5s net

Sequences and Series

J A Green

Library of Mathematics
Editor: Walter Ledermann

Sequences and Series J A Green RKP

Infinite series are an important tool in the mathematics of science, and this book is designed to provide students with a self-contained first account of the theory. Although the treatment is elementary (diagrams and illustrative numerical examples are used throughout), the necessary basic concepts, particularly that of the limit of a sequence of numbers, are developed and discussed fully. The second chapter introduces infinite series themselves, convergence, the standard convergence tests (comparison, ratio, integral tests, etc.), absolute convergence, power series, and multiplication of series. The third chapter deals with some further topics, including the numerical evaluation of the sum of a series, integration of power series, Dirichlet's test, and series of complex numbers.

Sequences and Series

SEQUENCES AND SERIES

BY

J. A. GREEN

LONDON: Routledge & Kegan Paul Ltd

NEW YORK: Dover Publications Inc

First published 1958
in Great Britain by
Routledge & Kegan Paul Limited
Broadway House, 68–74 Carter Lane
London, E.C.4
and in the U.S.A. by
Dover Publications Inc.
180 Varick Street
New York, 10014

Reprinted 1959, 1962, 1964, 1966

Library of Congress Catalog Card Number: 66-21238

Printed in Great Britain
by Butler & Tanner Limited
Frome and London

Preface

THIS book is intended primarily for students of science and engineering. Its aims are, first, to present the fundamental mathematical ideas which underlie the notion of a convergent series, and secondly to develop, as far as the small space allows, a body of technique and a familiarity with particular examples sufficient to make the reader feel at home with such applications of infinite series as he is likely to meet in his scientific studies.

I do not believe that these two aims are mutually antagonistic. It is true that a certain sophisticated skill is necessary for the construction of proofs of even quite elementary theorems involving, for example, the definition of the limit of a sequence, and that the acquisition of such skill would take more time than the non-specialist mathematician can spare. But this does not mean that either the fundamental definitions or the statements of the theorems cannot be clearly understood by the non-specialist; on the contrary, it is essential that they should be understood.

Accordingly it has been my policy to lay more emphasis on the illustration of basic ideas by numerical examples, than on formal proofs; the latter have often been relegated to small print, or omitted (such omissions are noted in the text). In particular the idea of convergence itself is directly involved in the practical problem of numerical calculation of the sum of a series, and I have devoted some space to this topic, traditionally neglected in elementary books on series.

It is a great pleasure to acknowledge my debt to my colleagues at Manchester, and especially to Dr. W. Ledermann, for their constructive comments at every stage.

J. A. GREEN

The University,
Manchester

Contents

CHAPTER ONE

Sequences

1. INFINITE SEQUENCES

A *sequence* is any succession of numbers $a_1, a_2, a_3, \ldots$; these numbers are called the *terms* of the sequence. A *finite sequence* $a_1, a_2, \ldots, a_N$ is one which has only a finite number of terms, but we shall be interested mainly in *infinite sequences* whose characteristic property is that they have no last term. For example, the sequence 1, 2, 3, ... of the positive integers is an infinite sequence; so is the sequence $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$ whose nth term is $(-1)^{n+1}/n$.

If we want to describe an infinite sequence, it is obviously impossible to write down all its terms. Instead we must give, as in the last example, a rule for calculating the nth term a_n. This rule for the 'general term' may take the simple form $a_n=f(n)$, where $f(n)$ is some easily evaluated function of n; in the two sequences just mentioned, for instance, we had $a_n=n$, and $a_n=(-1)^{n+1}/n$, respectively.

It is often useful to write (a_n) as an abbreviation for the sequence $a_1, a_2, a_3, \ldots$ whose nth term is a_n.

Example 1. Take any fixed number x. Then (x^n) is the sequence $x, x^2, x^3, \ldots$.

Example 2. $a_n=n^s$ (s any fixed number) is the general term of the sequence $(n^s)=1^s, 2^s, 3^s, \ldots$ For $s=1$ this is just the sequence (n) of positive integers.

Example 3. Take $a_n=1$, for all n. This defines the sequence $(1)=1, 1, 1, \ldots$ all of whose terms are equal to 1. (Notice that it is *not* necessary that all the terms of a sequence should be distinct.)

Example 4. Another example where the terms are not all distinct is the case $x=-1$ of Example 1. This sequence contains only the numbers 1 and -1, alternately, viz. $-1, 1, -1, 1, \ldots$

On the other hand many of the sequences which occur in practice are defined *recursively*; this means that a rule is given, by which the nth term a_n can be computed *when the earlier terms are known.*

Example 5. If $a_{n+1}=\sqrt{2a_n}$, we can calculate a_{n+1} as soon as a_n is known. The value of a_1 must be given to start with, and then any number of terms can be worked out in succession. If $a_1=1$ we have $a_2=\sqrt{2.1}=\sqrt{2}$, $a_3=\sqrt{2\sqrt{2}}$, $a_4=\sqrt{2\sqrt{2\sqrt{2}}}$, etc., or in decimal notation $(a_n)=1, 1{\cdot}414, 1{\cdot}682, 1{\cdot}834, 1{\cdot}915, \ldots$

Example 6. A similar 'recursive formula', but involving *two* previous terms, is $a_{n+2}=0{\cdot}2a_{n+1}-0{\cdot}1a_n$. Here we need to be given a_1 and a_2 to start with, and then the subsequent terms can be found. For instance, taking $a_1=0$, $a_2=1$, we get $a_3=0{\cdot}2a_2-0{\cdot}1a_1=0{\cdot}2$, $a_4=0{\cdot}2a_3-0{\cdot}1a_2=-0{\cdot}06$, $a_5=0{\cdot}2a_4-0{\cdot}1a_3=-0{\cdot}032$, etc.

2. SUCCESSIVE APPROXIMATIONS

One very important way in which sequences make their appearance in practice, is where the numerical solution of some problem is attempted by finding successive approximations. These approximations form a *sequence* whose terms approach the number which is being sought. To take an elementary example, suppose that it is required to find the numerical value of $\sqrt{2}$; that is to say, we are looking for a positive number A with the property that $A^2=2$. By the usual 'square root process' of elementary arithmetic, we learn how to make a sequence of (better and better) approximations to A. The first approximation given by this process, which we might call a_1, is 1. At the next stage we get $a_2=1{\cdot}4$, then $a_3=1{\cdot}41$, $a_4=1{\cdot}414$, and so on. None of the terms $1, 1{\cdot}4, 1{\cdot}41, 1{\cdot}414, \ldots$ of this sequence is equal to A. But they approach or 'tend to' $A=\sqrt{2}$, in the sense

that if we go sufficiently far along the sequence, we get numbers which differ from A by as little as we like. It should also be

	2·	1·414
	1·	
24	1·00	
	96	
281	400	
	281	
2824	11900	
	11296	
	604	

noticed that, in practice, this is all we require. For in any given practical application, in which for some reason the value of $\sqrt{2}$ is required, all that is really necessary is its value correct to a certain number of decimal places, and this we can secure by working out a *finite* number of steps of the square root process. It is this idea of a sequence which 'tends to' a limiting value, which we shall discuss in the next paragraphs.

3. GRAPHICAL REPRESENTATION OF A SEQUENCE

It helps in understanding this notion if we can represent sequences graphically. We shall do this in either of two ways; first we can simply mark the values of $a_1, a_2, a_3, \ldots$ as points on a single axis or scale. It is advisable to write above each point the name of the term to which it is equal (and a given point may correspond to many terms). For example, the sequence $\left(1-\frac{1}{n}\right)=0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ is represented in Fig. 1

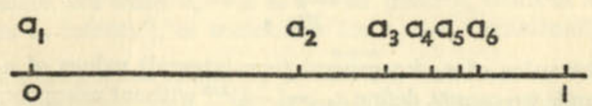

FIG. 1

while the diagram for $((-1)^n)=-1, 1, -1, 1, \ldots$ has only two points, each of which serves for infinitely many terms of the sequence.

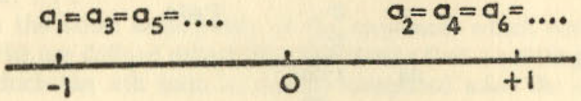

FIG. 2

Although this representation is very compact, it may be preferable to display the sequence more fully. Our second method of depicting a sequence (a_n) is to draw a 'graph', regarding a_n as a function of n. However this graph is not a continuous curve, but consists merely of a succession of isolated points, because a_n is supposed defined only for $n=1, 2, 3, \ldots$ Fig. 3 shows the graph of $((-\frac{1}{2})^n)$, for which $a_1=-\frac{1}{2}$, $a_2=\frac{1}{4}$, $a_3=-\frac{1}{8}$, $a_4=\frac{1}{16}$, etc. For this sequence, it is not possible to give a real meaning

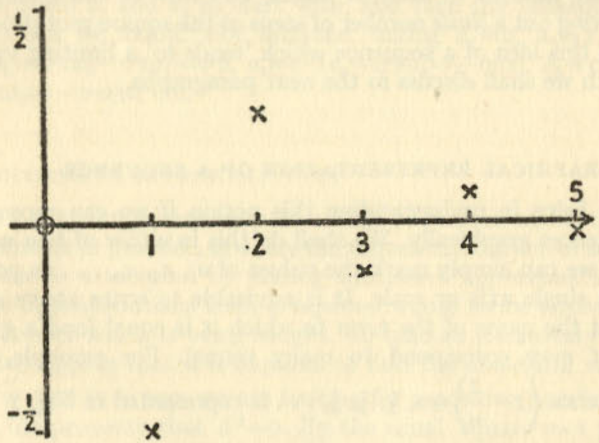

FIG. 3

to the value of a_n for general (non-integral) values of n—for example we cannot define $a_{3/2}=(-\frac{1}{2})^{3/2}$ without using complex numbers. But even in a case, such as $a_n=1-\frac{1}{n}$ (Fig. 4), where

it *would* be possible to 'fill in' the intermediate values, we refrain from doing so, on the grounds that it is only the *integral* values of n which interest us at the moment.

