ to two terms plus remainder. Estimate the remainder.

2. Use the expansion of Exercise 1 (discarding the remainder) to calculate $\sqrt{2}$. What is the degree of accuracy of the approximation?

3. What linear function best approximates to $\sqrt[3]{(1 + x)}$ in the neighborhood of $x = 0$? Between what values of x is the error of the approximation less than 0.01?

4. What quadratic function best approximates to $\sqrt[3]{(1 + x)}$ in the neighborhood of $x = 0$? What is the greatest error in the interval $-0.1 \le x \le 0.1$?

5. (*a*) What linear function and (*b*) what quadratic function best approximate to $\sqrt[n]{(1 + x)}$ in the neighborhood of $x = 0$? What are the greatest errors when $-0.1 \le x \le 0.1$?

6. Calculate sin (0.01) to 4 places.

7. Do the same for (*a*) cos (0.01). (*b*) $\sqrt[3]{126}$, (*c*) $\sqrt{97}$.

8. Expand $\sin(x + h)$ in a Taylor series in h. Use this to find $\sin 31°$ $[= \sin(30° + 1°)]$ to 3 places.

Expand the functions in Exercises 9–18 in the neighborhood of $x = 0$ to three terms plus remainder (writing the remainder in Lagrange's form).

9. $\sin^2 x$. **10.** $\cos^3 x$. **11.** $\log \cos x$.

12. $\tan x$. **13.** $\log \dfrac{1}{\cos x}$. **14.** e^{-x^2}.

15. $\dfrac{1}{\cos x}$. **16.** $\cot x - \dfrac{1}{x}$. **17.** $\dfrac{1}{\sin x} - \dfrac{1}{x}$.

18. $\dfrac{\log(1 + x)}{1 + x}$.

19. (*a*) Expand $e^{\sin x}$ to five terms plus remainder; (*b*) in the power series for e^x substitute for x the power series for $\sin x$, taking enough terms to secure that the coefficient of x^4 is correct. Compare with (*a*).

20. Find the polynomial of fourth degree which best approximates to $\tan x$ in the neighborhood of $x = 0$. In what interval does this polynomial represent $\tan x$ to within 5%?

21. Find the first six terms of the Taylor series for y in powers of x for the functions defined by

(*a*) $x^2 + y^2 = y$, $y(0) = 0$.
(*b*) $x^2 + y^2 = y$, $y(0) = 1$.
(*c*) $x^3 + y^3 = y$, $y(0) = 0$.

PROBLEMS

SECTION 5.5, page 453

1. Find the first four nonvanishing terms of the Taylor series for the following functions in the neighborhood of $x = 0$:

(*a*) $x \cot x$, (*b*) $\dfrac{\sqrt{\sin x}}{\sqrt{x}}$, (*c*) $\sec x$,

(*d*) $e^{\sin x}$, (*e*) e^{e^x}, (*f*) $\log \sin x - \log x$.

2. Find the Taylor series for arc $\sin x$ in the neighborhood of $x = 0$ by using

$$\text{arc}\sin x = \int_0^x \frac{dt}{\sqrt{1 - t^2}}.$$

(Compare Section 3.2, Problem 2.)

***3.** Find the first three nonvanishing terms of the Taylor series for $\sin^2 x$ in the neighborhood of $x = 0$ by multiplying the Taylor series for $\sin x$ by itself. Justify this procedure.

***4.** Find the first three nonvanishing terms of the Taylor series for $\tan x$ in the neighborhood of $x = 0$ by using the relation $\tan x = \sin x/\cos x$, and justify the procedure.

***5.** Find the first three nonvanishing terms of the Taylor series for $\sqrt{\cos x}$ in the neighborhood of $x = 0$ by applying the binomial theorem to the Taylor series for $\cos x$, and justify the procedure.

***6.** Find the Taylor series for $(\arcsin x)^2$. (Compare Section 3.2, Problem 2.)

7. Find the Taylor series for the following functions in the neighborhood of $x = 0$:

$$(a)\ \sinh^{-1} x, \qquad (b)\ \int_0^x e^{-t^2}\,dt, \qquad (c)\ \int_0^x \frac{\sin t}{t}\,dt.$$

***8.** Estimate the error involved in using the first n terms in the series in Problem 7.

9. The elliptic function $s(u)$ has been defined (Section 3.14a) as the inverse of the elliptic integral

$$u(s) = \int_0^s \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}\,.$$

Find the Taylor expansion of $s(u)$ to the term of degree 5.

10. Evaluate the following limits:

$$(a)\ \lim_{x\to\infty} x\left[\left(1 + \frac{1}{x}\right)^x - e\right], \qquad (b)\ \lim_{x\to\infty}\left\{\frac{e}{2}x + x^2\left[\left(1 + \frac{1}{x}\right)^x - e\right]\right\},$$

$$^*(c)\ \lim_{x\to\infty} x\left[\left(1 + \frac{1}{x}\right)^x - e\log\left(1 + \frac{1}{x}\right)^x\right],$$

$$(d)\ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}, \qquad (e)\ \lim_{x\to\infty}\left(\frac{\sin x}{x}\right)^{1/x^2}.$$

***11.** Find the first three terms of the Taylor series for $[1 + (1/x)]^x$ in powers of $1/x$.

***12.** Two oppositely charged particles $+e$, $-e$ situated at a small distance d apart form an electric dipole with moment $M = ed$. Show that the potential energy

(*a*) At a point situated on the axis of the dipole at a distance r from the center of the dipole is $(M/r^2)(1 + \epsilon)$, where ϵ is approximately equal to $d^2/4r^2$.

(*b*) At a point situated on the perpendicular bisector of the dipole is 0.

(*c*) At a point with polar coordinates r, θ relative to the center and axis of the dipole is $[M\cos(\theta/r^2)](1 + \epsilon)$, where ϵ is approximately equal to

$$(d^2/8r^2)(5\cos^2\theta - 3).$$

(The potential energy of a single charge q at a point at a distance r from the charge is q/r; the potential energy of several charges is the sum of the potential energies of the separate charges.)

Answers to Exercises

SECTION 5.5, page 453

1. $1 + \frac{1}{2}x - \frac{x^2}{8(1 + \theta x)^{3/2}}$;

for $x < 0$, $-\frac{x^2}{8(1 + x)^{3/2}} < R < -\frac{x^2}{8}$;

for $x > 0$, $-\frac{x^2}{8} < R < -\frac{x^2}{8(1 + x)^{3/2}}$.

2. 1.5; error just over 6%.

3. $1 + \frac{1}{3}x, \ -\frac{3}{10}\left(\frac{99}{100}\right)^{5/6} < x < \frac{3}{10}$.

4. $1 + \frac{1}{3}x - \frac{1}{9}x^2$; $\frac{5}{81} \times 9^{-3}$.

5. (*a*) $1 + \frac{x}{n}$; $\frac{1}{2n}\left(\frac{1}{n} - 1\right) \times 9^{-2}$.

(*b*) $1 + \frac{x}{n} + \frac{1}{2n}\left(\frac{1}{n} - 1\right)x^2$; $\frac{1}{6n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right) \times 9^{-3}$.

6. 0.0100. **7.** (*a*) 0.9999; (*b*) 5.0133; (*c*) 9.8489. **8.** 0.515.

9. $x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \frac{x^8}{8!}[-128 \cos(2\theta x)]$.

10. $1 - \frac{3x^2}{2} + \frac{7x^4}{8} - \frac{3}{4}\frac{x^6}{6!}[243 \cos(3\theta x) + \cos(\theta x)]$.

11. $-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6$

$- 16\frac{x^8}{8!}[17 + 248 \tan^2(\theta x) + 756 \tan^4(\theta x) + 840 \tan^6(\theta x)$

$+ 315 \tan^8(\theta x)]$.

12. $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$

$+ 16\frac{x^7}{7!}[17 + 248 \tan^2(\theta x) + 756 \tan^4(\theta x) + 840 \tan^6(\theta x)$

$+ 315 \tan^8(\theta x)]$.

13. $\frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6$

$+ 16\frac{x^8}{8!}[17 + 248 \tan^2(\theta x) + 756 \tan^4(\theta x) + 840 \tan^6(\theta x)$

$+315 \tan^8(\theta x)]$.

14. $1 - x^2 + \frac{1}{2}x^4 - \dfrac{x^6}{3!}\, e^{-\theta^2 x^2}.$

15. $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$

$$+ \frac{x^6}{6!}\,[720 \sec^7(\theta x) - 840 \sec^5(\theta x) + 182 \sec^3(\theta x) - \sec(\theta x)].$$

16. $-\frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \cdots.$

17. $\frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \cdots.$

18. $x - \frac{3}{2}x^2 + \frac{11}{6}x^3 + \dfrac{x^4}{4!}\dfrac{1}{(1+\theta x)^5}(-50 + 24 \log \overline{1 + \theta x}).$

19. $1 + x + \frac{1}{2}x^2 - \frac{3}{24}x^4 + \dfrac{x^5}{5!}\, e^{\sin \theta x}\,[\cos^5(\theta x) - 10\cos^3(\theta x) + \cos(\theta x)$

$$- 10 \sin(\theta x)\cos^3(\theta x) + 15 \sin(\theta x)\cos(\theta x) + 6\sin^2(\theta x)\cos(\theta x)].$$

20. $x + \frac{1}{3}x^3;\quad 0 < x < \dfrac{\pi}{6}.$

21. (*a*) $y = x^2 + x^4 + 2x^6 + \cdots$, (*b*) $y = 1 - x^2 - x^4 - 2x^6 - \cdots$,
(*c*) $y = x^3 + x^9 + \cdots.$

Solutions and Hints to Problems

SECTION 5.5, page 453

1. (*a*) $1 - \dfrac{x^2}{3} - \dfrac{x^4}{45} - \dfrac{2x^6}{945}\cdots$, (*b*) $1 - \dfrac{x^2}{12} + \dfrac{x^4}{1440} - \dfrac{x^6}{23712}\cdots$,

(*c*) $1 + \dfrac{x^2}{2} + \dfrac{5}{24}x^4 + \dfrac{61}{720}x^6 + \cdots$, (*d*) $1 + x + \dfrac{x^2}{2} - \dfrac{x^3}{8}\cdots$,

(*e*) $e + ex + ex^2 + \dfrac{5}{6}ex^3 + \cdots$, (*f*) $-\dfrac{x^2}{6} - \dfrac{x^4}{180} - \dfrac{x^6}{2835}\cdots.$

2. Expand $1/\sqrt{1 - t^2}$ in a Taylor series at $t = 0$ and integrate:

$$x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2^2\cdot 2!}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}\frac{x^7}{7} + \cdots.$$

3. $x^2 - \dfrac{1}{3}x^4 + \dfrac{2}{45}x^6 + \cdots$,

$$(\sin x)^2 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - x^7 R\right)$$

$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + x^8 R',$$

where R and R' remain bounded as $x \to 0$.

4. $x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots;$

$$\frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - x^7 R}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - x^6 S} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + x^7 T,$$

where R, S, T are bounded as $x \to 0$.

5. $1 - \frac{x^2}{4} - \frac{x^4}{96} - \cdots;$

$$\sqrt{\cos x} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - x^6 R\right)^{1/2}$$

$$= 1 + \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - x^6 R\right) - \frac{1}{8}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - x^6 R\right)^2$$

$$+ \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - x^6 R\right)^3 S = 1 - \frac{x^2}{4} - \frac{x^4}{96} + x^6 T,$$

where R, S, T are bounded as $x \to 0$.

6. Take the square of the series for arc $\sin x$.

$$\sum_{\tau=0}^{\infty}\left(\sum_{n=0}^{\tau}\binom{2n}{n}\binom{2\tau - 2n}{\tau - n}\frac{1}{(2n+1)(2\tau - 2n + 1)}\right)\frac{x^{2\tau+2}}{2^{2\tau}}.$$

7. (*a*) $\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2\nu - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2\nu}\frac{x^{2\nu+1}}{2\nu + 1}.$

(*b*) $\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\frac{x^{2\nu+1}}{2\nu + 1};$ (*c*) $\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu + 1)!}\frac{x^{2\nu+1}}{2\nu + 1}.$

8. (*a*) $\frac{(2n)!\,x^{2n+1}}{2^{2n}(n!)^2(2n + 1)},$ (*b*) $\frac{x^{2n+1}}{n!\,(2n + 1)},$ (*c*) $\frac{x^{2n+1}}{(2n + 1)!\,(2n + 1)}.$

9. $s(u) = u - (1 + k^2)\frac{u^3}{3!} + (1 + 14k + k^4)\frac{u^5}{5!} + \cdots.$

10. (*a*) $-\frac{e}{2}$, (*b*) $\frac{11e}{24}$, (*c*) 0, (*d*) $e^{-1/6}$, (*e*) 1.

11. $e - \frac{e}{2}\left(\frac{1}{x}\right) + \frac{11e}{24}\left(\frac{1}{x}\right)^2 - \cdots.$

12. The potential energy at (r, θ) is given exactly by

$$e[(r^2 + \tfrac{1}{4}d^2 + rd\cos\theta)^{-1/2} - (r^2 + \tfrac{1}{4}d^2 - rd\cos\theta)^{-1/2}],$$

where the charges e and $-e$ are placed at $(\frac{1}{2}d, 0)$ and $(-\frac{1}{2}d, 0)$, respectively. Expand in a series of powers of d/r.

EXERCISES

SECTION 5.6, page 457

1. What is the order of contact of the curves $y = e^x$ and $y = 1 + x + \frac{1}{2}\sin^2 x$ at $x = 0$?

2. What is the order of contact of $y = \sin^4 x$ and $y = \tan^4 x$ at $x = 0$?

3. Determine the constants a, b, c, d in such a way that the curves $y = e^{2x}$ and $y = a\cos x + b\sin x + c\cos 2x + d\sin 2x$ have contact of order 3 at $x = 0$.

4. What is the order of contact of the curves

$$x^3 + y^3 = xy, \qquad x^2 + y^2 = x$$

at their points of intersection? Plot the curves.

5. What is the order of contact of the curves

$$x^2 + y^2 = y, \qquad x^2 = y$$

at their points of intersection?

PROBLEMS

SECTION 5.6, page 457

1. Prove if $f(a) = 0$ and $f(x)$ has sufficiently many derivatives at $x = a$ that $f(x)^n$ has at least an $(n - 1)$th order contact with the x-axis.

2. The curve $y = f(x)$ passes through the origin 0 and touches the x-axis at 0. Show that the radius of curvature of the curve at 0 is given by

$$\rho = \lim_{x \to 0} \frac{x^2}{2y}.$$

***3.** K is a circle which touches a given curve at a point P and passes through a neighboring point Q of the curve. Show that the limit of the circle K as $Q \to P$ is the circle of curvature of the curve at P.

***4.** Show that the order of contact of a curve and its osculating circle is at least three at points where the radius of curvature is a maximum or minimum.