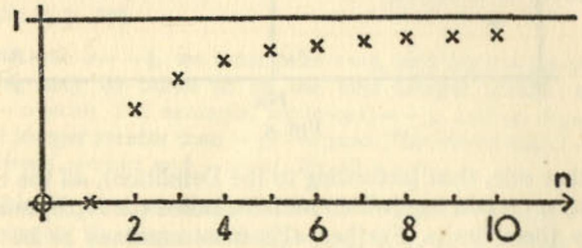

FIG. 4

4. THE LIMIT OF A SEQUENCE

Figures 3 and 4 display very clearly the fact that these sequences are 'tending' to 'limits' or limiting values, as n becomes large. It is clear, for example, that as n increases, $(-\frac{1}{2})^n$ becomes more and more nearly equal to 0, and that $1-\frac{1}{n}$ becomes nearer and nearer to 1. We express this by saying that $(-\frac{1}{2})^n$ *tends to* 0 *as n tends to infinity*, and that $1-\frac{1}{n}$ *tends to* 1 *as n tends to infinity*, respectively. The precise definition of this kind of statement is as follows.

Definition. A sequence (a_n) tends to a limit A as n tends to infinity, if, given any positive number h, however small, we can find an integer N_h such that all the terms a_n of the sequence after the N_hth lie between $A-h$ and $A+h$.

Notation. We write '$a_n \to A$ as $n \to \infty$' (read 'a_n tends to A as n tends to infinity'), or sometimes $\lim_{n\to\infty} a_n = A$. Occasionally the phrase 'as n tends to infinity' is omitted, for shortness; then we should write simply $a_n \to A$, or $\lim a_n = A$. We can think of the terms a_n as 'approximations' to A. If certain 'limits of tolerance' $\pm h$ are allowed, i.e. if we are satisfied when a_n lies within h of A

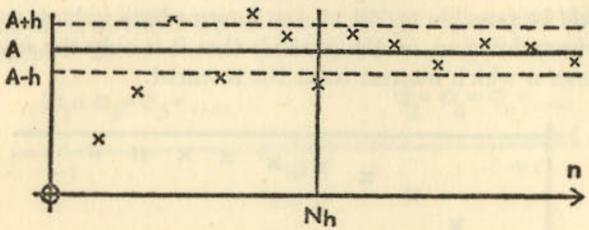

FIG. 5

on either side, then (according to the Definition), *all* the terms a_n after a certain one (which we have called the N_hth) must lie within these limits. Further, this must continue to be true, whatever value h has—e.g. if we set a narrower tolerance $\pm k$, say (with $k<h$), it must again be true that all the terms after a certain point are within the new limits (but of course it would usually be necessary to go farther along the sequence; i.e. the number N_k such that all the terms after the N_kth lie between $A-k$ and $A+k$, would usually be larger than N_h).

Example 1. To prove, directly from this definition, that $a_n = 1 - \frac{1}{n} \to 1$ as $n \to \infty$. The term a_n differs from 1 by $\frac{1}{n}$. Take $h=0{\cdot}001$ as an example; clearly a_n lies between $1-0{\cdot}001$ and $1+0{\cdot}001$, if n has any value >1000. Therefore $N_h=1000$ would satisfy the conditions in the definition, *for the special value* $h=0{\cdot}001$. This is not enough: we must prove that an N_h can be found corresponding to *any* h. This however is quite easy; take N_h=the first integer greater than $1/h$. Then it is clear that a_n lies between $1-h$ and $1+h$, whenever $n>N_h$.

Example 2. We prove next that $x^n \to 0$ *as* $n \to \infty$, *if* $-1<x<1$. (This includes the case $x=-\frac{1}{2}$ depicted in Fig. 3.) Let c be the numerical value of x, disregarding the sign. Then the numerical value of x^n is c^n (for example, the *numerical* value of $(-\frac{1}{2})^n$ is always $(\frac{1}{2})^n$, although $(-\frac{1}{2})^n$ is negative if n is odd). Now $c^n<h$ if $n \log c < \log h$. We must remember that $\log c$ is negative, because $c<1$, and when we divide an inequality by a negative number, its direction is reversed. Thus $n \log c < \log h$ is equivalent to $n > \log h/\log c$. Therefore if N_h is the first integer

greater than $\log h/\log c$, then x^n lies between $0-h$ and $0+h$, for every $n>N_h$. Since such an N_h can be found for *any* positive h, the conditions of the Definition are satisfied (with $a_n=x^n$, and $A=0$). (Another proof, not involving logarithms is given in Example 6, p. 17).

In the case $x=-\frac{1}{2}$, we must take $c=\frac{1}{2}$, and $\log c = -0{\cdot}3010$. Thus N_h may be taken to be the first integer greater than $\log h/(-0{\cdot}3010)$. For example, $\log(0{\cdot}001)=-3$, and so $N_h=10$, the first integer greater than $-3/-0{\cdot}3010$. This shows that $(-\frac{1}{2})^n$ lies between $+0{\cdot}001$ and $-0{\cdot}001$, for all $n>10$.

Example 3. Another useful limit is $n^s \to 0$ *as* $n \to \infty$, *if* $s<0$. For example, the sequence $(n^{-1}) = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ tends to 0 as $n \to \infty$.

Put $s=-k$, so that $k>0$. Then $n^s = \frac{1}{n^k} < h$ if $n^k > \frac{1}{h}$, i.e. if $n > \sqrt[k]{\frac{1}{h}}$. Therefore the conditions of the Definition will be satisfied by taking N_h=first integer greater than $\sqrt[k]{\frac{1}{h}}$.

Example 4. The 'constant sequence' $(a)=a, a, a, a, \ldots$, all of whose terms are equal to a fixed constant a, has a as limit, according to our definition. For whatever value h may have, *all* the terms are between $a-h$ and $a+h$.

Example 5. Any number can be written as a decimal, which may of course be an *infinite* decimal. This is equivalent to saying that the number is the limit of a sequence of finite decimals; for example $\pi = 3{\cdot}14159265\ldots$ is the limit of the sequence $3, 3{\cdot}1, 3{\cdot}14, 3{\cdot}141, \ldots$

A number which can be expressed as the quotient of one integer by another (such as $\frac{1}{3}, \frac{-5}{4}, \frac{6}{1}$, etc.) is called a *rational* number. Not every number is rational; for example it can be proved (this is not easy) that π is *not* rational. However every number is the limit of a sequence of rational numbers. For example, π is the limit of the sequence $3, 3{\cdot}1 = \frac{31}{10}, 3{\cdot}14 = \frac{314}{100}, 3{\cdot}141 = \frac{3141}{1000}, \ldots$, whose terms are all rational. Obviously the same argument could be applied to any number. (It should be remarked that there are always many different sequences of rational numbers tending to any given number.)

5. OTHER TYPES OF SEQUENCE

Not every sequence has a limit, as can be seen by considering the following examples.

Example 1. $(n)=1, 2, 3, 4, \ldots$

Example 2. $(-n^2)=-1, -4, -9, -16, \ldots$

Example 3. $((-2)^n)=-2, 4, -8, 16, \ldots$

Example 4. $((-1)^n)=-1, 1, -1, 1, \ldots$

In none of these cases does a_n tend to a limit A. In (1), for example, the terms become larger and larger without limit; we say that '(a_n) tends to $+\infty$ as $n\rightarrow\infty$'—a statement which is not covered by our definition on p. 5, because we assumed there that A was an ordinary number. The precise definition of the phrase '(a_n) tends to $+\infty$ as $n\rightarrow\infty$', or, as it is usually written, '$a_n\rightarrow+\infty$ as $n\rightarrow\infty$', is as follows:

A sequence (a_n) tends to $+\infty$ as $n\rightarrow\infty$, if, given any number K, however large, we can find an integer N_K such that all the terms a_n after the N_Kth are greater than K.

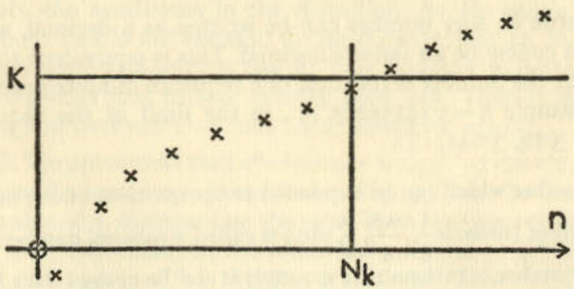

FIG. 6

Example 5. $n^s\rightarrow+\infty$ as $n\rightarrow\infty$, if $s>0$ (cf. Example 3, p. 7).

For given any K (which we may assume is positive), we have $n^s>K$ if $n>\sqrt[s]{K}$. Although $\sqrt[s]{K}$ will not usually be an integer, we just take $N_K=$the first integer greater than $\sqrt[s]{K}$, and the condition of our definition is satisfied.

Example 6. $x^n\rightarrow+\infty$ as $n\rightarrow\infty$, if $x>1$ (cf. Example 2, p. 6).

For $x^n>K$ provided that $n\log x>\log K$, and this time $\log x>0$, because $x>1$, so that we may divide by $\log x$ without reversing the direction of the inequality. Therefore $x^n>K$ if $n>\log K/\log x$, and thus we may take N_K to be the first integer greater than $\log K/\log x$.