***5.** Show that the osculating circle at a point where the radius of curvature is a maximum or minimum does not cross the curve, unless the contact is of higher than third order.

***6.** Find the maxima and minima of the following functions:

$$(a)\ \cos x \cosh x, \qquad (b)\ x + \cos x.$$

***7.** Determine the maxima and minima of the function $y = e^{-1/x^2}$ (see p. 242).

Answers to Exercises

SECTION 5.6, page 457

1. 2. **2.** 5. **3.** $a = , \frac{8}{3}b = \frac{16}{3}, c = -\frac{5}{3}, d = -\frac{5}{3}$.

4. Third order and also zero order at (0, 0); zero order at $(\frac{1}{2}, \frac{1}{2})$.

5. Third order at (0, 0).

Solutions and Hints to Problems

SECTION 5.6, page 457

1. Let $f(x)$ be twice differentiable. Then $f(x) = f'(a)(x - a) + R_1(x)$ where $R_1(x)$ is differentiable and

$$\lim_{x \to a} \frac{R_1(x)}{(x - a)^2} = \frac{1}{2} f''(a).$$

Thus $R_1(x)/(x - a)$ is differentiable and $f(x) = (x - a)g(x)$, where $g(x)$ is differentiable. Consequently, $f(x)^n = (x - a)^n g(x)^n$, where $g(x)^n$ is differentiable.

2. The radius of curvature is $1/f''(0)$ (see p. 357). Since the curve touches, or is tangent to the x-axis at the origin, $f(x) = (x^2/2) f''(0) + R_2$, where $\lim_{x \to 0} R_2/x^2 = 0$, or $R_2 = 0(x^2)$. Therefore

$$\rho = \frac{1}{f''(0)} = \lim_{x \to 0} \frac{x^2}{2y}.$$

3. Take P as origin and the tangent to the curve at P as x-axis. Let the coordinates of Q be (x, y). Then the center of the circle in question lies on the y-axis at the point $\eta = y/2 + x^2/2y$; then use Problem 2.

4. At a point P where $\rho = [(1 + y'^2)^{3/2}]/y''$ is a maximum or minimum, we necessarily have $y'' = 3y'y''/(1 + y'^2)$. Take axes as in Problem 3; then

$y'''(0) = 0$, so that the equation of the curve in the neighborhood of $x = 0$ is $y = \frac{1}{2}x^2 + ax^4 + \cdots$. The equation of the osculating circle is $y = \frac{1}{2}x^2 + bx^4 + \cdots$, and the contact is at least of order 3.

5. If the contact is exactly of order 3 then, in the notation of Problem 4, the difference in ordinates between the curve and its osculating circle is $(a - b)x^4 + \cdots$, which has constant sign if $a \neq b$.

6. (*a*) A maximum at $x = 0$; other extrema when $\tanh x = \tan x$, maxima when $(2n - 1)\pi < x < 2n\pi$, minima when $2n\pi < x < (2n + 1)\pi$.

(*b*) No extrema. Inflections at $x = \pi/2 + 2n\pi$.

7. Since all derivatives vanish at $x = 0$, the test employing the order of contact is not applicable. A minimum at $x = 0$.

EXERCISES

SECTION A.I.3, page 464

Evaluate the limits in Exercises 1–19.

1. $\lim_{x\to a} \dfrac{x^n - a^n}{x - a}$.

2. $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$.

3. $\lim_{x\to 0} \dfrac{24 - 12x^2 + x^4 - 24\cos x}{(\sin x)^6}$.

4. $\lim_{x\to 0} \dfrac{e^x - e^{-x}}{\sin x}$.

5. $\lim_{x\to 0} \dfrac{\arcsin x}{x}$.

6. $\lim_{x\to \pi/2} \dfrac{\tan 5x}{\tan x}$.

7. $\lim_{x\to 1} \left(\dfrac{2}{x^2 - 1} - \dfrac{1}{x - 1}\right)$.

8. $\lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$.

9. $\lim_{x\to 0} \left(\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}\right)$.

10. $\lim_{x\to 0} x^{\sin x}$.

11. $\lim_{x\to 0} (1 + x)^{1/x}$.

12. $\lim_{x\to 0} \dfrac{e^{2x} - 1}{\log(1 + x)}$.

13. $\lim_{x\to 0} \dfrac{x \tan x}{\sqrt{(1 - x^2)} - 1}$.

14. $\lim_{x\to 0} x \log x$.

15. $\lim_{x\to 1} x^{1/(1-x)}$.

16. $\lim_{x\to 0} \left(\dfrac{\tan x}{x}\right)^{1/x^2}$.

17. $\lim_{x\to 0} \dfrac{a^x - b^x}{x}$.

18. $\lim_{x\to \infty} x \log \dfrac{x - a}{x - b}$.

19. $\lim_{x\to 0} \dfrac{\log \cos ax}{\log \cos bx}$.

PROBLEMS

SECTION A.I.3, page 464

1. Prove if f is continuous on the interval [0, 1] that

$$\lim_{x\to 0} x\int_x^1 \frac{f(z)}{z^2}\,dz = f(0).$$

2. Prove that the function $y = (x^2)^x$, $y(0) = 1$ is continuous at $x = 0$.

Answers to Exercises

SECTION 5A.I.3, page 464

1. na^{n-1}. **2.** $\frac{1}{6}$. **3.** $\frac{1}{30}$. **4.** 2. **5.** 1.

6. Write expression as $\dfrac{\cot x}{\cot 5x} : \frac{1}{5}$.

7. $-\frac{1}{2}$. **8.** $\frac{1}{6}$. **9.** $\frac{1}{3}$. **10.** Take logarithms; 1.

11. e. **12.** 2. **13.** -2. **14.** 0. **15.** $\dfrac{1}{e}$. **16.** $\dfrac{1}{\sqrt[6]{e}}$.

17. $\log \dfrac{a}{b}$ **18.** $b - a$ **19.** $\dfrac{a^2}{b^2}$

Solutions and Hints to Problems

SECTION 5.A.3, page 464

1. Put in the indeterminate form ∞/∞:

$$\lim_{x\to 0} \int_x^1 \frac{f(z)}{z^2}\,dz \Big/ \left(\frac{1}{x}\right).$$

2. Take logarithms. Use

$$\lim_{x\to 0} (x^2)^x = \lim_{x\to 0} -\frac{2\log|x|}{\dfrac{1}{x}}.$$

6 *Numerical Methods*

EXERCISES

SECTION 6.1, page 482

1. From the formula $\pi/4 = \int_0^1 1/(1 + x^2)\, dx$, calculate π

(*a*) using the trapezoid formula with $h = 0.1$.
(*b*) using Simpson's rule with $h = 0.1$.

2. Calculate $\int_0^1 1/\sqrt{1 + x^4}\, dx$ numerically with an error less than 0.1.

3. Show that Simpson's rule gives the correct value for the volume of the sphere $x^2 + y^2 + z^2 = r^2$ when the values f_0, f_1, f_2 are taken as the areas of the cross sections made by the planes $x = -r$, 0, r, respectively. Explain.

4. Show that the length of the ellipse $x = a \cos t$, $y = b \sin t$ is

$$s = 4a\int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 t}\, dt, \text{ where } e^2 = \frac{a^2 - b^2}{a^2}.$$

Calculate the length of the ellipse for which $e = \frac{1}{2}$ to four significant figures, by using Simpson's rule with six divisions.

5. Expand the integral of Exercise 4 as a series, and estimate the number of terms necessary for accuracy to four significant figures.

6. Evaluate $\int_0^1 \frac{\log(1 + x)}{x}\, dx$, using Simpson's rule with $h = 0.1$.

PROBLEMS

SECTION 6.1, page 482

1. Prove if $f''(x) \geq 0$, that the trapezoid rule yields a greater value and the tangent rule a lesser value than the exact integral of f.

2. Estimate the value $h = (b - a)/n$ needed for a calculation by Simpson's rule accurate to p decimal places of

(*a*) $\log 2 = \int_1^2 \frac{1}{x}\, dx$, (*b*) $\pi = 4\int_0^1 \frac{1}{1 + x^2}\, dx$.

3. Estimate in terms of k and s, ($k < 1$ and $s < 1$) the number of points needed to calculate within an error ϵ the elliptic integral

$$u(s) = \int_0^s \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

4. Let $f(x)$ be a continuous function on the interval $\alpha \le x \le \alpha + h$, with a uniformly bounded derivative: $|f'(x)| \le M_1$ for M_1 a constant. Prove that for any fixed point ξ, $\alpha \le \xi \le \alpha + h$, the estimate

$$\left| \int_\alpha^{\alpha+h} f(x)\,dx - hf(\xi) \right| \le \frac{M_1 h^2}{2}$$

holds. (*Hint:* Use Taylor's theorem.)

5. Calculate $\int_0^\infty e^{-x^2}\,dx$ numerically to within $\frac{1}{100}$.

Answers to Exercises

SECTION 6.1, page 482

1. 3.14, 3.1416. **2.** 0.93.

3. The volume is expressible as

$$\int_{-r}^{r} A(x)\,dx,$$

where $A(x) = \pi x^2$ is the area of the cross section perpendicular to the x-axis; but Simpson's rule gives the integral of a quadratic function exactly.

4. $5.881a$. **5.** 11. **6.** 0.82247.

Solutions and Hints to Problems

SECTION 6.1, page 482

1. We have already proven that a convex function lies on or below its chord and on or above its tangent.

2. (*a*) $h < \frac{1}{2} \frac{\sqrt[4]{6}}{10^{(p-1)/4}}$,

(*b*) $h < \frac{1}{4} \frac{\sqrt[4]{2}}{10^{(p-1)/4}}$.

3. Since the derivatives are nondecreasing, take $M_1 = f'(s)$, $M_2 = f''(s)$, or $M_4 = f^{(4)}(s)$ in the estimates of the errors for the rectangle, trapezoid, and Simpson's rules, respectively.

4. For $x > \xi$,

$$-M_1(x - \xi) \le f(x) - f(\xi) \le M_1(x - \xi)$$

and for $x < \xi$,

$$-M_1(\xi - x) \le f(x) - f(\xi) \le M_1(\xi - x).$$

Integrate from ξ to $\alpha + h$ and from α to ξ and add

$$\left| \int_{\alpha}^{\alpha+h} f(x)\,dx - hf(\xi) \right| \le \tfrac{1}{2}M_1[(\alpha + h - \xi)^2 + (\xi - \alpha)^2] \le \tfrac{1}{2}M_1h^2$$

since the expression in brackets is maximized for $\xi = \alpha$ or $\xi = \alpha + h$.

5. Set

$$\int_0^\infty e^{-x^2}\,dx = \int_0^a e^{-x^2}\,dx + \int_a^\infty e^{-x^2}\,dx.$$

Take a so large that the second integral is smaller than 1/400 and estimate the first integral within the same tolerance of error. Thus, for the second integral, first make the transformation $x = \sqrt{u}$; then for $a = 3$,

$$\int_a^\infty e^{-x^2}\,dx = \int_{a^2}^\infty \frac{e^{-u}}{2\sqrt{u}}\,du \le \int_{a^2}^\infty \frac{e^{-u}}{2}\,du$$

$$< \tfrac{1}{2}e^{-a^2} < \tfrac{1}{2}2^{-9} < \frac{1}{1000}.$$

For $0 \le x \le 3$ it is easily verified for $f(x) = e^{-x^2}$ that the maximum of the absolute value is $M_4 = f^{(4)}(0) = 12$. Hence by the estimate of the text it is necessary to use Simpson's rule with a subdivision so fine that

$$\frac{h^4}{5} < \frac{1}{200},$$

which is certainly satisfied if $h \le \frac{1}{3}$. *Answer:* 0.89.

EXERCISES

SECTION 6.2, page 490

1. The hypotenuse of a right-angled triangle is measured accurately as 40, and one angle is measured as 30° with a possible error of $\frac{1}{2}$°. Find the possible error in the lengths of each of the sides and in the area of the triangle.

2. Estimate tan 46° and sin 29°.

3. Calculate $\log_e 2$ to three decimal places by means of an expansion in series.

4. Calculate $\log_e 5$ to six decimal places, using the values of $\log_e 2$ and $\log_e 3$ given in the text.

5. Calculate π to five decimal places, using any one of the formulas in Section 2 (p. 492).

PROBLEMS

SECTION 6.2, page 490

1. The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum. If the pendulum drives a clock which gains a minute per day determine the necessary correction in l.

2. To measure the height of a hill, a tower 100 meters high on top of the hill is observed from the plain. The angle of elevation of the base of the tower is 42° and the tower itself subtends an angle of 6°. What are the limits of error in the determination of the height if the angle 42° is subject to an error of 1°?

Answers to Exercises

SECTION 6.2, page 490

1. 0.175; 0.302; 3.490.

2. Use $\tan 46° = \tan(45° + 1°)$: $\tan 46° = 1.04$.
Use $\sin 29° = \sin(30° - 1°)$: $\sin 29° = 0.485$.

3. 0.693.

4. Use series for $\log p$; 1.609438.

5. 3.14159.

Solutions and Hints to Problems

SECTION 6.2, page 490

1. The error in T is one part in 1440.

$$\frac{\Delta T}{T} \approx \frac{\Delta l}{2l} = \frac{1}{1440}.$$

Hence lengthen the pendulum by approximately one part in 720.

2. Let x be the angle of elevation of the base of the tower, and α the angle subtended by the tower. The height of the hill h is then

$$h = \frac{100 \sin x \cos(\alpha + x)}{\sin \alpha},$$

which, for $x = 42°$, $\alpha = 6°$ is 427.8 meters. Since

$$\frac{dh}{dx} = \frac{100 \cos(2x + \alpha)}{\sin \alpha} = 0,$$

the error is higher than first order.

$$\frac{d^2h}{dx^2} = -\frac{200 \sin(2x + \alpha)}{\sin \alpha} = -1913.$$

The limit of error in meters is approximately

$$\frac{1913}{2}\left(\frac{\pi}{180}\right)^2 \approx 0.29,$$

or 7/100%.

EXERCISES

SECTION 6.3, page 494

1. Using Newton's method, find the positive root of $x^6 + 6x - 8 = 0$ to four decimal places.

2. Find to four places the root of $x = \tan x$ between π and 2π. *Prove* that the result is accurate to four places.

3. Using Newton's method, find the value of x for which

$$\int_0^x \frac{u^2}{1 + u^2}\, du = \frac{1}{2}.$$

4. Find the roots of the equation $x = 2 \sin x$ to two places.

5. Determine the positive roots of the equation $x^5 - x - 0.2 = 0$ by the method of iteration.

6. Determine the least positive root of $x^4 - 3x^3 + 10x - 10 = 0$ by the method of iteration.

7. Find the roots of $x^3 - 7x^2 + 6x + 20 = 0$ to four decimal places.

PROBLEMS

SECTION 6.3, page 494

1. (*a*) To solve the equation $x = f(x)$, show how best to choose the constant a so that the iteration scheme

$$x_{k+1} = x_k + a[x_k - f(x_k)]$$

converges as rapidly as possible in the neighborhood of the solution.

(*b*) Apply this method to solve the equation for $\sqrt{A}$,

$$x = \frac{A}{x}.$$

(*c*) Show if $A \geq 1$ that the number of accurate decimal places is at least doubled at each step of the iteration scheme obtained in (*b*).

2. (*a*) Show how best to choose a polynomial

$$g(x) = a + bx^2$$

so that the iteration scheme for $\sqrt{A}$,

$$x_{k+1} = x_k + g(x_k)\left(x_k - \frac{A}{x_k}\right),$$

converges most rapidly in the neighborhood of the solution.

(*b*) Estimate the rapidity of convergence.

(*c*) Show how to improve further the convergence by suitable choices of polynomials $g(x)$ which are of higher degree.

3. Investigate suitable schemes of the type of Problems 1 and 2 for the calculation of $\sqrt[n]{A}$.

Answers to Exercises

SECTION 6.3, page 494

1. 1.0755. **2.** 4.4934. **3.** 1.475.

4. 0, 1.90, -1.90. **5.** 1.045.

6. Write equation in form $x = 1 + 0.3x^3 - 0.1x^4$; 1.519.

7. -1.2361, 3.2361, 5.0000.

Solutions and Hints to Problems

SECTION 6.3, page 494

1. (*a*) Let $\xi = f(\xi)$ be the solution sought for. Expand $f(\xi + h)$ by Taylor's theorem:

$$f(\xi + h) = f(\xi) + hf'(\xi) + \tfrac{1}{2}h^2 f''(\xi + h).$$

Then if $x_k = \xi + h$,

$$x_{k+1} = \xi + [1 + a - af'(\xi)]h - \tfrac{1}{2}af'(\xi + \theta h)h^2.$$

The error will be quadratic in h if

$$a = \frac{1}{f'(\xi) - 1}.$$

(*b*) $x_{k+1} = \dfrac{1}{2}\left(x_k + \dfrac{A}{x_k}\right).$

(*c*) Suppose the number of accurate decimal places in x_k is ν_k. Then

$$|x_k - \sqrt{A}| < \tfrac{1}{2}(10)^{-\nu_k},$$

and, from part *a*,

$$|x_{k+1} - \sqrt{A}| < M\tfrac{1}{2}(10)^{-2\nu_k},$$

where $M \geq A/8x^2$ for all x satisfying

$$|x - \sqrt{A}| < |x_k - \sqrt{A}|. \text{ For } \nu_k \geq 1,\ |x_k| \geq \sqrt{A/8},$$

we have $M \leq 1$, from which the result follows.

2. (*a*) and (*b*). Proceed as above, but demand that the error in x_{k+1} be bounded by Ch^3 for some constant C;

$$a = -\frac{3}{10}, \qquad b = -\frac{1}{5A}.$$

(*c*) Use even polynomials,

$$g(x) = a_0 + a_1x^2 + \cdots + a_\nu x^{2\nu}$$

and require that the error be bounded by $Cx^{\nu+2}$. The coefficients will be rational.

3. Use $f(x) = A/x^{n-1}$ in the iteration scheme of Problem 1. Take $g(x)$ of the form

$$g(x) = a_0 + a_1x^n + a_2x^{2n} + \cdots + a_\nu x^{\nu n}.$$

Again the coefficients will be rational.

It is possible to generalize this technique to obtain an iteration scheme of this type for the root of any polynomial equation.

PROBLEMS

SECTION 6A.1, page 504

1. Prove that $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$.

***2.** By considering $\int_{1/2}^{n+1/2} \log(\alpha + x)\,dx$, $\alpha > 0$, show that

$$\alpha(\alpha + 1)\cdots(\alpha + n) = a_n n!\, n^\alpha,$$

where a_n is bounded below by a positive number. Show that a_n is monotonically decreasing for sufficiently large values of n. [The limit of a_n as $n \to \infty$ is $1/\Gamma(\alpha)$.]

3. Find an approximate expression for $\log \frac{n_1!\, n_2! \cdots n_l!}{n!}$, where $n_1 + n_2 + \cdots + n_l = n$.

4. Show that the coefficient of x^n in the binomial expansion of $\frac{1}{\sqrt{1-x}}$ is asymptotically given by $\frac{1}{\sqrt{\pi n}}$.

Solutions and Hints to Problems

SECTION 6.A.1, p. 504

1. Use the asymptotic estimates for $n!$. (Compare Problems 1.7, No. 9.)

2. Since $\log(\alpha + x)$ is convex downward, and $\alpha > 0$,

$$\log(\alpha + 1) + \cdots + \log(\alpha + n) > \int_{1/2}^{n+1/2} \log(\alpha + x)\,dx$$
$$= (n + \tfrac{1}{2} + \alpha)\log(n + \tfrac{1}{2} + \alpha) - (\alpha + \tfrac{1}{2})\log(\alpha + \tfrac{1}{2}) - n,$$

or $$\alpha(\alpha + 1)\cdots(\alpha + n) > \alpha \frac{(n + \frac{1}{2} + \alpha)^{n+1/2+\alpha}}{(\alpha + \frac{1}{2})^{\alpha+1/2}} e^{-n} > k(\alpha) n!\, n^\alpha,$$

where $k(\alpha)$ is a positive number depending on α. Furthermore,

$$\frac{a_n}{a_{n-1}} = \left(1 + \frac{\alpha}{n}\right)\left(1 - \frac{1}{n}\right)^\alpha = 1 - \frac{\alpha(\alpha + 1)}{2}\frac{1}{n^2} + \frac{R}{n^3},$$

where R remains bounded as $n \to \infty$. Therefore, for sufficiently large values of n, $a_n < a_{n-1}$ and the sequence is monotonically decreasing.

3. $\sum_{\nu=1}^{l} (n_\nu + \frac{1}{2}) \log n_\nu - (n + \frac{1}{2}) \log n + \text{constant}.$

4. The coefficient of x^n is $\dfrac{2n!}{2^{2n}(n!)^2} \approx \dfrac{1}{\sqrt{\pi n}}$.

7 Infinite Sums and Products

PROBLEMS

SECTION 7.1, page 511

1. Prove that

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + = 1,$$

(cf. Problems 1.6, No. 12*a*) and use the result to prove $\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}$ converges.

2. Use the result of Problem 1 to obtain upper and lower bounds for

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}.$$

3. Prove that $\sum_{\nu=0}^{\infty}(-1)^{\nu} \frac{2\nu+3}{(\nu+1)(\nu+2)} = 1.$

4. For what values of α does the series $1 - \frac{1}{2^{a}} + \frac{1}{3^{a}} - \frac{1}{4^{a}} + \cdots$ converge?

5. Prove that if $\sum_{\nu=1}^{\infty} a_{\nu}$ converges, and $s_n = a_1 + a_2 + \cdots + a_n$, then the sequence

$$\frac{s_1 + s_2 + \cdots + s_N}{N}$$

also converges, and has $\sum_{\nu=1}^{\infty} a_{\nu}$ as its limit.

6. Is the series $\sum_{n=1}^{\infty} \left(\frac{2n}{2n+1} - \frac{2n-1}{2n}\right)$ convergent?

7. Is the series $\sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{\nu}{\nu+1}$ convergent?

8. Prove that if $\sum_{\nu=1}^{\infty} a_{\nu}^{2}$ converges, so does $\sum_{\nu=1}^{\infty} \frac{a_{\nu}}{\nu}$.

9. (*a*) If a_n is a monotonic increasing sequence with positive terms, when does the series $\frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots$ converge?

(*b*) Give an example of a monotone decreasing sequence with $\lim_{n\to\infty} a_n = 1$ for which the series diverges.

(*c*) Show that if decreasing sequences are allowed, then it is possible to obtain convergent sums even when $\lim_{n\to\infty} a_n = 1$.

10. If the series $\sum_{\nu=1}^{\infty} a_\nu$ with decreasing positive terms converges, then $\lim_{n\to\infty} na_n = 0$.

11. Show that the series $\sum_{\nu=1}^{\infty} \sin \frac{\pi}{\nu}$ diverges.

12. Prove that if Σa_ν converges and if $b_1, b_2, b_3, \ldots$ is a bounded monotonic sequence of numbers, then $\Sigma a_\nu b_\nu$ converges.

Moreover, prove that if $S = \Sigma a_\nu b_\nu$ and if $\Sigma a_\nu \le M$, then $|S| \le Mb_1$.

13. A sequence $\{a_n\}$ is said to be of bounded variation if the series

$$\sum_{i=1}^{\infty} |a_{i+1} - a_i|$$

converges.

(*a*) Prove that if the sequence $\{a_n\}$ is of bounded variation, then the sequence $\{a_n\}$ converges.

(*b*) Find a divergent infinite series Σa_i whose elements a_i constitute a sequence which is of bounded variation.

(*c*) Prove the following generalization of Abel's convergence test (see p. 515) due to Dedekind.

The series $\Sigma a_i p_i$ is convergent if Σa_i oscillates between finite bounds and $\{p_i\}$ is a null sequence which is of bounded variation.

(*d*) Discuss the convergence of the following infinite series:

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin nx}{\log n} (-1)^n;$

(ii) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos nx}{\log n} (-1)^n$

for x any fixed real number.

14. Discuss the convergence or divergence of the following series:

(*a*) $\displaystyle\sum \frac{(-1)^\nu}{\nu}$, (*b*) $\displaystyle\sum \frac{(-1)^\nu \cos(\theta/\nu)}{\nu}$,

(*c*) $\displaystyle\sum \frac{\cos \nu\theta}{\nu}$, (*d*) $\displaystyle\sum \frac{\sin \nu\theta}{\nu}$,

(*e*) $\displaystyle\sum \frac{(-1)^\nu \cos \nu\theta}{\nu}$, (*f*) $\displaystyle\sum \frac{(-1)^\nu \sin \nu\theta}{\nu}$.

15. Find the sums of the following derangements of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

for log 2:

(*a*) $1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \tfrac{1}{12} + - - \cdots$.

(*b*) $1 + \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{6} + + + \cdots$.

16. Find whether the following series converge or diverge:

(*a*) $1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{1}{9} + + - \cdots$.

(*b*) $1 + \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{2}{6} + \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{2}{9} + + - \cdots$.

Solutions and Hints to Problems

SECTION 7.1, page 511

1. For the nth partial sum,

$$\sum_{\nu=1}^{n} \frac{1}{\nu(\nu+1)} = \sum_{\nu=1}^{n} \left(\frac{1}{\nu} - \frac{1}{\nu+1}\right) = 1 - \frac{1}{n+1}.$$

Convergence of $\Sigma \dfrac{1}{\nu^2}$ follows from the bounded monotonic character of the partial sums:

$$\sum_{\nu=1}^{n} \frac{1}{\nu^2} < 1 + \sum_{\nu=2}^{n} \frac{1}{(\nu-1)\nu} < 1 + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 2.$$

2. Upper bound obtained in Problem 1.
Lower bound:

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} > \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+1)} = 1.$$

3. Use $(2\nu + 3)/[(\nu + 1)(\nu + 2)] = 1/(\nu + 1) + 1/(\nu + 2)$ to obtain for the nth partial sum

$$s_n = 1 + \frac{(-1)^n}{n+2}.$$

4. By Leibnitz's test, converges for $a > 0$. For $a \leq 0$ the nth term does not approach zero; diverges.

5. Apply Problems 1.6, No. 11 to the sequence of partial sums.

6. Yes. Sequence of partial sums is bounded and monotonic:

$$\sum_{\nu=1}^{n}\left(\frac{2\nu}{2\nu+1}-\frac{2\nu-1}{2\nu}\right)=\sum_{\nu=1}^{n}\frac{1}{2\nu(2\nu+1)}$$
$$=\frac{1}{4}\sum_{\nu=1}^{n}\frac{1}{\nu(\nu+\frac{1}{2})}$$
$$<\frac{1}{4}\sum_{\nu=1}^{n}\frac{1}{\nu^2}$$
$$<\frac{1}{4}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2},$$

where the last series converges by the result of Problem 1.

7. No. The nth term does not approach zero.

8. The series is absolutely convergent since by Cauchy's inequality

$$\sum_{\nu=1}^{n}\left|\frac{a_\nu}{\nu}\right| \le \left[\left(\sum_{\nu=1}^{n} a_\nu^2\right)\left(\sum_{\nu=1}^{n}\frac{1}{\nu^2}\right)\right]^{1/2}.$$

9. (*a*) If $\lim a_n \le 1$, the terms do not tend to zero. If $\lim a_n > k > 1$, compare the series with $\Sigma \dfrac{1}{k^n}$.

(*b*) $a_n = \dfrac{n+1}{n}$.

(*c*) $a_n = \dfrac{(n+1)^2}{n^2}$ (cf. Problem 1).

10. For any ϵ, $\sum_{\nu=n}^{m} a_\nu < \epsilon$ for every n, m sufficiently large. But $\sum_{\nu=n+1}^{m} a_\nu >$ $(m-n)a_m$, or $ma_m < \epsilon + na_m$. Keeping n fixed, choose m so large that $na_m < \epsilon$; for every such m, $ma_m < 2\epsilon$.

11. Apply the result of Problem 10.

12. Let s_n denote the partial sums of $\sum_{\nu=1}^{\infty} a_\nu$, s the sum, and let $\sigma_n = s_n - s$. Then

$$\sum_{\nu=n}^{m} a_\nu b_\nu = \sum_{\nu=n}^{m}(\sigma_\nu - \sigma_{\nu-1})b_\nu = \sum_{\nu=n}^{m}\sigma_\nu(b_\nu - b_{\nu+1}) - \sigma_{n-1}b_n + \sigma_m b_{m+1}.$$

For every sufficiently large ν, $|\sigma_\nu| < \epsilon$, and

$$\left|\sum_{\nu=n}^{m} a_\nu b_\nu\right| < \epsilon\sum_{\nu=n}^{m}|b_\nu - b_{\nu+1}| + \epsilon\,|b_n| + \epsilon\,|b_{m+1}|$$
$$< \epsilon\,|b_n - b_{m+1}| + \epsilon\,|b_n| + \epsilon\,|b_{m+1}|.$$

This is in turn less than $4B\epsilon$, where B is a bound for $|b_\nu|$, and the series $\sum_{\nu=1}^{\infty} a_\nu b_\nu$ converges.

13. (*a*) Clearly, by definition, for $m > n$,

$$\lim_{n,m\to\infty} \sum_{i=n}^{m} |a_{i+1} - a_i| = 0.$$

The proof now follows by the Cauchy convergence criterion.

(*b*) $a_n = \dfrac{1}{n}$.

(*c*) Use the method of proof of Abel's theorem (p. 516) together with the fact that $p_n \to 0$ as $n \to \infty$.