By analogy, we say that a sequence (a_n) tends to $-\infty$ as $n\rightarrow\infty$ (written $a_n\rightarrow-\infty$ as $n\rightarrow\infty$), if, given any number L, we can find an integer N_L such that all the terms a_n after the N_Lth are less than L. This is equivalent to saying that $-a_n\rightarrow+\infty$. For instance the sequence $(-n^2)$ in Example 2 tends to $-\infty$.

We often write $\lim a_n=+\infty$, as an abbreviation for '$a_n\rightarrow+\infty$ as $n\rightarrow\infty$', and similarly $\lim a_n=-\infty$. But it should be remembered that '$+\infty$' and '$-\infty$' are not numbers, and cannot be treated as if they were. Some reasons for this caution will appear in § 7.

To complete the classification of sequences, we say that a sequence which does not (i) tend to a limit A (ii) tend to $+\infty$, nor (iii) tend to $-\infty$, is an *oscillating* sequence. Such sequences are, from our point of view, 'irregular', although they comprise a very wide class. Examples (3) and (4) ($a_n=(-2)^n$ and $a_n=(-1)^n$) are both oscillating sequences.

6. RULES FOR CALCULATING LIMITS

If (a_n) and (b_n) are two sequences, and if $a_n\rightarrow A$ as $n\rightarrow\infty$, while $b_n\rightarrow B$, then the 'sum' sequence $(a_n+b_n)=a_1+b_1, a_2+b_2, \ldots$ tends to $A+B$.

Naturally this statement requires rigorous proof, using the definition of § 4. We shall not, for reasons of space, give this proof, but it is really just a precise formulation of the following idea: a_n can be made as near A as we like, by making n large enough; similarly b_n can be made as near B as we like. When a_n is very near to A, and also b_n is very near to B, then a_n+b_n must be very near $A+B$. That means that we can make a_n+b_n as near $A+B$ as we like, by making n large enough.

In the same way we have rules for differences and products:

Rule 1. *If $a_n \to A$ and $b_n \to B$ as $n \to \infty$, then $a_n+b_n \to A+B$, $a_n-b_n \to A-B$, and $a_nb_n \to AB$. Furthermore, if c is any constant, then $ca_n \to cA$ (this is the special case $b_n=c$ of $a_nb_n \to AB$).*

Example 1. Find $\lim\,(1+(\frac{1}{3})^n)$. Take $a_n=1$ (this defines a *constant* sequence, see Example 4, p. 7) and $b_n=(\frac{1}{3})^n$. We know that $a_n \to 1$, and $b_n \to 0$ (Example 2, p. 6); therefore $a_n+b_n =1+(\frac{1}{3})^n \to 1+0=1$.

Example 2. $\left(1+\frac{1}{n}\right)\left(2-\frac{1}{n^2}\right) \to (1+0)(2-0)=2$. (We use here the fact that $\frac{1}{n}=n^{-1}$ and $\frac{1}{n^2}=n^{-2}$ both $\to 0$ (Example 3, p. 7)).

Rule 2. *If $a_n \to A$, and $b_n \to B$, as $n \to \infty$, and if each term $b_n \neq 0$, and also $B \neq 0$, then $a_n/b_n \to A/B$.*

We omit the proof of this rule. The conditions $b_n \neq 0$ and $B \neq 0$ are obviously essential, if the expressions a_n/b_n and A/B are to have any meaning.

Rule 3. *Let (a_n), (b_n) be two sequences.*

(*i*) *If $a_n \to +\infty$, or if $a_n \to -\infty$, then $1/a_n \to 0$.*
(*ii*) *If $a_n \to +\infty$, and $b_n \to$ a finite limit B, then $a_n+b_n \to +\infty$.*
(*iii*) *If $a_n \to -\infty$, and $b_n \to$ a finite limit B, then $a_n+b_n \to -\infty$.*
(*iv*) *If $a_n \to +\infty$, and $b_n \to$ a positive limit B, then*
$$a_nb_n \to +\infty.$$
(*v*) *If $a_n \to +\infty$, and $b_n \to$ a negative limit B, then*
$$a_nb_n \to -\infty.$$

We shall not give formal proofs of these facts, but the reader should try in each case to understand the general principles involved. For example (i) if $a_n \to +\infty$, it means that a_n is very large, for large values of n; hence $1/a_n$ is very near 0. In (ii), the term b_n

is near to B, for large n, while a_n is very large. Even if B is a large negative number, the sum a_n+b_n can be made positive and indeed as large as we like, by making n large enough. Therefore
$$a_n+b_n \to +\infty.$$

Example 3. $\left(n+\frac{1}{n}\right)=2,\ 2\frac{1}{2},\ 3\frac{1}{3},\ 4\frac{1}{4}, \ldots$ tends to $+\infty$. For $a_n=n \to +\infty$, while $b_n=\frac{1}{n} \to 0$.

Example 4. $4-n+(\frac{1}{2})^n \to -\infty$. For this is the sum of a term $-n$, which tends to $-\infty$, with a term $4+(\frac{1}{2})^n$, which tends to the finite limit 4.

Example 5. $(n+1)/(n^2+1)=n\left(1+\frac{1}{n}\right)\Big/ n^2\left(1+\frac{1}{n^2}\right)$
$$=\left(\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\Big/\left(1+\frac{1}{n^2}\right) \to 0\times 1/1=0, \text{ as } n \to \infty.$$

Example 6. $(n^2+1)/(3n^2+n+1)$
$$=\tfrac{1}{3}\left(1+\frac{1}{n^2}\right)\Big/\left(1+\frac{1}{3n}+\frac{1}{3n^2}\right) \to \tfrac{1}{3}.1/1=\tfrac{1}{3}, \text{ as } n \to \infty.$$

Example 7. $a_n=(1-n^2)/(1+2n)$. Take a factor $(-n^2)$ out of the numerator, and a factor $2n$ out of the denominator; we get $a_n=\left(-\frac{n}{2}\right)\left(1-\frac{1}{n^2}\right)\Big/\left(1+\frac{1}{2n}\right)$. The factors in the two last brackets both tend to 1 as $n \to \infty$. Therefore, for large n, a_n will be very nearly equal to $\left(-\frac{n}{2}\right)$ (in the sense that $a_n/\left(-\frac{n}{2}\right)$ will be very nearly 1). Therefore, since $-\frac{n}{2}$ clearly $\to -\infty$, $a_n \to -\infty$ as well.

Example 8. *Rational functions of n.* A *polynomial* in n is a function of the form $c_hn^h+c_{h-1}n^{h-1}+\ldots+c_1n+c_0$, where $c_0, c_1, \ldots, c_h$ are constants. If $c_h \neq 0$, h is called the *degree* of the polynomial, and c_hn^h is called the *leading term*. A *rational function* is the quotient of one polynomial by another; e.g. the functions in Examples 5, 6, 7 are all rational functions of n.

Let (a_n) be a sequence whose nth term is a rational function of n, say $a_n = \dfrac{c_h n^h + c_{h-1} n^{h-1} + \ldots + c_0}{d_k n^k + d_{k-1} n^{k-1} + \ldots + d_0}$ $(c_h, d_k \neq 0)$. We can write this

$$a_n = \frac{c_h n^h}{d_k n^k} \cdot \left\{1 + \frac{c_{h-1}}{c_h}\frac{1}{n} + \ldots + \frac{c_0}{c_h}\frac{1}{n^h}\right\} \Big/ \left\{1 + \frac{d_{k-1}}{d_k}\frac{1}{n} + \ldots + \frac{d_0}{d_k}\frac{1}{n^k}\right\}.$$

The terms in the brackets both tend to 1 as $n \to \infty$. Thus for large n, a_n *behaves like the quotient of its leading terms*, in the sense that the ratio of a_n to this quotient $c_h n^h / d_k n^k$ tends to 1 as $n \to \infty$. Thus for example, if $h < k$, then $c_h n^h / d_k n^k = (c_h/d_k)\dfrac{1}{n^{k-h}} \to 0$ as $n \to \infty$, and therefore a_n also tends to 0. If $h = k$, then $a_n \to c_h/d_k$, and if $h > k$, then $a_n \to +\infty$ or to $-\infty$, according as c_h/d_k is positive or negative.

It is sometimes useful to notice the following rule.

Rule 4. *If $a_n \to A$ and $b_n \to B$ as $n \to \infty$, and if $a_n < b_n$, for all n, then $A \leq B$.*

This is almost obvious. But it should be remarked that if $a_n < b_n$ for all n, it is not necessarily true that $A < B$—all we can be sure of is that $A \leq B$. For example, $1 - \dfrac{1}{n} < 1$ for all n, but $\lim \left(1 - \dfrac{1}{n}\right) = 1$.

7. SOME DANGEROUS EXPRESSIONS

It is as well to remember that Rules 1 and 2 are not intended to apply to sequences which tend to $+\infty$ or $-\infty$. Nor should it be expected that expressions such as $\infty - \infty$, $0/0$, ∞/∞ have any definite meaning. In general, we must accept the fact that '∞' *cannot be treated like an ordinary number, and that division by* 0 *is impossible.* The following examples make the position clear.

Example 1. If we take two sequences (a_n) and (b_n), which both tend to $+\infty$, we might expect the quotient a_n/b_n to have '∞/∞' as limit. Consider, however, the following three examples (i) $a_n = n^2$, $b_n = n$, (ii) $a_n = n$, $b_n = n$, (iii) $a_n = n$, $b_n = n^2$. In each case, a_n and b_n both tend to $+\infty$. But in (i) $a_n/b_n = n \to +\infty$, in (ii) $a_n/b_n = 1$, which $\to 1$ as $n \to \infty$, and in (iii) $a_n/b_n = 1/n \to 0$. What value are we to give to '∞/∞'? Our answer is that we do not give any meaning at all to this expression; we simply avoid it. (It might be remarked that we could construct examples like the three we have just mentioned, in which a_n/b_n has any given positive number we like as limit, or again, in which a_n/b_n has no limit at all, but oscillates.)