(*d*) (i) By the equation on p. 135,

$$S_N = \sum_{n=2}^{N} \sin(nx)(-1)^n = \sum_{n=2}^{N} \sin nx \cos n\pi$$
$$= \frac{1}{2} \sum_{n=2}^{N} [\sin n(x - \pi) + \sin n(\pi + x)]$$

is uniformly bounded for all N, when x is not an odd integer multiple of π, whereas in this case $S_N = 0$. The sequence $a_n = 1/(\log n)$ is of bounded variation since it is monotonic, so that the series converges for all x.

(ii) By the method of (i), the series is convergent when x is not an odd integer multiple of π.

14. (*a*) and (*b*). Convergent by Leibnitz's test.

(*c*) If $\theta = 2n\pi$, diverges. If $\theta \neq 2n\pi$, then from the formula of Section 2.2e, for $a = \frac{1}{2}\pi$, $h = \theta$,

$$\sum_{\nu=1}^{n} \cos \nu\theta = \frac{\sin \frac{1}{2}n\theta \cos \frac{1}{2}(n+1)\theta}{\sin \frac{1}{2}\theta};$$

now apply the Abel convergence test (p. 515); converges.

(*d*) Compare the solution of *c*; converges.

(*e*) $(-1)^\nu \cos\theta = \cos \nu\pi \cos \nu\theta$
$= \cos(\theta + \pi)$.

Converges unless $\theta = (2n - 1)\pi$.

(*f*) Converges; compare the solution of (*e*).

15. (*a*) $$s_{3n} = \sum_{\nu=1}^{n} \left\{\frac{1}{2\nu - 1} - \frac{1}{2(2\nu - 1)} - \frac{1}{4\nu}\right\}$$
$$= \frac{1}{2} \sum_{\nu=1}^{n} \left\{\frac{1}{2\nu - 1} - \frac{1}{2\nu}\right\}$$
$$= \tfrac{1}{2}\{1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots\};$$

converges to $\frac{1}{2}\log 2$.

(*b*) The sum to $6n$ terms is the same as that for the original series; converges to log 2.

16. (*a*)
$$S_{3n} = 1 + (\tfrac{1}{2} - \tfrac{1}{3}) + \tfrac{1}{4} + (\tfrac{1}{5} - \tfrac{1}{6}) + \cdots + \frac{1}{3n-2} + \left(\frac{1}{3n-1} - \frac{1}{3n}\right)$$
$$\geq 1 + \tfrac{1}{4} + \tfrac{1}{7} + \cdots + \frac{1}{3n-2}$$
$$\geq \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{9} + \cdots + \frac{1}{3n}$$
$$\geq \tfrac{1}{3}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{n}\right);$$
diverges.

(*b*)
$$S_{3n+1} = 1 + (\tfrac{1}{2} - \tfrac{1}{3}) - (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) - (\tfrac{1}{6} - \tfrac{1}{7}) + \cdots + \left(\frac{1}{3n-1} - \frac{1}{3n}\right) - \left(\frac{1}{3n} - \frac{1}{3n+1}\right);$$
converges by Leibnitz's test.

EXERCISES

SECTION 7.2, page 520

Find whether the series in Exercises 1–6 are convergent or not:

1. $\displaystyle\sum_{\nu=1}^{\infty} \frac{1}{1+\nu^2}$.

2. $\displaystyle\sum_{\nu=1}^{\infty} \frac{\nu!}{\nu^\nu}$.

3. $\displaystyle\sum_{\nu=1}^{\infty} \frac{1}{\sqrt{\nu(\nu+1)}}$.

***4.** $\displaystyle\sum_{\nu=2}^{\infty} \frac{1}{(\log \nu)^\alpha}$, α fixed.

5. $\displaystyle\sum_{\nu=2}^{\infty} \frac{1}{(\log \nu)^{\log \nu}}$.

6. $\displaystyle\sum_{\nu=1}^{\infty} \frac{\nu}{2^\nu}$.

Estimate the error after n terms of the series in Exercises 7–10:

7. $\displaystyle\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu^2}$.

8. $\displaystyle\sum_{\nu=1}^{\infty} \frac{1}{\nu!}$.

9. $\displaystyle\sum_{\nu=1}^{\infty} \frac{1}{\nu^\nu}$.

10. $\displaystyle\sum_{\nu=1}^{\infty} \frac{\nu}{2^\nu}$.

11. Prove that $\displaystyle\sum_{\nu=1}^{\infty} \sin^2\left[\pi\left(\nu + \frac{1}{\nu}\right)\right]$ converges.

12. Does $\displaystyle\sum_{\nu=-\infty}^{\infty} e^{-\nu^2}$ $\left(\text{that is, } 1 + 2\displaystyle\sum_{\nu=1}^{\infty} e^{-\nu^2}\right)$ converge?

13. Prove that if $u_i \geq 0$ ($i = 1, 2, 3, \ldots$) and $\sum_{i=1}^{\infty} u_i$ converges, then $\sum_{i=1}^{\infty} u_i^2$ also converges.

14. Show that if $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ both converge, then $\sum_{k=1}^{\infty} a_k b_k$ also converges.

15. Prove that

$$1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \cdots + \frac{1}{3n+1} + \frac{1}{3n+2} - \frac{2}{3n+3} + \cdots = \log 3.$$

16. Show that $\sum_{\nu=1}^{\infty} \frac{\nu!}{(2\nu)!}$ converges.

PROBLEMS

SECTION 7.2, page 520

1. Prove that $\sum_{\nu=2}^{\infty} \frac{1}{\nu(\log \nu)^{\alpha}}$ converges when $\alpha > 1$ and diverges when $\alpha \leq 1$.

2. Prove that $\sum_{\nu=3}^{\infty} \frac{1}{\nu \log \nu (\log \log \nu)^{\alpha}}$ converges when $\alpha > 1$ and diverges when $\alpha \leq 1$.

3. Prove that if n is an arbitrary integer greater than 1

$$\sum_{\nu=1}^{\infty} \frac{a_\nu^n}{\nu} = \log n,$$

where a_ν^n is defined as follows:

$$a_\nu^n = \begin{cases} 1 \text{ if } n \text{ is not a factor of } \nu, \\ -(n-1) \text{ if } n \text{ is a factor of } \nu. \end{cases}$$

4. Show that $\sum_{\nu=2}^{\infty} \frac{\log(\nu+1) - \log \nu}{(\log \nu)^2}$ converges.

5. Show that $\sum_{\nu=1}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots \nu}{(\alpha+1)(\alpha+2)\cdots(\alpha+\nu)}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$.

***6.** By comparison with the series $\sum_{\nu=1}^{\infty} \frac{1}{\nu^{\alpha}}$, prove the following test: If $\frac{\log(1/|a_n|)}{\log n} > 1 + \epsilon$ for some fixed number $\epsilon > 0$ independent of n, and for every sufficiently large n, the series Σa_ν converges absolutely; if

$\dfrac{\log(1/|a_n|)}{\log n} < 1 - \epsilon$ for every sufficiently large n and some number $\epsilon > 0$, independent of n, the series Σa_ν does not converge absolutely.

7. Show that the series $\sum_{\nu=1}^{\infty}\left(1 - \dfrac{1}{\sqrt{\nu}}\right)^\nu$ converges.

8. For what values of α do the following series converge?

(*a*) $1 - \dfrac{1}{2^\alpha} + \dfrac{1}{3} - \dfrac{1}{4^\alpha} + \dfrac{1}{5} - \dfrac{1}{6^\alpha} + - \cdots.$

(*b*) $1 + \dfrac{1}{3^\alpha} - \dfrac{1}{2^\alpha} + \dfrac{1}{5^\alpha} + \dfrac{1}{7^\alpha} - \dfrac{1}{4^\alpha} + + - \cdots.$

9. By comparison with the series $\Sigma \dfrac{1}{\nu(\log \nu)^\alpha}$, prove the following test:

The series $\Sigma\,|a_\nu|$ converges or diverges according as

$$\frac{\log(1/n\,|a_n|)}{\log\log n}$$

is greater than $1 + \epsilon$ or less than $1 - \epsilon$ for every sufficiently large n.

10. Derive the nth root test from the test of Problem 6.

11. Prove the following comparison test: if the series Σb_ν of positive terms converges, and

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{b_{n+1}}{b_n}$$

from a certain term onward, the series Σa_ν is absolutely convergent; if Σb_ν diverges and

$$\left|\frac{a_{n+1}}{a_n}\right| > \frac{b_{n+1}}{b_n}$$

from a certain term onwards, the series Σa_ν is not absolutely convergent.

***12.** By comparison with $\sum_{\nu=1}^{\infty} \dfrac{1}{\nu^\alpha}$, prove "Raabe's" test:

The series $\Sigma\,|a_\nu|$ converges or diverges according as

$$n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right)$$

is greater than $1 + \epsilon$ or less than $1 - \epsilon$ for every sufficiently large n and for some $\epsilon > 0$ independent of n.

13. By comparison with $\Sigma \dfrac{1}{\nu(\log \nu)^\alpha}$, prove the following test:

The series $\Sigma\,|a_\nu|$ converges or diverges according as

$$n \log n\left(\frac{|a_n|}{|a_{n+1}|} - 1 - \frac{1}{n}\right)$$

is greater than $1 + \epsilon$ or less than $1 - \epsilon$ for every sufficiently large n.

14. Prove Gauss's test:

If
$$\frac{|a_n|}{|a_{n+1}|} = 1 + \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}},$$

where $|R_n|$ is bounded and $\epsilon > 0$ is independent of n, the $\Sigma\, |a_\nu|$ converges if $\mu > 1$, diverges if $\mu \leq 1$.

15. Test the following hypergeometric series for convergence or divergence:

(*a*) $$\frac{\alpha}{\beta} + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1)} + \frac{\alpha(\alpha + 1)(\alpha + 2)}{\beta(\beta + 1)(\beta + 2)} + \cdots.$$

(*b*) $$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha(\alpha + 1) \cdot \beta(\beta + 1)}{1 \cdot 2 \cdot \gamma(\gamma + 1)} + \frac{\alpha(\alpha + 1)(\alpha + 2) \cdot \beta(\beta + 1)(\beta + 2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma + 1)(\gamma + 2)} + \cdots.$$

Answers to Exercises

SECTION 7.2, page 520

1. Convergent.

2. Prove first that $n!/n^n \leq 2/n^2$ when $n > 2$: convergent.

3. Divergent. **4.** Compare Section 3.7, p. 248: divergent.

5. Note that $(\log n)^{\log n} = n^{\log(\log n)}$ and $\log(\log n) > 2$ when n is large; convergent.

6. Convergent. **7.** $\frac{1}{(n+1)^2}$.

8. $$\text{Error} = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right)$$
$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right)$$
$$< \frac{1}{(n+1)!}\frac{1}{1 - \frac{1}{n+1}} < \frac{1}{n \cdot n!}.$$

9. $$\text{Error} = \frac{1}{(n+1)^{n+1}} + \frac{1}{(n+2)^{n+2}} + \cdots$$
$$< \frac{1}{(n+1)^{n+1}} + \frac{1}{(n+1)^{n+2}} + \frac{1}{(n+1)^{n+3}} + \cdots < \frac{1}{n(n+1)^n}.$$

10. Error $= \dfrac{n+1}{2^{n+1}} + \dfrac{n+2}{2^{n+2}} + \cdots$. Now for $n > 1$,

$$n + 2 < \tfrac{3}{2}(n+1),\ n + 3 < \tfrac{3}{2}(n+2) < (\tfrac{3}{2})^2(n+1), \ldots,$$

hence

$$\text{Error} < \frac{n+1}{2^{n+1}}\left(1 + \tfrac{3}{4} + (\tfrac{3}{4})^2 + \cdots\right) < \frac{n+1}{2^{n-1}}.$$

11. Observe that

$$\sin^2 \pi\left(\nu + \frac{1}{\nu}\right) = \sin^2 \frac{\pi}{\nu} < \frac{\pi^2}{\nu^2}$$

by the inequality $\sin x < x$ for $x > 0$.

12. Convergent.

13. For i sufficiently large $u_i < 1$ and hence $u_i^2 < u_i$.

14. Use Cauchy's inequality.

15. $1 + \tfrac{1}{2} - \tfrac{2}{3} + \cdots - \dfrac{2}{3n+3} = \displaystyle\sum_{\nu=1}^{3\nu+3} \frac{1}{\nu} - 3\sum_{\nu=1}^{n+1} \frac{1}{3\nu} = \sum_{\nu=n+2}^{3n} \frac{1}{\nu}$;

then use the formula on p. 526,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{n} = \log n + C + \epsilon_n,$$

where $\lim_{n\to\infty} \epsilon_n = 0$.

16.
$$\frac{\nu!}{(2\nu)!} < \left(\frac{1}{\nu+1}\right)\left(\frac{1}{\nu+2}\right)\cdots\left(\frac{1}{2\nu}\right) < \frac{1}{2^\nu}.$$

Solutions and Hints to Problems

SECTION 7.2, page 520

1. Compare with $\displaystyle\int \frac{dx}{x(\log x)^\alpha}$.

2. Compare with $\displaystyle\int \frac{dx}{x \log x(\log\log x)^\alpha}$.

3. Take the sum from $\nu = 1$ to $\nu = mn$:

$$\sum_{\nu=1}^{mn} \frac{a_\nu{}^n}{\nu} = \sum_{\nu\neq kn} \frac{1}{\nu} - \sum_{\nu=kn} \frac{n-1}{\nu} = \sum_{\nu=1}^{mn} \frac{1}{\nu} - \sum_{k=1}^{m} \frac{n}{kn} = \sum_{\nu=m+1}^{mn} \frac{1}{\nu}.$$

4. Use $\log(1 + 1/\nu) < 1/\nu$ and integral test.

5. Set $S_n = \sum_{\nu=1}^{n} c_\nu$, where

$$c_\nu = \frac{1 \cdot 2 \cdot 3 \cdots \nu}{(\alpha + 1)(\alpha + 2) \cdots (\alpha + \nu)}.$$

Compare $\log c_\nu$ to $\int \log \frac{x}{\alpha + x}\, dx$:

$$\log \frac{1}{\alpha + 1} + \int_1^\nu \log \frac{x}{\alpha + x}\, dx \leq \log c_\nu \leq \log \frac{\nu}{\alpha + \nu} \int_1^\nu \log \frac{x}{\alpha + x}\, dx.$$

Use $\int \log \frac{x}{\alpha + x}\, dx = x \log x - (\alpha + x) \log(\alpha + x)$

to show

$$\frac{k_1}{\left(1 + \frac{\alpha}{\nu}\right)^\nu (\alpha + \nu)^\alpha} \leq c_\nu \leq \frac{k_2}{\left(1 + \frac{\alpha}{\nu}\right)^\nu (\alpha + \nu)^\alpha},$$

where k_1 and k_2 are constants. Observe that $\lim_{\nu \to \infty} \left(1 + \frac{\alpha}{\nu}\right)^\nu = e^\alpha$. Consequently, by the integral test, S_n converges for $\alpha > 1$, diverges for $\alpha \leq 1$.