Example 2. The situation with '$0/0$' is the same. Consider the following three examples: (i) $a_n = 1/n$, $b_n = 1/n^2$ (ii) $a_n = 1/n$, $b_n = -2/n$, (iii) $a_n = 1/n^2$, $b_n = 1/n$. In each case, $a_n \to 0$ and also $b_n \to 0$, as $n \to \infty$. But for $\lim (a_n/b_n)$ we find (i) $+\infty$, (ii) $-\frac{1}{2}$, (iii) 0. Thus '$0/0$' cannot be given any meaning; it is another expression which has to be avoided.

Example 3. '$\infty - \infty$.' Take (i) $a_n = n$, $b_n = n^2$, (ii) $a_n = n + 4$ $b_n = n$, (iii) $a_n = (-\frac{1}{2})^n + n$, $b_n = n$. In each case a_n and $b_n \to +\infty$ as $n \to \infty$. Thus if '$\infty - \infty$' has any definite meaning, we should expect to find it by considering the limit of $a_n - b_n$ in these examples. However in (i), $\lim (a_n - b_n) = \lim n(1 - n) = -\infty$, in (ii) $\lim (a_n - b_n) = 4$, while in (iii), $a_n = b_n = (-\frac{1}{2})^n \to 0$.

The reader should not imagine that, for example, $a_n = \dfrac{n+1}{n^2+1}$ has no limit because it 'becomes $\dfrac{\infty + 1}{\infty^2 + 1}$' on 'substituting $n = \infty$'. What the discussion above does show is that *we cannot calculate limits by 'substituting $n = \infty$'*. We must work instead by applying Rules 1, 2 and 3 to reduce the given expression to a combination of functions whose limits are either known already, or can be found from first principles, i.e. from the definitions given in §§ 4 and 5.

8. SUBSEQUENCES

If $(a_n) = a_1, a_2, a_3, \ldots$ is a sequence, then any infinite succession of its terms, picked out in any way (but preserving the original order) is called a *subsequence* of (a_n).

Example 1. $a_2, a_4, a_6, a_8, \ldots$ is a subsequence of (a_n). Its nth term is a_{2n}.

Example 2. $a_1, a_4, a_9, a_{16}, \ldots$ is the subsequence (a_{n^2}) of (a_n).

Example 3. An important type of subsequence is that obtained by removing a finite number of terms from the beginning of (a_n), but leaving the rest of the sequence unchanged. For example $a_7, a_8, a_9, \ldots$ is the subsequence obtained by omitting the first six terms of (a_n). The nth term of this subsequence is a_{6+n}.

The behaviour of a subsequence is very easy to describe, in the cases where (a_n) tends to a (finite) limit,[1] or to $+\infty$ or $-\infty$. We have the following rule.

Rule 5. *If $a_n \to A$ as $n \to \infty$, then any subsequence of (a_n) also tends to A. Similarly, if $a_n \to +\infty(-\infty)$ then any subsequence of (a_n) also tends to $+\infty(-\infty)$.*

For example, if $a_n \to A$, it means that, given h, we can be sure that all the terms a_n after some point are within h of A. But if this is so, then all the terms of the subsequence (which are just *some* of the terms of (a_n)) are within h of A, after this point.

Example 4. The sequence $(x^{n^2+n}) = x^2, x^6, x^{12}, x^{20}, \ldots$ is a subsequence of (x^n). Therefore it tends to 0, if $-1 < x < 1$, since $x^n \to 0$ in this case.

Example 5. (see Example 3). If $(a_n) = a_1, a_2, a_3, \ldots$ tends to A as $n \to \infty$, then any sequence obtained by removing a finite number of terms from the beginning of (a_n) tends to the same limit. For example, the sequence $(a_{n+1}) = a_2, a_3, a_4, \ldots \to A$.

It is also clear that the introduction of a finite number of new terms at the beginning does not alter the limit of a sequence. For example, we know that $(1/n) = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ tends to 0. Then the sequence $42, -4, 5011, 8, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$, obtained by putting in four extra terms at the start, still has limit 0. In

[1] When we say that (a_n) tends to a *finite* limit, this means that $a_n \to$ some number A, as defined on p. 5. The word 'finite' is introduced to distinguish this from '$a_n \to +\infty$' and '$a_n \to -\infty$', but it should be remembered that these have quite different definitions from '$a_n \to A$' (p. 8).

fact, the behaviour of a sequence, of any type, is unaffected by removing or inserting or altering *a finite number of terms.*

Example 6. An interesting exercise in the application of the principles of this section is the following. Let $a_n = \sin nx$, x being a fixed angle which is not an integral multiple of π radians (if x is a multiple of π, then $a_n = 0$ for all n). We shall prove that (a_n) *does not tend to a limit.* For suppose, to the contrary, that $a_n \to A$ as $n \to \infty$. Let $b_n = \cos nx$. Put $\alpha = (n+1)x$, $\beta = x$ in the trigonometric identity $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$; this gives $a_{n+2} - a_n = 2b_{n+1}\sin x$, hence $b_{n+1} = (a_{n+2} - a_n)/2\sin x$. Now if $a_n \to A$, it follows (Example 5) that $a_{n+2} \to A$ as well. Therefore $b_{n+1} \to (A-A)/2\sin x = 0$, as $n \to \infty$. Therefore $b_n \to 0$ also (for $(b_n) = b_1, b_2, b_3, \ldots$ is obtained by simply adding one extra term b_1 to the beginning of $(b_{n+1}) = b_2, b_3, \ldots$). Consequently the subsequence $(b_{2n}) = b_2, b_4, b_6, \ldots$ has the same limit 0. But $b_{2n} = \cos 2nx = 2\cos^2 nx - 1 = 2b_n^2 - 1$. Take the limits of both sides, and we get a contradiction $0 = 2\cdot 0^2 - 1$. This can only mean that our original assumption that (a_n) tends to a limit A, is false. (We have proved only that (a_n) does not tend to a *finite* limit. However it is trivial that (a_n) does not tend to $+\infty$ or $-\infty$ either, since $-1 \leq \sin nx \leq 1$ for all n.)

9. MONOTONE SEQUENCES AND BOUNDED SEQUENCES

(a_n) is called an *increasing* sequence if $a_n \leq a_{n+1}$ for all n; that is, if $a_1 \leq a_2 \leq a_3 \leq \ldots$ Similarly, a *decreasing* sequence (a_n) is one for which $a_n \geq a_{n+1}$ for all n; thus $a_1 \geq a_2 \geq a_3 \geq \ldots$ A sequence which is either increasing or decreasing is sometimes called a *monotone* sequence.

Example 1. (n), $\left(1 - \frac{1}{n}\right)$ are increasing sequences.

Example 2. If (a_n) is increasing, then $(-a_n)$ is decreasing, and *vice versa.*

A sequence (a_n) is *bounded above* if there is a number H such that $a_n \leq H$ for all n. It is *bounded below* if there is a number G such that $a_n \geq G$ for all n.

Example 3. (n) is bounded below, but not above. (Every term is $\geqslant 0$).

Example 4. $\left(1-\frac{1}{n}\right)=0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ is bounded above and below. For $0 \leqslant 1-\frac{1}{n} \leqslant 1$, for all n.

We come now to an important general principle.

Fundamental Theorem on Monotone Sequences. *(i) If an increasing sequence (a_n) is bounded above, then it must tend to a finite limit. (ii) If the increasing sequence (a_n) is not bounded above, then $a_n \to +\infty$ as $n \to \infty$.*

Similar results hold for a decreasing sequence (b_n): if (b_n) is bounded below, then $b_n \to$ a finite limit; if (b_n) is not bounded below, then $b_n \to -\infty$. (This means, it is impossible for the increasing sequence (a_n) to oscillate or tend to $-\infty$, and it is impossible for (b_n) to oscillate or tend to $+\infty$.)

Let us represent the terms a_n by points on a line. Each a_n is to the right of all the preceding terms, because (a_n) is an increasing sequence.

Case (ii) is illustrated by Fig. 7. For any given number K we must have $a_n > K$ for *some* n (otherwise $a_n < K$ for all n, and that would mean that (a_n) was bounded above). Then, because the

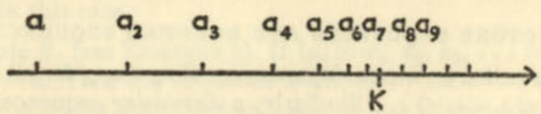

FIG. 7

sequence is increasing, *all* the subsequent terms are $>K$. This shows that $a_n \to \infty$.

In case (i), there is some number H such that $a_n < H$ for all n (Fig. 8). A rigorous proof that a_n tends to a limit is beyond the scope of this book, but we can see how this comes about, as follows.

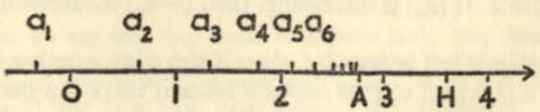

FIG. 8

All the points a_n are to the left of H. There must be some *greatest* integer m, which has points of the sequence to its right. In Fig. 8, $m=2$, and all the a_n after a_4 are between 2 and 3. Similarly, there must be a greatest one among the numbers 2·0, 2·1, 2·2, . . . , 2·9,

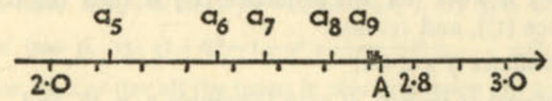

FIG. 9

which has any points a_n to its right. In Fig. 9, which is an 'enlargement' of the relevant part of Fig. 8, this 'greatest number' is 2·7. All the a_n after a_8 lie between 2·7 and 2·8. In the same way, we should find that all the a_n after a certain one lie in a particular interval of length 0·01; between 2·71 and 2·72, perhaps. We can go on with this process as long as we like. The result is a certain infinite decimal A (which in our case starts off 2·71 . . .), and A is the limit of the sequence. For all the a_n after a certain point are within as small a distance as we like of A.