6. If $\frac{\log(1/|a_n|)}{\log n} > 1 + \epsilon$, then

$$|a_n| < 1/n^{1+\epsilon}.$$

7. Apply the result of Problem 6.

8. (*a*) Set $a_{2n-1} = 1/(2n - 1)$ and $a_{2n} = 1/(2n)^\alpha$. Converges for $\alpha = 1$. If $\alpha > 1$, $\sum_{k=1}^{2m} a_k = \sum_{n=1}^{m} a_{2n-1} - \sum_{n=1}^{m} a_{2n}$ where $\Sigma\, a_{2n-1}$ diverges and $\Sigma\, a_{2n}$ converges; the original series diverges. If $\alpha < 1$

$$\sum_{k=1}^{2m+1} a_k = 1 - \sum_{k=1}^{m} \left[\frac{1}{(2n)^\alpha} - \frac{1}{2n + 1}\right]$$

$$< 1 - \sum_{k=1}^{m} \left[\frac{1}{(2n)^\alpha} - \frac{1}{2n}\right]$$

$$< -\sum_{k=1}^{m} \frac{1}{(2n)^\alpha}\left[1 - \frac{1}{(2n)^{1-\alpha}}\right].$$

For n sufficiently large $1/(2n)^{1-\alpha} < \frac{1}{2}$ and $\Sigma\, a_k < -\frac{1}{2} \Sigma \frac{1}{(2n)^\alpha}$. The series diverges.

(*b*) Converges for $\alpha = 1$.

9. Proceed as in Problem 6.

10. The nth root test may be written as follows: if $\dfrac{\log 1/|a_n|}{n} > \epsilon$, the series converges; if $< -\epsilon$, the series diverges. Write

$$\frac{\log 1/|a_n|}{\log n} = \frac{n}{\log n}\,\frac{\log 1/|a_n|}{n}.$$

11. If $\left|\dfrac{a_{n+1}}{a_n}\right| < \dfrac{b_{n+1}}{b_n}$ for every $n \geq N$, then

$$|a_{n+1}| < \frac{b_{n+1}}{b_n}\,|a_n| < \frac{b_{n+1}}{b_n}\,\frac{b_n}{b_{n-1}}\,|a_{n-1}| < \cdots < \frac{|a_N|}{b_N}\,b_{n+1};$$

therefore $\Sigma\,|a_\nu|$ converges if Σb_ν does. Similarly for divergence.

12. Use the result of Problem 10, comparing with $\sum\limits_{\nu=1}^{\infty} \dfrac{1}{\nu^\alpha}$. The series $\Sigma\,|a_\nu|$ converges if

$$\frac{|a_n|}{|a_{n+1}|} > \left(1 + \frac{1}{n}\right)^\alpha > 1 + \frac{\alpha}{n} + \frac{R}{n^2},$$

where $\alpha > 1$. Then

$$n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > \alpha + \frac{R}{n} > 1 + \epsilon.$$

Reverse the argument:

$$n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > 1 + \epsilon$$

implies the convergence of $\Sigma\,|a_\nu|$. Similarly for divergence.

13. $\Sigma\,|a_\nu|$ converges if

$$\frac{|a_n|}{|a_{n+1}|} > \left(1 + \frac{1}{n}\right)\left(1 + \frac{\log\left(1 + \frac{1}{n}\right)}{\log n}\right)^\alpha > 1 + \frac{1}{n} + \frac{\alpha}{n \log n} + \frac{R}{n^2 \log n},$$

where $\alpha > 1$. Then

$$n \log n\left(\frac{|a_n|}{|a_{n+1}|} - 1 - \frac{1}{n}\right) > \alpha + \frac{R}{n} > 1 + \epsilon.$$

Reversal of this argument gives the convergence test; similarly for divergence.

14. Apply Raabe's test (Problem 12) when $\mu \neq 1$. When $\mu = 1$, apply the result of Problem 13.

15. Apply Gauss's test (Problem 14).

(*a*) Converges if $\beta \geq \alpha + 1$; diverges otherwise.

(*b*) Converges if $\gamma > \alpha + \beta$; diverges otherwise.

EXERCISES

SECTION 7.4, page 529

1. Show by comparison with a series of constant terms that the following series converge uniformly in the intervals stated:

(*a*) $x - x^2 + x^3 - x^4 + \cdots (-\frac{1}{2} \le x \le \frac{1}{2})$.

(*b*) $\frac{1}{2}\sqrt{1 - x^2} + \frac{1}{4}\sqrt{1 - x^4} + \frac{1}{8}\sqrt{1 - x^8} + \cdots + \frac{1}{2^n}\sqrt{1 - x^{2n}} + \cdots$ $(-1 \le x \le 1)$.

(*c*) $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \cdots + \frac{\sin nx}{n^2} + \cdots$.

(*d*) $e^x + e^{2x} + \cdots + e^{nx} + \cdots (-2 \le x \le -1)$.

2. Prove that $\lim f_n(x) = 0$, where $f_n(x) = nx/(1 + n^2x^2)$, $-1 \le x \le 1$. Prove that the convergence is nonuniform.

***3.** (*a*) Find $\lim_{n\to\infty} f_n(x)$, where $f_n(x) = n^2x^2/(1 + n^2x^2)$, $-1 \le x \le 1$. Prove that the convergence is nonuniform. Prove that nevertheless

$$\lim_{n\to\infty}\int_{-1}^{1} f_n(x)\,dx = \int_{-1}^{1}\lim_{n\to\infty} f_n(x)\,dx.$$

(*b*) Discuss the behavior of the sequence given by $f_n(x) = n^a x^2/(1 + n^a x^2)$ with regard to convergence, uniform convergence, and term-by-term integrability.

***4.** Sketch the curves $y = f_n(x) = x^{2n}/(1 + x^{2n})$, $-2 \le x \le 2$, for $n = 1$, 3, 10. Find $\lim_{n\to\infty} f_n(x)$. Prove that the convergence is nonuniform.

5. Show that $\sum_{\nu=-\infty}^{\infty} e^{-(x-\nu)^2}$ converges uniformly in any fixed interval $a \le x \le b$.

6. Show that in the interval $0 \le x \le \pi$ the following sequences converge, but not uniformly:

(*a*) $\sqrt[n]{\sin x}$, (*b*) $(\sin x)^n$, (*c*) $\sqrt[n]{x \sin x}$,

(*d*) $[f(x)]^n$, where $f(x) = \frac{\sin x}{x}$, $f(0) = 1$,

(*e*) $\sqrt[n]{f(x)}$, where $f(x) = \frac{\sin x}{x}$, $f(0) = 1$.

7. Sketch the curves $x^{2n} + y^{2n} = 1$ for $n = 1, 2, 4$. To what limit do these curves tend as $n \to \infty$?

PROBLEMS

SECTION 7.4, page 529

1. The sequence $f_n(x)$, $n = 1, 2, \ldots$, is defined in the interval $0 \leq x \leq 1$ by the equations

$$f_0(x) \equiv 1, \quad f_n(x) = \sqrt{x f_{n-1}(x)}.$$

(*a*) Prove that in the interval $0 \leq x \leq 1$ the sequence converges to a continuous limit.

*(*b*) Prove that the convergence is uniform.

***2.** Let $f_0(x)$ be continuous in the interval $0 \leq x \leq a$. The sequence of functions $f_n(x)$ is defined by

$$f_n(x) = \int_0^x f_{n-1}(t)\, dt, \qquad n = 1, 2, \ldots.$$

Prove that in any fixed interval $0 \leq x \leq a$ the sequence converges uniformly to 0.

***3.** Let $f_n(x)$, $n = 1, 2, \ldots$, be a sequence of functions with continuous derivatives in the interval $a \leq x \leq b$. Prove that if $f_n(x)$ converges at each point of the interval and the inequality $|f_n'(x)| < M$ (where M is a constant) is satisfied for all values of n and x, then the convergence is uniform.

4. (*a*) Show that the series $\sum_{\nu=1}^{\infty} \frac{1}{\nu^x}$ converges uniformly for $x \geq 1 + \epsilon$ with $\epsilon > 0$ any fixed number.

(*b*) Show that the derived series $-\Sigma \frac{\log \nu}{\nu^x}$ converges uniformly for $x \geq 1 + \epsilon$ with ϵ a fixed positive number.

***5.** Show that the series $\Sigma \frac{\cos \nu x}{\nu^\alpha}$, $\alpha > 0$, converges uniformly with ϵ any small positive value, for $\epsilon \leq x \leq 2\pi - \epsilon$.

6. The series

$$\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots$$

converges uniformly for $\epsilon \leq x \leq N$ when ϵ, N are fixed positive numbers.

7. Find the regions in which the following series are convergent:

(*a*) $\sum x^{\nu!}$, (*b*) $\sum \frac{(\nu!)^2 x^\nu}{(2\nu)!}$,

(*c*) $\sum \frac{a^\nu}{\nu^x}$, $0 < a < 1$, (*d*) $\sum \frac{a^\nu}{\nu^x}$, $a > 1$,

(*e*) $\sum \frac{\log \nu}{\nu^x}$, (*f*) $\sum \frac{x^\nu}{1 - x^\nu}$,

***8.** Prove that if the Dirichlet, series $\Sigma \frac{a_\nu}{\nu^x}$ converges for $x = x_0$, it converges for any $x > x_0$; if it diverges for $x = x_0$, it diverges for any $x < x_0$. Thus there is an "abscissa of convergence" such that for any greater value of x the series converges, and for any smaller value of x the series diverges.

9. If $\Sigma \frac{a_\nu}{\nu^x}$ converges for $x = x_0$, the derived series $-\Sigma \frac{a_\nu \log \nu}{\nu^x}$ converges for any $x > x_0$.

Answers to Exercises

SECTION 7.4, page 529

1. (*a*) $|x|^\nu \leq \frac{1}{2^\nu}$ for $|x| \leq \frac{1}{2}$.

(*b*) $\frac{\sqrt{1 - x^{2n}}}{2^n} \leq \frac{1}{2^n}$.

(*c*) $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$.

(*d*) For $x \leq -1$,

$$\sum_{\nu=1}^{\infty} e^{\nu x} \leq \sum_{\nu=1}^{\infty} e^{-\nu}.$$

2. For $x = 0, f_n(x) = 0$.

For $x \neq 0, f_n(x) = \frac{1}{nx}\left(1 + \frac{1}{n^2x^2}\right)^{-1}$.

Convergence is nonuniform since $f\left(\pm \frac{1}{n}\right) = \pm \frac{1}{2}$.

3. (*a*) $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$

(*b*) $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0 \end{cases}$ $(a > 0)$.

Convergence is nonuniform, and $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\, dx = \int_{-1}^{1} \lim_{n\to\infty} f_n(x)\, dx$.

4. $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } |x| < 1, \\ \frac{1}{2} & \text{if } |x| = 1, \\ 1 & \text{if } |x| > 1. \end{cases}$

5. Observe that

$$\sum_{\nu=-\infty}^{\infty} e^{-(x-\nu)^2} = e^{-x^2} \sum_{\nu=-\infty}^{\infty} e^{-\nu^2+2\nu x},$$

and for $|\nu| > 4(|a| + |b|) = M$, $|2\nu x| < \nu^2/2$. Thus

$$\sum_{|\nu|>M} e^{-(x-\nu)^2} \leq 2 \sum_{\nu=M+1}^{\infty} e^{-\nu^2/2}.$$

6. In each case, although the terms of the sequence are continuous, the limit is not.

7. The square with vertices at $(\pm 1, \pm 1)$.

Solutions and Hints to Problems

SECTION 7.4, page 529

1. (*a*) $f_n(x) = x^{\sigma_n}$, where

$$\sigma_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$
$$= 1 - \frac{1}{2^n}.$$

Consequently, $\lim_{n\to\infty} f_n(x) = x$.

(*b*) $f_n(x) - x = \phi(2^{-n})$, where $\phi(z) = x^{1-z} - x$, and we demand $0 < z < \delta < 1$. From

$$\phi'(z) = -x^{1-z}\log x < -x^{1-\delta}\log x$$

we have for the first term of the Taylor expansion of $\phi(z)$ in the neighborhood of $z = 0$,

$$|\phi(z)| = |R_1(z)| < \delta x^{1-\delta}\log\frac{1}{x}.$$

However, for a fixed $\delta < 1$, the factor $x^{1-\delta}\log(1/x)$ reaches its maximum at $x = e^{-1/(1-\delta)}$. Thus

$$|\phi(z)| < \frac{\delta}{e(1-\delta)} < \epsilon$$

for sufficiently small δ, hence sufficiently large n. Since this bound is independent of x, uniform convergence is proved.

2. From Taylor's theorem with remainder

$$f_n(x) = \frac{1}{n!}\int_0^x (x-t)^{n-1} f_0(t)\,dt$$
$$\leq \frac{a^n}{n!} M,$$

where M is an upper bound for $f(t)$ on the interval $0 \leq t \leq a$.

3. Let $\epsilon > 0$. Divide up the interval by points $x_0 = a, x_1, \ldots, x_m = b$ into subintervals of length less than $\epsilon/3M$. At each point x_i we can choose n_i so large that $|f_n(x_i) - f_m(x_i)| < \epsilon/3$ when n and $m > n_i$. Let N be the greatest of $n_0, n_1, \ldots, n_m$. Then prove by the mean value theorem that in each subinterval the inequality $|f_n(x) - f_m(x)| < \epsilon$ holds when n and $m > N$.

4. (*a*) If $x \geq 1 + \epsilon, \sum_{\nu=1}^{\infty} \frac{1}{\nu^x} \leq \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1+\epsilon}}$. Similarly for (*b*).

5. Prove the theorem analogous to Abel's theorem, p. 515, for uniform convergence.

6. If x lies in the interval $\epsilon \leq x \leq N$, then $y = (x - 1)/(x + 1)$ lies in the interval $-1 + 2\epsilon/(1 + \epsilon) \leq y \leq 1 - 2/(N + 1)$.

7. (*a*) $-1 < x < 1$; (*b*) $-4 < x < 4$; (*c*) any x, (*d*) no x, (*e*) $x > 1$, (*f*) $-1 < x < 1$.

8. If $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^{x_0}}$ converges, write $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^x} = \sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^{x_0}} \cdot \frac{1}{\nu^{x-x_0}}$, and use Problems 7.1, No. 12 or Abel's theorem, p. 515.

If $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^{x_0}}$ diverges, $\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu^x}$ cannot converge for $x < x_0$ by what has just been proved.

9. Write $\Sigma \frac{a_\nu \log \nu}{\nu^x} = \Sigma \frac{a_\nu}{\nu^{x_0}} \cdot \frac{\log \nu}{\nu^{x-x_0}}$.

EXERCISES

SECTION 7.5, page 540

Determine the intervals of convergence of the series $\sum_{n=1}^{\infty} a_n x^n$, where a_n is given by the formula in Exercises 1–20:

1. $\frac{1}{n}$. **2.** n. **3.** $\frac{1}{\sqrt{n}}$.