Example 5. $a_n = 1-\frac{1}{n}$ is increasing, and bounded above. As we have already seen (Example 1, p. 6), a_n does tend to a limit, namely 1.

Example 6. Let x be a number between 0 and 1. Then (x^n) is a decreasing sequence (since $x^{n+1}=x.x^n<x^n$, for all n), and it is bounded below, because $x^n>0$ for all n. Therefore it must tend to a limit, X, say. Now $(x^{n+1})=x^2, x^3, \ldots$ must have the same limit X (Example 5, p. 14). But (x^{n+1}) can also be obtained by multiplying every term of (x^n) by the constant x. It follows by Rule 1 (p. 10) that its limit is xX. Hence $xX=X$, from which $(x-1)X=0$, and, since we know that $x-1\neq 0$, it follows that $X=0$. This is of course the result which we had already found in § 4 (Example 2, p. 6).

10. THE FUNCTIONS x^n, n^s AND $n^s x^n$

We first collect for reference some of the results obtained in this chapter.

(i) *$x^n \to 0$ as $n \to \infty$, if $-1<x<1$* (Example 2, p. 6),
$x^n \to +\infty$ as $n \to \infty$, if $x>1$ (Example 6, p. 9).

To complete the list, we remark that

$x^n \to 1$ if $x=1$ (for the sequence (x^n) is then the constant sequence (1)), and

x^n oscillates if $x \leqslant -1$.

(ii) *$n^s \to 0$ as $n \to \infty$, if $s<0$* (Example 3, p. 7), and
$n^s \to +\infty$ as $n \to \infty$, if $s>0$ (Example 5, p. 8).

There is also the trivial case $s=0$, which gives the constant sequence $(n^0)=(1)$; thus $n^0 \to 1$.

Functions of the type $n^s x^n$. Consider the sequence whose nth term is $a_n = n^3(\frac{1}{2})^n = n^3/2^n$. This can be regarded as the quotient of the two functions n^3 and 2^n, which both tend to $+\infty$ as $n \to \infty$. As we saw in § 7, the expression '∞/∞' gives us no indication of what the limit of (a_n) might be, or indeed of whether it has a limit at all. On the other hand, it is always worth while to consider some numerical values—these often give an indication of the behaviour of a sequence, although any guesses arrived at in this way have to be verified by a rigorous argument based on our definitions.

n	1	2	4	10	20	50
n^3	1	8	64	1000	8×10^3	$1{\cdot}25 \times 10^5$
2^n	2	4	16	1024	$1{\cdot}05 \times 10^6$	$1{\cdot}1 \times 10^{15}$
$n^3/2^n$	0·5	2	4	1·0	$7{\cdot}6 \times 10^{-3}$	$1{\cdot}1 \times 10^{-10}$

These rough values suggest that, starting with $n=10$, the values of 2^n soon become very much larger than those of n^3 (although they are smaller for some initial values of n) and that in fact $n^3/2^n \to 0$ as $n \to \infty$. This is confirmed by the following general result: *if $y>1$, then $n^s/y^n \to 0$ as $n \to \infty$, for any value of s.*

First, if $s<0$, then both n^s and $(1/y)^n \to 0$, so that their product n^s/y^n tends to 0. If $s=0$, then $n^s/y^n = (1/y)^n$, and again this tends

to zero. We may assume now that $s>0$. Because $y>1$, the $2s$-th root of y is also >1, and so we can write it in the form $\sqrt[2s]{y} = 1+t$, where t is a positive number. If $a_n = n^s/y^n$, then, taking $2s$-th roots of each side, $\sqrt[2s]{a_n} = \sqrt{n}/(\sqrt[2s]{y})^n = \sqrt{n}/(1+t)^n$. By the binomial theorem (see p. 25) $(1+t)^n = 1 + nt + \dfrac{n(n-1)}{2}t^2 + \ldots + t^n$, which is greater than nt (for all the terms in this expression are *positive*). Therefore $\sqrt[2s]{a_n} < \sqrt{n}/nt = 1/t\sqrt{n}$. Take $2s$-th powers of both sides; this gives $a_n < 1/(t^2 n)^s$. We can make $1/(t^2n)^s$ as small as we like by making n large enough. Thus a_n (which is always >0) can be made as near 0 as we like by making n large enough, and this shows that $a_n \to 0$.

This can be expressed in a slightly different way by saying: $n^s x^n \to 0$ as $n \to \infty$, if $0<x<1$. For if $0<x<1$, then $y=1/x>1$, and we can apply the previous case, writing $n^s x^n = n^s/(1/x)^n = n^s/y^n$. Furthermore, if $-1<x<0$, then the *numerical* value of $n^s x^n$ (i.e. apart from sign) has the same as that of $n^s c^n$, where c is the numerical value of x. Thus $n^s x^n \to 0$ as $n \to \infty$ in this case also. To summarize:

(iii) *$n^s x^n \to 0$ as $n \to \infty$, for any s, and any x such that $-1<x<1$.*

Example 1. The result $n^s/y^n \to 0$ (in the case $s>0$ and $y>1$) can be interpreted as meaning that, although n^s and y^n both tend to $+\infty$, y^n does so 'faster' than n^s. This is true even if s is very large and y is only slightly larger than 1. For example, one would at first sight imagine that n^{1000} was very much larger than $(1{\cdot}0001)^n$, and of course this is so *for a finite number of values of n*. However eventually $(1{\cdot}0001)^n$ is much larger than n^{1000}, and the quotient $n^{1000}/(1{\cdot}0001)^n \to 0$ as $n \to \infty$.

Example 2. Find $\lim\,(n^5 - (1{\cdot}5)^n)$. We can write $a_n = n^5 - (1{\cdot}5)^n = (1{\cdot}5)^n\{n^5/(1{\cdot}5)^n - 1\}$. The expression in the brackets {} tends to $0-1=-1$, while $(1{\cdot}5)^n \to +\infty$. Therefore $a_n \to -\infty$ (Rule 3 (v), p. 10). (In a less precise way, one could say that $(1{\cdot}5)^n$ tends to $+\infty$ faster than n^5, as in Example 1, and that therefore for large n the term n^5 is negligible in comparison with $-(1{\cdot}5)^n$.)

11. SOLUTION OF EQUATIONS BY ITERATION

There are several 'iterative methods' for solving equations; that is to say, methods whereby the solution is obtained as the limit of a sequence of successive approximations. We shall describe here the simplest of these methods, for details of others the reader is referred to books on numerical methods (e.g. Whittaker and Robinson, *Calculus of Observations*, Blackie & Co.).

Suppose that $f(x)$ is any continuous[1] function of x. Take any number x_1, and define the sequence (x_n) recursively by the rule $x_{n+1}=f(x_n)$, for every n. Thus $x_2=f(x_1)$, $x_3=f(x_2)$, etc. Suppose that x_n tends to a limit a, as $n\to\infty$. Then $f(x_n)$ must tend to $f(a)$,[2] but on the other hand, $f(x_n)=x_{n+1}$, and (x_{n+1}) has the same limit as (x_n) (Example 5, p. 14). Therefore $a=f(a)$. In other words, *if x_n tends to a limit, then this limit is a root of the equation $x=f(x)$.*

The sequence (x_n) does depend on the starting value x_1. If we take a different x_1, the resulting sequence might tend to a different limit b (which would, by the same argument as before, be another root of $x=f(x)$), or it might have no limit at all. It is necessary to devise a working procedure which can guide us in choosing a 'good' x_1.

It will usually be possible to fix the positions of the roots approximately (e.g. by drawing a rough graph). Let $f'(x)$ denote the derivative of $f(x)$. We have then the following

Criterion. *Suppose (i) that there is known to be a root a of $x=f(x)$ somewhere between $x=\alpha$ and $x=\beta$, and (ii) that there exists a positive number $k<1$ such that $0<f'(x)<k$, for all x between α and β. Then the sequence (x_n) defined by $x_1=\alpha$, $x_{n+1}=f(x_n)$, tends to a as $n\to\infty$. Furthermore, the difference between the nth term x_n and a is less than $k^{n-1}|\beta-\alpha|$,[3] for any n.*

[1] See e.g. P. J. Hilton, *Differential Calculus*, in this series.

[2] We are using here the fact that $f(x)$ is a *continuous* function. The characteristic property of a continuous function can be expressed in just this way: that if a sequence (x_n) tends to a limit a, then the sequence $(f(x_n))$ of function values must tend to $f(a)$. All the ordinary functions of elementary mathematics are continuous.

[3] We remind the reader that $|\beta-\alpha|$ is the numerical value of the

Example 1. Let $f(x)=\sqrt{2x}$. The equation $x=f(x)$ can be solved easily in the ordinary way; its roots are $x=0$ and $x=2$. The derivative is $f'(x)=\frac{1}{2}.2.x^{-\frac{1}{2}}=1/\sqrt{2x}$, so that $0<f'(x)<1/\sqrt{2}$, for all x between 1 and 3, say. We can apply the Criterion (with $a=2$, $\alpha=1$, $\beta=3$, $k=1/\sqrt{2}$). It shows that the sequence defined by $x_1=1$, $x_{n+1}=\sqrt{2x_n}$ (some of the terms of this sequence are given in Example 5, p. 2) tends to 2 as $n\to\infty$.