4. $\sqrt{n}$. **5.** $\frac{1}{n^2}$. **6.** $\frac{n}{n!}$.

7. $\frac{1}{a + n}$. **8.** $\frac{1}{an + b}$. **9.** $\frac{1}{\log (n + 1)}$.

10. $\frac{1}{\log \log 10n}$. **11.** $\frac{1}{\sqrt[n]{n}}$. **12.** a^n.

13. $a^{\sqrt{n}}$. **14.** $a^{\log n}$. **15.** $(\sqrt[n]{n} - 1)^n$.

16. $\dfrac{(n!)^2}{(2n)!}$. **17.** $\dfrac{n + \sqrt{n}}{n^2 + n}$. **18.** $\dfrac{1}{1 + a^n}$, $(a \neq -1)$.

19. $\dfrac{1}{\sqrt{n}} + \dfrac{(-1)^n}{n}$. **20.** $\dfrac{1}{n^{1+1/n}}$.

21. Obtain approximations in series for the following integrals by expanding the integrand in a power series and integrating:

(*a*) $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$, (*b*) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1 - x^4)}}$,

(*c*) $\displaystyle\int_0^1 \frac{\log(1 + x)}{x}\,dx$, (*d*) $\displaystyle\int_5^{10} \frac{dx}{\sqrt{(1 + x^4)}}$.

22. By multiplication of power series obtain the expansions of the following up to the terms in x^4:

(*a*) $e^x \sin x$, (*b*) $[\log(1 + x)]^2$,

(*c*) $\dfrac{\arcsin x}{\sqrt{1 - x}}$, (*d*) $\sin^2 x$,

PROBLEMS

SECTION 7.5, page 540

1. If the interval of convergence of the power series $\Sigma a_n x^n$ is $|x| < \rho$, and that of $\Sigma b_n x^n$ is $|x| < \rho'$, where $\rho < \rho'$, what is the interval of convergence of $\Sigma(a_n + b_n)x^n$?

2. If $a_\nu > 0$ and Σa_ν converges, then

$$\lim_{x\to 1-0} \sum a_\nu x^\nu = \sum a_\nu.$$

3. If $a_\nu > 0$ and Σa_ν diverges,

$$\lim_{x\to 1-0} \sum a_\nu x^\nu = \infty.$$

***4.** Prove Abel's theorem:
If $\Sigma a_\nu X^\nu$ converges, then $\Sigma a_\nu x^\nu$ converges uniformly for $0 \leq x \leq X$.

***5.** If $\Sigma a_\nu X^\nu$ converges, then $\lim\limits_{n\to X-0} \Sigma a_\nu x^\nu = \Sigma a_\nu X^\nu$.

***6.** By multiplication of power series prove that

(*a*) $e^x e^y = e^{x+y}$. (*b*) $\sin 2x = 2 \sin x \cos x$.

7. Using the binomial series, calculate $\sqrt{2}$ to four decimal places.

8. Let a_n be any sequence of real numbers, and S the set of all limit points of the a_n. We denote the least upper bound p of S by $p = \overline{\lim}\, a_n$. Show that the power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < \rho$ and diverges for $|x| > \rho$, where

$$\rho = \frac{1}{\overline{\lim}\, \sqrt[n]{|c_n|}}.$$

Answers to Exercises

SECTION 7.5, page 540

Note on Exercises 1–20: in most of these problems the ratio test is effective, but for Exercises 12–15 the root test is preferable.

1. $|x| < 1$. **2.** $|x| < 1$. **3.** $|x| < 1$.
4. $|x| < 1$. **5.** $|x| < 1$. **6.** $-\infty < x < +\infty$.
7. $|x| < 1$. **8.** $|x| < 1$. **9.** $|x| < 1$.
10. $|x| < 1$. **11.** $|x| < 1$. **12.** $|x| < 1/|a|$
13. $|x| < 1$. **14.** $|x| < 1$. **15.** $-\infty < x < +\infty$.
16. $|x| < 4$. **17.** $|x| < 1$. **18.** $|x| < 1$ or a, whichever is the greater.
19. $|x| < 1$.
20. Note that $1/n^{1+1/n}$ lies between n^{-1} and n^{-2}: $|x| < 1$.

21. (*a*) $1 - \dfrac{1}{3 \cdot 3!} + \dfrac{1}{5 \cdot 5!} - \dfrac{1}{7 \cdot 7!} + - \cdots$.

(*b*) $\dfrac{1}{2} + \dfrac{1}{320} + \dfrac{1}{3 \cdot 2^{12}} + \cdots$.

(*c*) $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dfrac{1}{5^2} - + \cdots$.

(*d*) Put $x = \dfrac{1}{t}$: $\dfrac{1}{10} - \dfrac{2^5 - 1}{10^6} + \dfrac{2^9 - 1}{24 \cdot 10^9} - + \cdots$.

22. (*a*) $x + x^2 + \dfrac{x^3}{3}$. (*b*) $x^2 - x^3 + \dfrac{11x^4}{12}$.

(*c*) $x + \dfrac{x^2}{2} + \dfrac{13x^3}{24} + \dfrac{19x^4}{48}$. (*d*) $x^2 - \dfrac{x^4}{3}$.

Solutions and Hints to Problems

SECTION 7.5, page 540

1. If $\rho < \rho'$ the radius of convergence is ρ, for if $\Sigma(a_n + b_n)x^n$ converges for some x for which $\rho < x < \rho'$, then it follows from the convergence of $\Sigma a_n x^n$ that $\Sigma b_n x^n$ converges also.

If $\rho = \rho'$, then $\Sigma(a_n + b_n)x^n$ may have any radius of convergence $\geq \rho$ whatever; for example, take $a_n = \alpha^n$, where $0 \leq \alpha$, $b_n = -\alpha^n(1 - \beta^n)$, where $0 \leq \beta \leq 1$. The radius of convergence is then $1/\alpha\beta$.

2. Clearly $\sum_{\nu=0}^{\infty} a_\nu x^\nu < \sum_{\nu=0}^{\infty} a_\nu$ for $x < 1$. On the other hand,

$$\lim_{x\to 1} \sum_{\nu=0}^{\infty} a_\nu x^\nu > \lim_{x\to 1} \sum_{\nu=0}^{N} a_\nu x^\nu = \sum_{\nu=0}^{N} a_\nu; \text{ or } \lim_{x\to 1} \sum_{\nu=0}^{\infty} a_\nu x^\nu \geq \sum_{\nu=0}^{\infty} a_\nu.$$

3. As in Problem 2, $\lim_{x\to 1} \sum_{\nu=0}^{\infty} a_\nu x^\nu \geq \sum_{\nu=0}^{\infty} a_\nu$ and hence is ∞.

4. Write $\sum_{\nu=0}^{\infty} a_\nu x^\nu = \sum_{\nu=0}^{\infty} a_\nu X^\nu \left(\frac{x}{X}\right)^\nu$. Then prove the theorem analogous to Problem 12, Section 7.1; for uniform convergence: if $\sum_{\nu=0}^{\infty} a_\nu$ converges, and if the sequence $b_0(x), b_1(x), \ldots, b_n(x), \ldots$ is monotonic for every x and uniformly bounded for every x in a certain interval, then $\sum_{\nu=0}^{\infty} a_\nu b_\nu(x)$ converges uniformly in that interval.

5. This follows from the uniform convergence of the series $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ in the interval $0 \leq x \leq X$. For then $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ is continuous in that interval.

6. (*a*)
$$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{x^\nu y^\mu}{\nu!\,\mu!} = \sum_{k=0}^{\infty} \sum_{\mu=0}^{k} \frac{x^{k-\mu} y^\mu}{(k-\mu)!\,\mu!}$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\mu=0}^{k} \binom{k}{\mu} x^{k-\mu} y^\mu$$
$$= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = e^{x+y}.$$

(*b*)
$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu x^{2\nu+1}}{(2\nu+1)!} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu x^{2\mu}}{(2\mu)!}$$
$$= \sum_{k=0}^{\infty} (-1)^k x^{2k+1} \sum_{\mu=0}^{k} \frac{1}{[2(k-\mu)+1]!\,(2\mu)!}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \sum_{\mu=0}^{k} \binom{2k+1}{2\mu}.$$

$$\text{But } \sum_{\mu=0}^{k}\binom{2k+1}{2\mu} = \sum_{\mu=0}^{k}\binom{2k+1}{2(k-\mu)+1} = \sum_{\nu=0}^{k}\binom{2k+1}{2\nu+1}.$$

$$\text{Since } \sum_{\mu=0}^{k}\left\{\binom{2k+1}{2\mu} + \binom{2k+1}{2\mu+1}\right\} = \sum_{j=0}^{2k+1}\binom{2k+1}{j} = 1,$$

it follows that

$$\sum_{\mu=0}^{k}\binom{2k+1}{2\mu} = \frac{1}{2},$$

which yields the result.

7. The binomial expansion of $(1 + x)^{1/2}$ converges very slowly for $x = 1$: more than 100 million terms are needed to obtain four place accuracy. Instead begin with any rational number p which has a square slightly larger than $\frac{1}{2}$, say $p = \frac{3}{4}$, and take $q = 2p^2 - 1$ and expand

$$\frac{1}{p}(1 + q)^{1/2} = \sqrt{2}.$$

Thus, to the desired accuracy,

$$\begin{aligned}\tfrac{4}{3}(1 + \tfrac{1}{8})^{1/2} &= \tfrac{4}{3}[1 + \tfrac{1}{2}(\tfrac{1}{8}) - \tfrac{1}{8}(\tfrac{1}{8})^2 + \tfrac{1}{16}(\tfrac{1}{8})^3 - \tfrac{5}{128}(\tfrac{1}{8})^4] \\ &= \tfrac{1}{3}[4 + 0.25 - 0.007812 + 0.000482 - 0.000038] \\ &= 1.41421.\end{aligned}$$

8. If $p = \overline{\lim}\, a_n$, then no limit point of a_n is greater than p. We conclude for every positive ϵ that $a_n < p + \epsilon$ for all but a finite number of values of n. Now, if $\rho = 1/\overline{\lim}\, \sqrt[n]{|c_n|}$ then for some N and all $n > N$

$$|c_n| < \left(\frac{1}{\rho} + \epsilon\right)^n.$$

Consequently,

$$\sum |c_n x^n| < \sum \left|\left(\frac{1}{\rho} + \epsilon\right)x\right|^n$$

and the series, on comparison with the geometric series, is seen to converge absolutely for $|x| < \rho/(1 + \epsilon\rho)$.

On the other hand, since every interval $[p - \epsilon, p]$ must contain a limit point of a_n, it follows that $|a_n| > p - \epsilon$ for infinitely many values of n. Thus

$$|c_n| > \left(\frac{1}{\rho} - \epsilon\right)^n$$

for infinitely many values of n, and if $|x| > \rho$, say $|x| > \rho/(1 - \epsilon\rho)$, then $|c_n x^n| > 1$ so that the terms of the power series cannot have the limit 0.

EXERCISES

SECTION 7.6, p. 546

Expand the functions in Exercises 1-6 in power series:

1. a^x.

2. $\dfrac{x + \log(1 - x)}{x^2}$.

3. $\sin^2 x$.

4. $\cos^2 x$.

5. $\sin^6 x$.

6. $\arcsin x^3$.

7. Using the method of undetermined coefficients, find the function $f(x)$ which satisfies the following conditions:

$$(a)\ f(0) = 3, \qquad (b)\ f'(x) = f(x) + x.$$

8. Find the rational functions represented by the following Taylor series:

(*a*) $x + x^2 - x^3 - x^4 + x^5 + x^6 - - + + \cdots$.

(*b*) $1 + 2x - 4x^3 - 5x^4 + 7x^6 + 8x^7 - - + + \cdots$.

9. Show that

(*a*) $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots = 1$.

(*b*) $\dfrac{1}{2} + \dfrac{1 \cdot 3}{2 \cdot 4 \cdot 6} + \dfrac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} + \cdots = \dfrac{1}{2}\sqrt{2}$.

Answers to Exercises

SECTION 7.6, page 546

1. $\displaystyle\sum_{\nu=0}^{\infty} \frac{(\log a)^\nu}{\nu!} x^\nu$.

2. $\displaystyle -\frac{1}{2} - \frac{x}{3} - \frac{x^2}{4} - \cdots - \frac{x^n}{n+2} - \cdots = -\frac{1}{x^2}\sum_{\nu=2}^{\infty} \frac{x^\nu}{\nu}$.

3. Write $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$: $\displaystyle\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}2^{2\nu-1}}{(2\nu)!} x^{2\nu}$.

4. $\displaystyle 1 + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu 2^{2\nu-1}}{(2\nu)!} x^{2\nu}$.

5. $\displaystyle\sum_{\nu=3}^{\infty} \frac{(-1)^{\nu-1}(2x)^{2\nu}}{32(2\nu)!}(15 + 3^{2\nu} - 6 \cdot 2^{2\nu})$.

6. $x^3 + \frac{1}{2}\frac{x^9}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^{15}}{5} + \cdots = x^3 + \sum_{\nu=2}^{\infty} \frac{(x^3)^{2\nu-1}}{2\nu - 1} \cdot \frac{1 \cdot 3 \cdots (2\nu - 3)}{2 \cdot 4 \cdots (2\nu - 2)}$.

8. (*a*) $x(1 + x)/(1 + x^2)$; (*b*) $(1 - x^2)/(1 - x + x^2)^2$.

9. (*a*) The series is equal to $\left.\frac{d}{dx}\left(\frac{e^x - 1}{x}\right)\right|_{x=1}$;

(*b*) The series is equal to $\left.\frac{\sqrt{(1 + x)} - \sqrt{(1 - x)}}{2}\right|_{x=1}$.

EXERCISES

SECTION 7.7, page 551

1. Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. From the expansion $\frac{1}{1 - z} = \sum_{\nu=0}^{\infty} z^\nu$, show that

$$\frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} = \sum_{\nu=0}^{\infty} r^\nu \cos\nu\theta$$

and

$$\frac{r\sin\theta}{1 - 2r\cos\theta + r^2} = \sum_{\nu=0}^{\infty} r^\nu \sin\nu\theta.$$

Answers to Exercises

SECTION 7.7, page 551

$$\mathbf{1.}\ \frac{1}{1 - z} = \sum_{\nu=0}^{\infty} r^\nu(\cos\nu\theta + i\sin\nu\theta)$$
$$= \frac{1 - re^{-i\theta}}{(1 - re^{-i\theta})(1 - re^{i\theta})}$$
$$= \frac{1 - r\cos\theta + i\sin\theta}{1 - 2r\cos\theta + r^2}.$$

PROBLEMS

APPENDIX I, page 555

1. Prove that the power series for $\sqrt{(1 - x)}$ still converges when $x = 1$.

2. Prove that for every positive ϵ there is a polynomial in x which represents $\sqrt{(1 - x)}$ in the interval $0 \le x \le 1$ with an error less than ϵ.

3. By setting $x = 1 - t^2$ in Problem 2, prove that for every positive ϵ there is a polynomial in t which represents $|t|$ in the interval $-1 \leq t \leq 1$ with an error less than ϵ.

4. (*a*) Prove that if $f(x)$ is continuous for $a \leq x \leq b$, then for every $\epsilon > 0$ there exists a polygonal function $\varphi(x)$ (that is, a continuous function whose graph consists of a finite number of rectilinear segments meeting at corners) such that $|f(x) - \varphi(x)| < \epsilon$ for every x in the interval.

(*b*) Prove that every polygonal function $\varphi(x)$ can be represented by a sum $\varphi(x) = a + bx + \Sigma c_i |x - x_i|$, where the x_i's are the abscissae of the corners.

5. Weierstrass' Approximation Theorem. Prove on the basis of the last statement that if $f(x)$ is continuous in $a \leq x \leq b$, then for every positive ϵ there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon$ for all values of x in the interval $a \leq x \leq b$.

Hint: Approximate $f(x)$ by linear combinations of the form $(x - x_r) + |x - x_r|$.

6. Prove that the following infinite products converge:

(*a*) $\displaystyle\prod_{n=1}^{\infty}\left[1 + \left(\frac{1}{2}\right)^{2n}\right]$,

(*b*) $\displaystyle\prod_{n=2}^{\infty}\frac{n^3 - 1}{n^3 + 1}$,

(*c*) $\displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n!}\right)$,

if $|z| < 1$.

7. Prove by the methods of the text that $\displaystyle\prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)$ diverges.

8. Prove the identity

$$\prod_{\nu=1}^{\infty}(1 + x^{2\nu}) = \frac{1}{1 - x}$$

for $|x| < 1$.

***9.** Consider all the natural numbers which represented in the decimal system have no 9 among their digits. Prove that the sum of the reciprocals of these numbers converge.

10. (*a*) Prove that for $s > 1$,

$$1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots = (1 - 2^{1-s})\zeta(s),$$

where $\zeta(s)$ is the zeta function defined on p. 560.

(*b*) Use this identity to show that

$$\lim_{s \to 1+0}(s - 1)\zeta(s) = 1.$$

11. *Integral Test for Convergence.*

(*a*) Let $f(x)$ be positive and decreasing for $x \geq 1$. Prove that the improper integral $\int_1^{\infty} f(x)\,dx$ and the infinite series $\sum_{k=1}^{\infty} f(k)$ either both converge or both diverge.

(*b*) Prove that in either case the limit

$$\lim_{n\to\infty}\left[\int_1^n f(x)\,dx - \sum_{k=1}^{n} f(k)\right]$$

exists.

(*c*) Apply this test to prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log^{\alpha} n}$$

converges for $\alpha > 1$ and diverges for $\alpha \leq 1$.

Solutions and Hints to Problems

APPENDIX I, page 555

1. Break off the series at the nth term; then

$$\frac{1}{2}x + \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 + \cdots + \frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots 2n}x^n < 1 - \sqrt{(1-x)} \leq 1.$$

Put $x = 1$: the partial sums are all ≤ 1.

Alternatively, show for the coefficient of x^n that

$$|a_n| = \frac{1}{2n-1}\prod_1^n\left(1 - \frac{1}{2k}\right) < \frac{1}{(2n-1)n^{1/3}},$$

where the estimate for the product can be verified by mathematical induction. Otherwise, by Stirling's formula (page 504), show that the product is $0(n^{-1/2})$.

2. In Problem 1, show that the greatest error occurs when $x = 1$ and that it can be made less than ϵ.

3. Use $|t| = \sqrt{t^2} = \sqrt{1 - (1 - t^2)}$.

4. (*a*) Divide the interval into a finite number of subintervals by points $a = x_0, x_1, x_2, \ldots, x_n = b$ so close that $|f(x) - f(\bar{x})| < \epsilon$ if x and $\bar{x}$ lie in the same subinterval. Join adjacent points $x = x_i$, $y = f(x_i)$ by straight lines.

(*b*) The expression $-\frac{k}{2}|x - x_i| + \frac{k}{2}|x - x_{i-1}|$ has the slope zero outside the interval (x_{i-1}, x_i). Add suitable terms of this kind.

5. Approximate $f(x)$ by a polygonal function

$$\phi(x) = a + bx + \sum c_i |x - x_i|$$

as in Problem 4 and approximate $\phi(x)$ by a polynomial using Problem 3.

6. Put each product in the form $\Pi(1 + a_n)$. Verify convergence of $\Sigma |a_n|$.

(*a*) $a_n = \dfrac{1}{2^{2n}}$; converges.

(*b*) $a_n = \dfrac{-2}{n^3 + 1}$; converges.

(*c*) $a_n = \dfrac{-z^n}{n}$; converges if $|z| < 1$ by ratio test.

7. For the nth partial product

$$P_n = \prod_{\nu=1}^{n} \left(1 + \frac{1}{\nu}\right) = \prod_{\nu=1}^{n} \left(\frac{\nu + 1}{\nu}\right) = n + 1;$$

diverges.

8. First prove by induction that

$$(1 - x) \prod_{\nu=0}^{n-1} (1 + x^{2^\nu}) = 1 - x^{2^n}.$$

9. The number of integers of n digits which have no 9's in their decimal expansions is $8 \cdot 9^{n-1}$. (the first digit cannot be zero either). Thus the sum of their reciprocals is less than $8 \cdot 9^{n-1}/10^{n-1}$. Sum over n to verify convergence.

10. (*b*) As $s \to 1 + 0$, the series on the left tends to log 2, while

$$\begin{aligned} \lim_{s \to 1+0} (1 - 2^{1-s})\zeta(s) &= \lim_{s \to 1+0} \frac{(s - 1)}{(s - 1)} (1 - 2^{1-s})\zeta(s) \\ &= \log 2 \lim_{s \to 1+0} (s - 1)\zeta(s), \end{aligned}$$

the latter result being found by differentiating 2^{1-s} with respect to s at $s = 1$.

11. (*a*) For x in the interval $n \leq x \leq n + 1$, $f(n) \geq f(x) \geq f(n + 1)$, and so, for any n,

$$\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)\, dx \leq \sum_{k=1}^{n-1} f(k),$$

from which the assertion follows immediately.

(*b*) Let
$$w_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)\, dx.$$

Then we have immediately from (*a*),

$$0 < f(n) < w_n < f(1),$$

and

$$w_{n+1} - w_n = f(n + 1) - \int_n^{n+1} f(x)\,dx < 0,$$

since f is decreasing. Thus the sequence w_n is decreasing and bounded from below (by 0), and so is convergent.

(*c*) See Problem 1, Section 7.2.

8 *Trigonometric Series*

EXERCISES

SECTION 8.1, page 572

1. Sketch the curves $y = \sum_{n=1}^{N} \frac{\sin nx}{n}$ for $N = 3, 5, 6$.

2. Sketch the curves $y = \sum_{n=1}^{N} \frac{\cos nt}{n^4}$ for $N = 3, 6, 8$.

PROBLEMS

SECTION 8.1, page 572

1. The *fundamental period* T of a periodic function f is defined as the greatest lower bound of the positive periods of f. Prove:

(*a*) If $T \neq 0$, then T is a period.

(*b*) If $T \neq 0$, then every other period is an integral multiple of T.

(*c*) If $T = 0$ and if f is continuous at any point, then f is a constant function.

2. Show that if f has incommensurable periods T_1 and T_2, then the fundamental period T is zero. Give an example of a nonconstant function with incommensurable periods.

3. Let f and g have fundamental periods a and b respectively. If a and b are commensurable, say $a/b = q/p$, where p and q are relatively prime integers, then show by example that $f + g$ can have as its fundamental period any value m/n, where $m = aq = bp$ and n is any natural number.

Solutions and Hints to Problems

SECTION 8.1, page 572

1. (*a*) Since T is the infinum of the positive periods, if T were not a period, then for every positive ϵ there must be a period τ satisfying $T < \tau < T + \epsilon$. By the same argument there would exist a period σ such that $T < \sigma < \tau$. Thus $\tau - \sigma < \epsilon$ would be a period; but ϵ can be taken smaller than T since

$T > 0$. There would then be a period smaller than T contradicting the fact that T is the infimum.

(*b*) If $T \neq 0$, then any period τ can be expressed in the form

$$\tau = \nu T + \sigma,$$

where ν is an integer and $0 \leq \sigma < \tau$. If $\sigma \neq 0$, then it is also a period and $\sigma < \tau$ contradicts the fact that τ is the infinum of periods. It follows that $\sigma = 0$.

(*c*) If $T = 0$, then for every positive ϵ there exists a period τ such that $0 < \tau < \epsilon$. Consequently, every interval of length ϵ contains a point $n\tau$ where n is an integer; but $f(n\tau) = f(0)$. Since ϵ can be chosen arbitrarily small, it follows that every interval contains points ξ such that $f(\xi) = f(0)$. From continuity, $f(x) = f(0)$ for all x.

2. Otherwise by Problem 1*b*, $T_1 = n_1 T$ and $T_2 = n_2 T$, where n_1 and n_2 are integers and the two would be commensurable.

No such function could be continuous at any point (cf. Solution of Problem 1*b*). Example

$$f(x) = \begin{cases} 0, & x = m + n\sqrt{2}, \\ 1, & \text{for all other values of } x \end{cases}$$

has the incommensurable periods 1 and $\sqrt{2}$.

3. Since p and q are relatively prime, there exist integers s, t such that $sp + tq = 1$. Set $\mu = 2ns$, $\nu = 2nt$; then $\mu p + \nu q = 2n$. Since $\mu p - n\nu q = 2n - 2n\nu q = 2n(1 - \nu q)$, it follows that $\mu p - \nu q$ is divisible by $\mu p + \nu q$. Now

$$\begin{aligned} \sin \frac{2\pi \nu x}{a} &+ \sin \frac{2\pi \mu\, x}{b} \\ &= \sin \frac{2\pi\nu\, qx}{m} + \sin \frac{2\pi\mu p}{m} \\ &= \left[\sin \pi \frac{x}{m}(\nu q + \mu p)\right]\left[\cos \frac{\pi x}{m}(\nu q - \mu p)\right] \\ &= \left[\sin \frac{2\pi\, nx}{m}\right]\left[\cos \frac{2n\pi}{m}(1 - \nu q)x\right], \end{aligned}$$

where both factors have the period m/n. It is easily established that this is the fundamental period of the product since $|1 - \nu q| > 1$.

EXERCISES

SECTION 8.2, page 576

1. Evaluate the sum $\sin \alpha + \sin 2\alpha + \cdots + \sin n\alpha$.

2. Evaluate the sum $\displaystyle\sum_{\nu=1}^{n} \frac{\sin n\alpha}{2^n}$.

Answers to Exercises

SECTION 8.2, page 576

1. $\sum_{\nu=1}^{n} \sin \nu\alpha$ = imaginary part of $\sum_{\nu=0}^{n} e^{i\nu\alpha}$: $\dfrac{\sin\left(\dfrac{n+1}{2}\right)\alpha \sin \dfrac{n}{2}\alpha}{\sin \frac{1}{2}\alpha}$

2. $\dfrac{2\sin(n+1)\alpha + \sin n\alpha + 2^{n+1}\sin\alpha - 2^n}{2^n(5 + 4\cos\alpha)}$

EXERCISES

SECTION 8.4, page 587

Obtain the Fourier expansions in $[-\pi, \pi]$ of the following functions.

1. $f(x) = \begin{cases} 1, |x| < \dfrac{\pi}{2} \\ 0, |x| > \dfrac{\pi}{2} \end{cases}$

2. $f(x) = \begin{cases} 1, x \geq 0 \\ 1 + x, x \leq 0, \end{cases}$

3. $f(x) = x^3.$

4. $f(x) = \begin{cases} 1 + x, x \leq 0 \\ 1 - x, x \geq 0. \end{cases}$

5. $f(x) = e^{-|x|}.$

6. $f(x) = \sin^2 x.$

7. $f(x) = \begin{cases} 0, x \leq 0 \\ \sin x, x \geq 0. \end{cases}$

8. $e^{ax}.$

9. $(x^2 - \pi^2)^2.$

10. $\sin ax(1 + \cos x).$

11. $f(x) = 1(a \leq x \leq b), f(x) = 0(-\pi < x < a), f(x) = 0(b < x \leq \pi).$

12. The function $f(t)$ is periodic with period 1, and in $0 \leq t < 1$ it is given by $f(t) = t$. Prove that

$$f(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi t}{n}.$$

Answers to Exercises

SECTION 8.4, page 587

1. $\dfrac{1}{2} + \dfrac{2}{\pi} \sum_{\nu=1}^{\infty} \dfrac{(-1)^{\nu+1} \cos (2\nu - 1)x}{2\nu - 1}$.

2. $1 - \dfrac{\pi}{4} + \sum \left[\dfrac{(-1)^{n+1} \sin nx}{n} - \dfrac{(1 + [-1]^{n+1}) \cos nx}{\pi n^2} \right]$,

3. $\sum_{n=1}^{\infty} \dfrac{(-1)^n[12 - (\pi n)^2]}{n^3} \sin nx$.

4. $1 - \dfrac{\pi}{2} + \dfrac{4}{\pi} \sum_{\nu=1}^{\infty} \dfrac{\cos (2\nu - 1)x}{(2\nu - 1)^2}$.

5. $\dfrac{1 - e^{-\pi}}{\pi} + \dfrac{2}{\pi} \sum_{\nu=1}^{\infty} \dfrac{1 - (-1)^{\nu} e^{-\pi}}{1 + \nu^2} \cos \nu x$.

6. $\dfrac{1}{2} - \dfrac{1}{2} \cos 2x$.

7. $\dfrac{1}{\pi} + \dfrac{1}{2} \sin x - \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{\sin 2nx}{4n^2 - 1}$.

8. $\dfrac{e^{a\pi} - e^{-a\pi}}{\pi} \left\{ \dfrac{1}{2a} + \sum_{\nu=1}^{\infty} \dfrac{(-1)^{\nu}}{a^2 + \nu^2} (a \cos \nu x - \nu \sin \nu x) \right\}$.

9. $\dfrac{8}{15} \pi^4 - 48 \sum_{\nu=1}^{\infty} \dfrac{(-1)^{\nu}}{\nu^4} \cos \nu x$.

10. $\dfrac{\sin a\pi}{a} \sum_{\nu=1}^{\infty} (-1)^{\nu} \nu \left[\dfrac{1}{\nu^2 - (a+1)^2} + \dfrac{1}{\nu^2 - (a-1)^2} - \dfrac{2}{\nu^2 - a^2} \right] \sin \nu x$,
if a is not an integer; $\frac{1}{2} \sin (a - 1)x + \sin ax + \frac{1}{2} \sin (a + 1)x$, if a is an integer.

11. $\dfrac{b - a}{2\pi} + \dfrac{1}{\pi} \sum_{\nu=1}^{\infty} \left(\dfrac{\sin \nu b - \sin \nu a}{\nu} \cos \nu x - \dfrac{\cos \nu b - \cos \nu a}{\nu} \sin \nu x \right)$.