Proof of the Criterion. For the purposes of this proof, we shall take $\alpha<\beta$. The case $\beta>\alpha$ is easily treated on exactly the same lines. We assume that $f(x)$ can be differentiated, and that its derivative $f'(x)$ is continuous. For such a function the 'mean value theorem' of the calculus[1] applies. This says that if x and a are any numbers, then $f(a)-f(x)=(a-x)f'(\xi)$, where ξ has some value between x and a. If a and x are both between α and β, then so is ξ, and hence $0<f'(\xi)<k$. Therefore if $x<a$,

$$0<f(a)-f(x)<(a-x)k \quad . \quad . \quad . \quad (1)$$

Put $x=x_1(=\alpha)$ in (1), remembering that $f(x_1)=x_2$ and that $f(a)=a$. This gives $0<a-x_2<(a-x_1)k$. Next put $x=x_2$ in (1), remembering that $f(x_2)=x_3$; we have $0<a-x_3<(a-x_2)k$. Since the previous inequality tells us that $a-x_2<(a-x_1)k$, this yields $0<a-x_3<(a-x_1)k^2$. Continuing in this way, we find

$$0<a-x_n<(a-x_1)k^{n-1},$$

for all n. Now $a-x_1<\beta-\alpha$ (see Fig. 10), so that

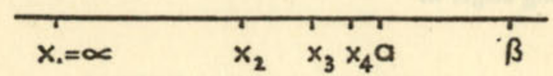

Fig. 10

$$0<a-x_n<(\beta-\alpha)k^{n-1},$$

for all n. As $n\to\infty$, $k^{n-1}\to 0$, because $k<1$. Therefore x_n must tend to a.

Working procedure. Suppose the equation to be solved is given in the form $F(x)=0$. This can be put in the form $x=f(x)$

difference $\beta-\alpha$, disregarding the sign. Notice that it is not assumed that $\alpha<\beta$.

[1] See e.g. P. J. Hilton, *Differential Calculus*, in this series.

in many ways, e.g. by taking $f(x)=x+cF(x)$, with any non-zero constant for c. We must (1) locate the root a required as nearly as possible, i.e. find α and β between which a lies, and (2) choose c so that the Criterion applies to $f(x)=x+cF(x)$. This technique is illustrated below.

Example 2. Calculate $\sqrt[3]{2}$. The equation to be solved is $x^3-2=0$, equivalent to $x=f(x)=x+c(x^3-2)$. (1) To locate the root. We soon find that $a=\sqrt[3]{2}$ lies between $\alpha=1{\cdot}2$ and $\beta=1{\cdot}3$, for $(1{\cdot}2)^3=1{\cdot}728<2$, while $(1{\cdot}3)^3=2{\cdot}197>2$. (2) Choose c so that $0<f'(x)=1+3cx^2<k(k<1)$, for all x between $1{\cdot}2$ and $1{\cdot}3$. A suitable value is $c=-0{\cdot}2$, for then $f'(x)=1-0{\cdot}6x^2$, which decreases from $0{\cdot}136$ to $0{\cdot}026$ as x varies from $1{\cdot}2$ to $1{\cdot}3$. Thus $0<f'(x)<k=0{\cdot}136$, for all x between $1{\cdot}2$ and $1{\cdot}3$.

According to the Criterion, $a=\sqrt[3]{2}$ will be the limit of the sequence defined by $x_1=1{\cdot}2$, $x_{n+1}=f(x_n)=x_n-0{\cdot}2(x_n{}^3-2)$. The first four terms are $x_1=1{\cdot}2$, $x_2=1{\cdot}2544$, $x_3=1{\cdot}2596$, $x_4=1{\cdot}2599$. The accuracy of these approximations can be estimated from the formula $0<a-x_n<(\beta-\alpha)k^{n-1}=(0{\cdot}1)(0{\cdot}136)^{n-1}$ (see p. 21). In particular, x_4 differs from $a=\sqrt[3]{2}$ by less than $(0{\cdot}1)(0{\cdot}136)^3$, which a rough calculation shows to be less than $0{\cdot}0003$. In this way we have *proved* that $\sqrt[3]{2}$ is between $x_4=1{\cdot}2599$ and $x_4+0{\cdot}0003=1{\cdot}2602$, which guarantees that the first four significant figures of $\sqrt[3]{2}$ are $1{\cdot}260$. Greater accuracy can be secured by taking larger n.

EXERCISES ON CHAPTER I

1. Write down the first five terms of each of the sequence defined below (use decimal notation).
   (i) $a_n=1-(0{\cdot}2)^n$. (ii) $a_n=1-(-0{\cdot}2)^n$. (iii) $a_n=(n^2+1)/(n+1)$. (iv) $a_n=\sqrt{n+1}-\sqrt{n}$. (v) $a_n=(-1)^{n+1}n$. (vi) $a_n=(\sin n\pi/2)^n$. (vii) $a_{n+1}=3/a_n$, $a_1=-1$. (viii) $2a_{n+2}=a_{n+1}+a_n(a_1=1, a_2=2)$. (ix) $a_{n+1}=a_n+(\frac{1}{2})^n$, $a_1=1$. (x) $a_{n+1}=\dfrac{1}{n}(a_1+a_2+\ .\ .\ .\ +a_n)$, $a_1=1$.
2. Make graphs (see § 3) for the sequences (i) to (vi) of Exercise 1.

3. State, in the cases (i) to (vi) of Exercise 1, whether (a_n) tends to a limit A, or to $+\infty$, or to $-\infty$, or if (a_n) oscillates.
4. How large must n be, for $(\frac{1}{3})^n$ to be less than (i) $0{\cdot}01$, (ii) 10^{-6}?
5. How large must n be, for $(1{\cdot}1)^n$ to be greater than (i) 10^3, (ii) 10^6?
6. Find $\lim a_n$, for each of the following a_n. (Note: $\lim a_n$ may be $+\infty$ or $-\infty$). (i) $2-(0{\cdot}2)^n$. (ii) $2+(-0{\cdot}2)^n$. (iii) $(n+1)/(n^2+1)$. (iv) $(4+n)/(3n-2)$. (v) $(2+n+n^2)/(1+4n+5n^2)$. (vi) $(1+n^3)/(2-3n^3)$. (vii) $(n^2+1)/(n^4+1)$. (viii) $(-1)^n n/(n^2+2)$. (ix) $2^n+\left(-\dfrac{1}{2}\right)^{n^2}$. (x) $\left\{-2+\dfrac{1}{n}+\left(\dfrac{1}{2}\right)^n\right\}^3$. (xi) $(3^n+1)/(2^n+1)$. (xii) $(2^n+1)/(2^{n+1}+1)$. (xiii) $2^n n^2/5^n$. (xiv) $(1+2^n)/(1-n)$. (xv) $(n-2^n)/(n^4-3^n)$. (xvi) $(5^n n^2+1)/(5^{n+1}n+1)$.
7. Find a number N such that $n^2/2^n<0{\cdot}001$ if $n>N$.
8. For what values of b is $a_n=(n+b)/(n+1)$ increasing?
9. Show that $(b_n)=(\cos nx)$ does not tend to a limit as $n\rightarrow\infty$, unless x is an integral multiple of 2π radians. (See Example 6, p. 15.)
10. Let $a_n=\sqrt[n]{x}(x>1)$. Show that (a_n) is (i) decreasing, and (ii) bounded below, and hence prove that $a_n\rightarrow$ a limit A as $n\rightarrow\infty$.
11. Find the limit A of Exercise 10, by considering the sequence (a_{2n}^2).
12. Apply the method of Example 2, p. 22 (§ 11) to find the (real) root of the equation $x^3+x-1=0$.

CHAPTER TWO
Infinite Series

1. FINITE SERIES

A 'series' is the sum of the terms of a sequence. For example, if $u_1, u_2, \ldots, u_N$ is a finite sequence, the sum $u_1+u_2+\ldots+u_N$ is the corresponding *finite series*. A convenient notation for $u_1+u_2+\ldots+u_N$ is $\sum_{n=1}^{N} u_n$. For instance, we could write $1^2+2^2+\ldots+N^2$ as $\sum_{n=1}^{N} n^2$. (Notice that we could write it equally well as $\sum_{r=1}^{N} r^2$, or $\sum_{s=1}^{N} s^2$; the 'summation variable' (n, or r, or s, respectively) is sometimes called a 'dummy variable', because it does not appear when the series is written out in full.) More generally, $\sum_{n=M}^{N} u_n$ denotes the sum $u_M+u_{M+1}+\ldots+u_N$.

Some special finite series will be useful to us later on.

Example 1. If the general term u_n of a finite series can be expressed in the form $f(n)-f(n+1)$, for some suitable function $f(n)$, then it is very easy to calculate $\sum_{n=1}^{N} u_n$. For $u_1+u_2+\ldots+u_N=(f(1)-f(2))+(f(2)-f(3))+(f(3)-f(4))+\ \ldots\ +(f(N)-f(N+1))=f(1)-f(N+1)$.

For example, $n=-\frac{1}{2}n(n-1)+\frac{1}{2}n(n+1)=f(n)-f(n+1)$, where $f(n)=-\frac{1}{2}n(n-1)$. Therefore $\sum_{n=1}^{N} n=1+2+\ldots+N=\frac{1}{2}N(N+1)$.

Example 2. The sum of the series $\sum_{n=1}^{N} 1/n(n+1)$ can also be evaluated in this way. For $u_n=1/n(n+1)=(1/n)-(1/n+1)$. Therefore $\sum_{n=1}^{N} 1/n(n+1)=1-\dfrac{1}{N+1}$.

Example 3. *The geometric series.* The series $1+x+x^2+\ldots+x^N=\sum_{n=0}^{N} x^n$ (notice that a term $x^0=1$ is included) is called a (finite) geometric series.