12. Apply the transformation $x = -\pi + 2\pi t$ to the function $\phi(x)$ given in Eq. (23*b*), p. 592.

PROBLEMS

SECTION 8.5 page 598

1. Obtain the Fourier series for the function $f(x) = \pi x$ on the interval $0 \le x \le 1$ as a pure sine series and as a pure cosine series.

2. Show how to represent a function defined on an arbitrary bounded interval as a Fourier series.

3. Obtain the infinite product for the cosine from the relation

$$\cos \pi x = \frac{\sin 2\pi x}{2 \sin \pi x}.$$

4. Using the infinite products for the sine and cosine, evaluate

(*a*) $\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{10}{9} \cdot \frac{10}{11} \cdot \frac{14}{13} \cdots$;
(*b*) $2 \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{14}{15} \cdot \frac{16}{15}$.

5. Express the hyperbolic cotangent in terms of partial fractions.

6. Determine the special properties of the coefficients of the Fourier expansions of even and odd functions for which $f(x) = f(\pi - x)$.

Solutions and Hints to Problems

SECTION 8.5, page 598

1. $2\left\{\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right\}$.

$$\frac{\pi}{2} - \frac{4}{\pi}\left\{\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right\}.$$

2. Let $f(x)$ be given for $a \leq x \leq b$. Set $c = \frac{1}{2}(a + b)$, $l = \frac{1}{2}(b - a)$. For $z = \pi(x - c)/l$, expand $f(lz/\pi + c)$ on the interval $-\pi \leq z \leq \pi$.

3. $\cos \pi x = \prod_{\nu=1}^{\infty}\left[1 - \frac{x^2}{(\nu - \frac{1}{2})^2}\right]$.

4. (*a*) $\sqrt{2}$; (*b*) $\sqrt{3}$.

5. $\coth \pi x = \frac{1}{\pi x} + \frac{2x}{\pi}\left(\frac{1}{1^2 + x^2} + \frac{1}{2^2 + x^2} + \frac{1}{3^2 + x^2} + \cdots\right)$.

6. For the sine series all coefficients of even order; for the cosine series all terms of odd order vanish.

PROBLEMS

SECTION 8.6, page 604

1. Investigate the convergence of the Fourier expansion

$$\cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots$$

of the function $-\log 2\left|\sin \frac{x}{2}\right|$.

Solutions and Hints to Problems

SECTION 8.6, page 604

1. The Fourier expansion of

$$f(x) = -\log 2\,|\sin \tfrac{1}{2}x|$$

is not covered by the convergence theorems since there is an infinite discontinuity at $x = 2n\pi$. Nonetheless, the Fourier series converges when $x \neq 2n\pi$ (Problems 7.1, No. 14*c*). To prove that the series converges to $f(x)$ verify that

$$-\log(1 - re^{ix}) = \sum \left(\frac{\cos nx}{n} + i\,\frac{\sin nx}{n}\right) r^n$$

for $r \leq 1$, $x \neq 2n\pi$. Take $r = 1$ and compute $\text{Re}\,\{-\log(1 - z)\}$.

PROBLEMS

SECTION 8.7, page 608

1. Prove Parseval's equation for a piecewise smooth function f where f may have a number of discontinuities.

Solutions and Hints to Problems

SECTION 8.7, page 608

1. In the neighborhood of a discontinuity replace $f(x)$ by a continuous function: if $f(x)$ has a jump at $x = \xi$, replace $f(x)$ in $[\xi - \epsilon, \xi + \epsilon]$ by the straight line joining the points $(\xi - \epsilon, f(\xi - \epsilon))$ and $(\xi + \epsilon, f(\xi + \epsilon))$. Parseval's equation holds for the redefined function and as ϵ approaches zero the terms in the equation approach their values for the original function.

PROBLEMS

APPENDIX II.1, page 619

1. Prove that

$$\phi_n(t) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} Bt_k^{*\,n-k}$$

2. Prove for $n \geq 1$ that

$$\phi_n(t) = (-1)^n \phi_n(1 - t).$$

3. Using the expression for the cotangent in partial fractions, expand $\pi x \cot \pi x$ as a power series in x. By comparing this with the series given on p. 624 show that

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^{2m}} = (-1)^{m-1} \frac{(2\pi)^{2m}}{2 \cdot (2m)!} B_{2m}^*.$$

4. Show that

$$\sum_{\nu=1}^{\infty} \frac{1}{(2\nu - 1)^{2m}} = \frac{(-1)^{m-1}(2^{2m} - 1)\pi^{2m}}{2(2m)!} B_{2m}^*.$$

5. Show that

$$\sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu^{2m}} = \frac{(-1)^m(2^{2m} - 2)\pi^{2m}}{2 \cdot (2m)!} B_{2m}^*.$$

6. Using the infinite products for the sine and cosine, show that

(*a*) $$\log\left(\frac{\sin x}{x}\right) = -\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} 2^{2\nu-1} B_{2\nu}^*}{(2\nu)!\, \nu} x^{2\nu};$$

(*b*) $$\log \cos x = -\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} 2^{2\nu-1}(2^{2\nu} - 1) B_{2\nu}^*}{(2\nu)!\, \nu} x^{2\nu}.$$

7. Prove that

(*a*) $$\int_0^1 \frac{\log x}{1 - x}\, dx = -\frac{\pi^2}{6},$$

(*b*) $$\int_0^1 \frac{\log x}{1 + x}\, dx = -\frac{\pi^2}{12}.$$

Solutions and Hints to Problems

APPENDIX II.1, page 619

1. Since $\phi_n(x)$ is obtained by n-fold integration of $\phi_0(x) = 1$, $\phi_n(x)$ is a polynomial of degree at most n:

$$\phi_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Differentiate $n - k$ times to obtain

$$\phi_n^{(n-k)}(x) = \phi_k(x) = (n - k)!\, a_{n-k} + \cdots$$

and take $x = 0$:

$$\phi_k(0) = b_k = (n - k)!\, a_{n-k}.$$

Thus only those terms appear for which k is even, $k = 2m$. Then

$$a_{n-2m} = \frac{(-1)^{m-1}B_m}{(n-2m)!\,2m!}.$$

Thus

$$\phi_n(t) = \frac{1}{n!}\sum_{k=0}^{\lambda}\binom{n}{2k}B_{2k}^{*}t^{n-2k},$$

where λ is the integral part of $\frac{1}{2}(n+1)$; $\lambda = [\frac{1}{2}(n+1)]$.

2. The result clearly holds for $\phi_0(t) = 1$. If the result holds for $\phi_n(t)$, then

$$\begin{aligned}\phi_{n+1}(t) &= \int_0^t \phi_n(s)\,ds + b_{n+1}\\ &= -\int_1^{1-t}\phi_n(1-q)\,dq + b_{n+1}\\ &= (-1)^{n+1}\int_0^{1-t}\phi_n(q)\,dq + b_{n+1},\\ &= (-1)^{n+1}\phi_{n+1}(1-t)\end{aligned}$$

where we have used

$$\int_0^{1-t}\phi_n(q)\,dq + \int_{1-t}^{1}\phi_n(q)\,dq = 0.$$

3. $\pi x \cot \pi x = 1 - 2x^2\sum_{\nu=1}^{\infty}\frac{1}{\nu^2 - x^2} = 1 - 2x^2\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left(\sum_{m=0}^{\infty}\frac{x^{2m}}{\nu^{2m}}\right)$

$$= 1 - 2\sum_{m=1}^{\infty}\left(\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2m}}\right)x^{2m}.$$

4. Observe that

$$\sum_{\nu=1}^{\infty}\frac{1}{\nu^n} = \sum_{\nu=1}^{\infty}\frac{1}{(2\nu)^n} + \sum_{\nu=1}^{\infty}\frac{1}{(2\nu-1)^n}.$$

Hence

$$\sum_{\nu=1}^{\infty}\frac{1}{(2\nu-1)^n} = \frac{2^n - 1}{2^n}\sum_{\nu=1}^{\infty}\frac{1}{\nu^n}.$$

Now employ the result of Problem 3.

5. From

$$\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu^n} = \sum_{\nu=1}^{\infty}\frac{1}{(2\nu)^n} - \sum_{\nu=1}^{\infty}\frac{1}{(2\nu-1)^n}$$

it follows from the preceding argument that

$$\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu^n} = -\frac{2^n - 2}{2^n}\sum_{\nu=1}^{\infty}\frac{1}{\nu^n}.$$

6. (*a*)
$$\log\left(\frac{\sin x}{x}\right) = \sum_{\nu=1}^{\infty}\log\left(1-\frac{x^2}{\nu^2\pi^2}\right)$$
$$= -\sum_{\nu=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{2m}}{m\nu^{2m}\pi^{2m}}$$
$$= -\sum_{m=1}^{\infty}\left(\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2m}}\right)\frac{x^{2m}}{m\pi^{2m}}$$
$$= -\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}(2\pi)^{2m}}{2\cdot(2m)!}B_{2m}^{*}\right)\frac{x^{2m}}{m\pi^{2m}}.$$

(*b*) Proceed as in part *a*.

7. (*a*)
$$\int_0^1 \frac{\log x}{1-x}\,dx = -\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}$$

(*b*)
$$\int_0^1 \frac{\log x}{1+x}\,dx = \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}}{\nu^2}.$$

9 *Differential Equations for the Simplest Types of Vibration*

EXERCISES

SECTION 9.2, page 636

For the equations in Exercises 1–5 find the general solution, and also the solution for which $x(0) = 0$, $\dot{x}(0) = 1$:

1. $\ddot{x} - 3\dot{x} + 2x = 0$.

2. $\ddot{x} + 3\dot{x} + 2x = 0$.

3. $2\ddot{x} + \dot{x} - x = 0$.

4. $\ddot{x} + 4\dot{x} + 4x = 0$.

5. $4\ddot{x} + 4\dot{x} + x = 0$.

6. Find the general solution, and also the solution for which $x(0) = 0$, $\dot{x}(0) = 1$, of the equation

$$\dot{x} + \ddot{x} + x = 0.$$

Determine the frequency ν, the period T, the amplitude a and the phase δ of the solution.

7. Find the solution of

$$2\ddot{x} + 2\dot{x} + x = 0$$

for which $x(0) = 1$, $\dot{x}(0) = -1$. Calculate the amplitude a, the phase δ, and the frequency ν of the solution.

Answers to Exercises

SECTION 9.2, page 636

1. $c_1e^t + c_2e^{2t}$; $e^{2t} - e^t$.

2. $c_1e^{-t} + c_2e^{-2t}$; $e^{-t} - e^{-2t}$.

3. $c_1e^{\frac{1}{2}t} + c_2e^{-t}$; $\frac{2}{3}(e^{\frac{1}{2}t} - e^{-t})$.

4. $c_1e^{-2t} + c_2te^{-2t}$; te^{-2t}.

5. $c_1e^{-\frac{1}{2}t} + c_2te^{-\frac{1}{2}t}$; $te^{-\frac{1}{2}t}$.

6. $e^{-\frac{1}{2}t}\left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t\right) = ae^{-(\frac{1}{2})t} \cos \frac{\sqrt{3}}{2}(t - \delta);$

$\frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t; \ \nu = \sqrt{\tfrac{3}{2}}, \ T = \frac{4\pi}{\sqrt{3}}, \ a = \frac{2}{\sqrt{3}}, \ \delta = \frac{\pi}{\sqrt{3}}.$

7. $\sqrt{2}e^{-(\frac{1}{2})t} \cos \frac{1}{2}(t + \frac{1}{2}\pi); \ a = \sqrt{2}, \ \delta = -\frac{\pi}{2}, \ \nu = \frac{1}{2}.$

EXERCISES

SECTION 9.3, page 640

For the equations in Exercises 1–5 find the solution satisfying the initial conditions $x(0) = 0$, $\dot{x}(0) = 0$. For Eqs. 1 to 4 state also the amplitude, the phase, and the value of ω for which the amplitude is a maximum.

1. $\ddot{x} + 3\dot{x} + 2x = \cos \omega t.$
2. $\ddot{x} + \dot{x} + x = \cos \omega t.$
3. $\ddot{x} + \dot{x} + x = \sin \omega t.$
4. $2\ddot{x} + 2\dot{x} + x = \cos \omega t.$
5. $\ddot{x} + 4\dot{x} + 4x = \cos \omega t.$
6. Integrate the equation for the electric circuit

$$\mu\dot{I} + \rho I = E,$$

where $E = E_0 \sin \omega t$, and μ, ρ, E_0, ω are constants.

Answers to Exercises

SECTION 9.3, page 640

1. $-\frac{e^{-t}}{1 + \omega^2} + \frac{2e^{-2t}}{4 + \omega^2} + \frac{(2 - \omega^2)\cos \omega t + 3\omega \sin \omega t}{(1 + \omega^2)(4 + \omega^2)};$

$$\alpha = \frac{1}{\sqrt{(1 + \omega^2)(4 + \omega^2)}}, \ \tan \omega\delta = \frac{3\omega}{2 - \omega^2}, \ \omega = 0.$$

2. $$\frac{e^{-\frac{1}{2}t}\left((\omega^2 - 1)\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}(\omega^2 + 1)\sin\frac{\sqrt{3}}{2}t\right)}{1 - \omega^2 + \omega^4} + \frac{(1 - \omega^2)\cos\omega t + \omega\sin\omega t}{1 - \omega^2 + \omega^4};$$

$$\alpha = \frac{1}{\sqrt{(1 - \omega^2 + \omega^4)}}, \tan\omega\delta = \frac{\omega}{1 - \omega^2}, \omega = \frac{1}{\sqrt{2}}.$$

3. $$\frac{e^{-\frac{1}{2}t}\left(\omega\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\omega(2\omega^2 - 1)\sin\frac{\sqrt{3}}{2}t\right)}{1 - \omega^2 + \omega^4} + \frac{(1 - \omega^2)\sin\omega t - \omega\cos\omega t}{1 - \omega^2 + \omega^4};$$

α, $\tan\omega\delta$, ω as in Exercise 2.

4. $$\frac{-e^{-\frac{1}{2}t}((1 - 2\omega^2)\cos\frac{1}{2}t + (1 + 2\omega^2)\sin\frac{1}{2}t)}{1 + 4\omega^4} + \frac{(1 - 2\omega^2)\cos\omega t + 2\omega\sin\omega t}{1 + 4\omega^4};$$

$$\alpha = \frac{1}{\sqrt{(1 + 4\omega^4)}}, \tan\omega\delta = \frac{2\omega}{1 - 2\omega^2}, \omega = 0.$$

5. $$e^{-2t}\left(\frac{\omega^2 - 4}{(\omega^2 + 4)^2} - \frac{2t}{\omega^2 + 4}\right) + \frac{(4 - \omega^2)\cos\omega t + 4\omega\sin\omega t}{(\omega^2 + 4)^2}$$