Let $\quad S=1+x+x^2+\ldots+x^{N-1}+x^N,$

then $\quad xS=\quad x+x^2+\ldots\quad +x^N+x^{N+1},$

and by subtraction, $(1-x)S=1-x^{N+1}$. Therefore

$$S=(1-x^{N+1})/1-x$$

provided that $x\neq 1$. If $x=1$, then $S=1+1+\ldots+1$ (there are $N+1$ terms here) $=N+1$.

Example 4. The *binomial theorem* of algebra is an 'expansion' of the polynomial $(x+y)^a$ in the form of a finite series. For the theorem states that for any positive integer a,

$$(x+y)^a=\sum_{n=0}^{a}\binom{a}{n}x^n y^{a-n},$$

where $\dbinom{a}{n}=\dfrac{a(a-1)\ \ldots\ (a-n+1)}{1.\ 2.\ \ldots\ n}$ if $n\geqslant 1$, and $\dbinom{a}{0}=1$.

2. INFINITE SERIES

If $u_1, u_2, u_3, \ldots$ is an infinite sequence, with nth term u_n, then the 'sum' $u_1+u_2+u_3+\ldots$ of all the terms is called an *infinite series*, and we denote this series by $\sum_{n=1}^{\infty} u_n$. For example, $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ Often it is convenient to start a series with some term u_M other than u_1, and we write then $\sum_{n=M}^{\infty} u_n = u_M + u_{M+1} + u_{M+2} + \ldots$ In particular, we may include a term u_0 at the beginning, for example $\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + \ldots = 1 + x + x^2 + \ldots$ When it is clear from the context where the series starts, the abbreviated notation Σu_n is often used.

Evidently we cannot just 'add up' an infinite number of terms in the ordinary way, and in fact it is not obvious that this kind of sum has any meaning at all. The following two examples show how an infinite 'sum' might arise.

Example 1. A *recurring decimal* is one of the most familiar infinite series. For instance, the statement that $\frac{1}{3} = 0{\cdot}\dot{3} = 0{\cdot}3333 \ldots$ really means that $\frac{1}{3}$ is the sum of the infinite number of terms $0{\cdot}3 + 0{\cdot}03 + 0{\cdot}003 + 0{\cdot}0003 + \ldots$, which we could also express in our notation $\sum_{n=1}^{\infty} 3/(10)^n$.

Example 2. *Achilles and the tortoise.* In this famous problem, it is supposed that a race is staged between a tortoise, on the one hand, and Achilles on the other. Achilles can run m times as fast as the tortoise ($m > 1$), so the tortoise is given a start, say of one minute. Thus the distance between the positions A_1 and T_1 of Achilles and the tortoise, respectively, at the moment when Achilles begins to run, is the distance which the tortoise can cover in one minute; we may as well take this as our unit of length. It takes Achilles only $1/m$th of a minute to reach T_1 (Achilles' position A_2), but the tortoise has by now advanced

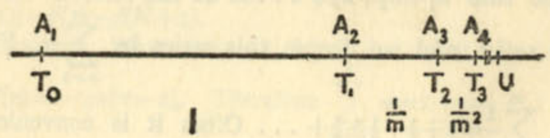

FIG. 11

a further distance $1/m$, to T_2. Achilles covers the stretch T_1T_2 in $1/m^2$th of a minute, but the tortoise is now at T_3, a distance $1/m^2$ away. The ancient paradox was this: How can Achilles ever overtake the tortoise, since he reaches each new position T_n only to find his opponent leading, by the small—but positive—distance $1/m^n$?

The answer is that the whole infinite sequence of stages T_0T_1, T_1T_2, T_2T_3, ... takes place in a *finite time*, or equivalently, the total distance $T_0T_1 + T_1T_2 + T_2T_3 + \ldots$ is finite, even though it is composed of an infinity of parts. At the end of this finite length a point U is reached where the runners are level. After that, Achilles takes the lead. The distance T_0U is given by the infinite sum $1 + (1/m) + (1/m)^2 + \ldots$, and we shall see in § 5 how to evaluate this.

3. CONVERGENT AND DIVERGENT SERIES

If we want to find the 'sum' of the infinite series $\sum_{n=1}^{\infty} u_n$ (assuming for the moment that such a thing exists), it is natural to begin by working out the 'partial sums' $s_1 = u_1$, $s_2 = u_1 + u_2$, $s_3 = u_1 + u_2 + u_3$, and so on. This gives a *sequence of partial sums* $s_1, s_2, s_3, \ldots$, whose nth term is $s_n = u_1 + u_2 + \ldots + u_n$; we can regard these partial sums as approximations to the 'full' infinite sum $\sum_{n=1}^{\infty} u_n$. If this sequence of partial sums does tend to a limit S as $n \to \infty$, then S is what we take as the *sum of the infinite series* $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots$ If, however, s_n does *not* tend to a limit, we must take it that the sum of the finite series $\sum_{n=1}^{\infty} u_n$ simply does not exist. Series of the first type are called *convergent series*, and those of the second type, which have no sum, are called *divergent series*. We repeat these definitions; they are essential for all the theory which follows.

I. The series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots$ is *convergent* if the sequence $s_1, s_2, s_3, \ldots$ of partial sums (the nth partial sum being $s_n = u_1 + u_2 + \ldots + u_n$) tends to a limit S as $n \to \infty$. The number S is then called the *sum* of the series.

II. The series $\sum_{n=1}^{\infty} u_n$ is *divergent* if the sequence of partial sums does not tend to a limit as $n \to \infty$ (that is, if $s_n \to +\infty$, or if $s_n \to -\infty$, or if s_n oscillates).

4. SOME EXAMPLES OF INFINITE SERIES

By far the most important, as well as one of the simplest series, is the infinite geometric series (i) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ (x is a constant). The nth partial sum (i.e. the sum of the first n terms) is $s_n = 1 + x + x^2 + \ldots + x^{n-1} = (1 - x^n)/(1 - x)$ (if $x \neq 1$), or $s_n = n$, if $x = 1$ (see Example 3, p. 25). If $-1 < x < 1$, we know, from the last chapter (see § 10, p. 18) that $x^n \to 0$ as $n \to \infty$, and therefore $s_n = (1 - x^n)/(1 - x) \to 1/(1 - x)$ as $n \to \infty$. Hence

If $-1 < x < 1$, *the geometric series* $\sum_{n=0}^{\infty} x^n$ *is convergent, and its sum is* $1/1 - x$.

If x has any other value, except $x = 1$, then x^n does not tend to a limit as $n \to \infty$. Therefore s_n does not tend to a finite limit as $n \to \infty$, which means that the series is *divergent*. It is also divergent in the case $x = 1$, for then $s_n = n$, which $\to +\infty$. Hence

If $x \leqslant -1$, *or if* $x \geqslant 1$, *then* $\sum_{n=0}^{\infty} x^n$ *is divergent.*

Example 1. The recurring decimal $0{\cdot}\dot{3} = 0{\cdot}3 + 0{\cdot}03 + 0{\cdot}003 + \ldots = 0{\cdot}3\{1 + (0{\cdot}1) + (0{\cdot}1)^2 + \ldots\}$. The series inside the brackets *converges* and has sum $1/(1 - 0{\cdot}1) = 1/0{\cdot}9 = 10/9$. Therefore $0{\cdot}\dot{3} = (0{\cdot}3) \times \frac{10}{9} = \frac{1}{3}$.

Any recurring decimal can be expressed as a fraction by this technique: e.g. $0{\cdot}\dot{1}3\dot{5} = 0{\cdot}135135135 \ldots = 0{\cdot}135\{1 + (0{\cdot}001) + (0{\cdot}001)^2 + \ldots\} = \frac{135}{1000} \times \frac{1}{1 - 0{\cdot}001} = \frac{135}{1000} \times \frac{1000}{999} = \frac{135}{999} = \frac{5}{37}$.

Example 2. In the problem of Achilles and the tortoise, we met the series $1 + (1/m) + (1/m)^2 + \ldots = \sum_{n=0}^{\infty} (1/m)^n$. This is convergent, because $1/m < 1$, and its sum is $1/(1 - (1/m)) = m/(m - 1)$. This is the distance T_0U from the start to the point U where Achilles overtakes the tortoise.

(ii) $\sum_{n=1}^{\infty} 1/n(n+1) = \frac{1}{1{\cdot}2} + \frac{1}{2{\cdot}3} + \frac{1}{3{\cdot}4} + \ldots$ is another series whose partial sums can be easily evaluated. For we saw in Example 2, p. 25, that $s_n = \frac{1}{1{\cdot}2} + \frac{1}{2{\cdot}3} = \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$. Therefore $s_n \to 1$ as $n \to \infty$, and it follows that the infinite series $\sum_{n=1}^{\infty} 1/n(n+1)$ is convergent, and its sum is 1.

(iii) $\sum_{n=1}^{\infty} 1/n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is sometimes called the 'harmonic series'. There is no simple formula for the partial sum $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + 1/n$, nevertheless we can prove that *the harmonic series is divergent*, as follows.

Consider $s_8 = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8})$, which we bracket in the manner shown. Clearly $(\frac{1}{3} + \frac{1}{4}) > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$. Therefore $s_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{5}{2}$. Similarly $s_{16} = s_8 + (\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{16}) > s_8 + (\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \ldots + \frac{1}{16}) > \frac{5}{2} + \frac{1}{2} = \frac{6}{2}$, and we find that $s_{32} > \frac{7}{2}$, $s_{64} > \frac{8}{2}$, and, in general, $s_{2^k} > \frac{k+2}{2}$. This shows that we can make s_{2^k} as large as we like, by making k large enough; in other words the subsequence $s_2, s_4, s_8, \ldots$ of (s_n) tends to $+\infty$. By Rule 5 (p. 14), (s_n)

cannot tend to a finite limit, for if it did, any subsequence of (s_n) would tend to the same limit. Therefore $\sum_{n=1}^{\infty} 1/n$ is divergent.

5. SOME RULES FOR CONVERGENT SERIES

We give next some simple rules.

Rule 6. *If $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ are two convergent series, with sums S, T respectively, then $\sum_{n=1}^{\infty} (u_n+v_n)$ converges and its sum is $S+T$, $\sum_{n=1}^{\infty} (u_n-v_n)$ converges and its sum is $S-T$, and if c is any constant, then $\sum_{n=1}^{\infty} cu_n$ converges and its sum is cS.*

Let $s_n=u_1+u_2+\ldots+u_n$, and let $t_n=v_1+v_2+\ldots+v_n$. Then, for example, s_n+t_n is the nth partial sum of $\sum_{n=1}^{\infty} u_n+v_n$. Since $s_n \to S$ and $t_n \to T$ as $n \to \infty$, it follows that $s_n+t_n \to S+T$, and therefore $\sum_{n=1}^{\infty}(u_n+v_n)$ converges, and has sum $S+T$. The other cases can be established in a similar manner.

Example 1. The series $\sum_{n=1}^{\infty} x^n=x+x^2+x^3+\ldots$ can be obtained by multiplying the series $\sum_{n=0}^{\infty} x^n=1+x+x^2+\ldots$ by the constant x. Therefore its sum is $x.\left(\frac{1}{1-x}\right)$, if $-1<x<1$.

Example 2. By subtracting $\sum_{n=0}^{\infty} y^n$ from $\sum_{n=0}^{\infty} x^n$, we find, in the case that x, y are both between -1 and 1, that $\sum_{n=0}^{\infty} (x^n-y^n)$ is convergent and has sum

$$(1/1-x)-(1/1-y)=(x-y)/(1-x)(1-y).$$

Rule 7. *If $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ are two convergent series, with sums S, T respectively, and if $u_n \leqslant v_n$ for all n, then $S \leqslant T$.*

For it follows that $s_n \leqslant t_n$, all n. Therefore $S=\lim s_n \leqslant \lim t_n=T$.

Rule 8. *Let $\sum_{n=1}^{\infty} u_n$ be a convergent series, with sum S.*

(i) If a finite number of terms are removed from the series, then the resulting series is still convergent, and its sum is $S-P$, where P is the sum of the terms removed.

(ii) If a finite number of terms are added to $\sum_{n=1}^{\infty} u_n$, then the resulting series is still convergent, and its sum is $S+P$, where P is the sum of the terms added.

A sketch of the proof of (i) is as follows: let $\sum_{n=1}^{\infty} u_n'$ be the new series, and let s_n' be its nth partial sum. It is clear that s_n'+(sum of the removed terms) = a partial sum, say s_{n+k}, of the original series. Since $(s_{n+k})=s_{k+1}, s_{k+1}, \ldots$ is a subsequence of (s_n), it has the same limit S as (s_n). That means that $s_n'+P \to S$, whence $s_n' \to S-P$. (ii) can be proved similarly.

Example 3. The series $x+x^2+\ldots$ of Example 1 can be obtained by removing the first term of the series $\sum_{n=0}^{\infty} x^n=1+x+x^2+\ldots$ This gives an alternative proof that the sum is $(1/1-x)-1=x/1-x$.

Remainder of a series. It is useful to have a name for the infinite series $\sum_{n=N+1}^{\infty} u_n = u_{N+1} + u_{N+2} + \ldots$, which starts after the Nth term of the series $\sum_{n=1}^{\infty} u_n$; we call it the *remainder after N terms.* (An alternative notation, which is quite equivalent, would be $\sum_{n=1}^{\infty} u_{N+n}$.) Rule 8 tells us that if the original series $\sum_{n=1}^{\infty} u_n$ is convergent, then the remainder $\sum_{n=N+1}^{\infty} u_n$ is also convergent, for it is obtained by removing the first N terms of $\sum_{n=1}^{\infty} u_n$. Furthermore, the sum $R_N = u_{N+1} + u_{N+2} + \ldots$ is equal to $S-(u_1+u_2+\ldots+u_N) = S - s_N$. We have therefore the formula

$$S = s_N + R_N,$$

which holds for all N. It is often possible to find an 'estimate' for R_N even when we cannot calculate its value exactly, and this is essential if we wish to compute S numerically; for example if we can prove that R_5 lies between 0 and 0·001, then we know that S is between s_5 and $s_5 + 0{\cdot}001$. In other words, we can get to within 0·001 of the full sum S, by using the partial sum s_5. This question will be discussed in Chapter 3.

Example 4. If $\sum_{n=1}^{\infty} u_n$ is convergent, then $R_N \to 0$ as $N \to \infty$.

For we have $R_N = S - s_N$ for every N, and $s_N \to S$ as $N \to \infty$ by the definition of S. Therefore $R_N \to S - S = 0$.

6. A TEST FOR DIVERGENCE

The following very simple test can sometimes be used to show that a series is divergent.

Test 1. *A series $\sum_{n=1}^{\infty} u_n$ is divergent if its nth term u_n does not tend to zero as $n \to \infty$.*

For if $s_n = u_1 + u_2 + \ldots + u_n$, then we have $u_n = s_n - s_{n-1}$. If the series $\sum_{n=1}^{\infty} u_n$ were convergent, then (s_n) would tend to a limit S, hence (s_{n-1}) would also tend to S. (Note: s_{n-1} is not defined for $n=1$; we have to think of this sequence as starting with $n=2$.) Therefore $u_n = s_n - s_{n-1} \to S - S = 0$. It follows that if u_n does *not* tend to 0, then Σu_n is not convergent, i.e. it is divergent.

Example 1. $\sum_{n=1}^{\infty} n/n+1 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots$ diverges, because $u_n = n/n+1 \to 1$.

Example 2. $\sum_{n=0}^{\infty} (-1)^n$ diverges, because $u_n = (-1)^n$ does not tend to a limit at all.

Example 3. $\sum_{n=1}^{\infty} 2^n/n^{18}$ diverges, because $u_n = 2^n/n^{18}$ does not tend to zero as $n \to \infty$ (it tends to $+\infty$, see p. 18).

Warning. Test 1 can *never* prove that a series is convergent. It is possible that $u_n \to 0$ even for a divergent series, for example we saw (§ 4 (iii), p. 29) that $\sum_{n=1}^{\infty} 1/n$ is divergent, although the nth term $1/n \to 0$ as $n \to \infty$.

7. THE COMPARISON TEST

One of our main tasks is to find 'convergence tests'; that is, tests which enable us to decide whether a given series $\sum_{n=1}^{\infty} u_n$ is convergent or not (without necessarily finding its sum). Test 1 is a very easy preliminary test, but it is of no use for proving that a series *is* convergent; it can only tell us that certain series are divergent.

In this paragraph, and in §§ 8, 9, we shall be mainly concerned with series whose terms are *positive* (or at any rate non-negative, i.e. positive or possibly zero); we reserve the notations p_n, q_n for the terms of such series. The nth partial sum $s_n = p_1 + p_2 + \ldots + p_n$ of a series $\sum_{n=1}^{\infty} p_n$ of positive terms is an *increasing* sequence. For $s_{n+1} = s_n + p_{n+1} \geqslant s_n$, all n. Therefore, by the Fundamental Theorem on p. 16, (s_n) either tends to a finite limit or to $+\infty$. This fact makes series of positive terms easier to deal with.

Test 2. *Comparison Test (first form). Let* $\sum_{n=1}^{\infty} p_n$, $\sum_{n=1}^{\infty} q_n$ *be two series of non-negative terms (i.e.* $p_n \geqslant 0$, *and* $q_n \geqslant 0$, *for all* n). *Then*

(*i*) *If* $p_n \leqslant q_n$ *for all* n, *and if* Σq_n *converges, then* Σp_n *converges also.*

(*ii*) *If* $p_n \geqslant q_n$ *for all* n, *and if* Σq_n *diverges, then* Σp_n *diverges also.*

As we have said, the sequences (s_n) and (t_n) of partial sums $s_n = p_1 + p_2 + \ldots + p_n$, $t_n = q_1 + q_2 + \ldots + q_n$ are both *increasing*.

(i) If Σq_n is convergent, it means that $t_n \to$ a limit T. All the t_n are $\leqslant T$. But $p_n \leqslant q_n$ for all n, hence $s_n \leqslant t_n \leqslant T$, for all n, i.e. (s_n) is *bounded*. Therefore (s_n), being a bounded increasing sequence, must tend to a limit, S, say (see p. 16), and so Σp_n converges. Incidentally the sum S of Σp_n is $\leqslant$ the sum T of Σq_n, by Rule 4, p. 12.

(ii) If Σq_n is divergent, the sequence (t_n) must tend to $+\infty$. However we are given that $p_n \geqslant q_n$, for all n, hence $s_n \geqslant t_n$ for all n, and thus s_n must tend to $+\infty$ as well. Therefore Σp_n is divergent.

Example 1. $\sum_{n=1}^{\infty} 1/\sqrt{n}$. We 'compare' with $\sum_{n=1}^{\infty} 1/n$, which is divergent. Since $1/\sqrt{n} \geqslant 1/n$, for all n (because $\sqrt{n} \leqslant n$), part (ii) of the Comparison Test shows that $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges as well.

Example 2. In general, *the series* $\sum_{n=1}^{\infty} 1/n^s$ *diverges, if* $s \leqslant 1$. For if $s \leqslant 1$, then $n^s \leqslant n$, for all n, hence $1/n^s \geqslant 1/n$, for all n. Thus $\Sigma 1/n^s$ diverges, as before.

Example 3. $\sum_{n=1}^{\infty} 1/n^2$ *is convergent.* The idea is to compare this with the series $\sum_{n=1}^{\infty} 1/n(n+1)$, which we know (§ 4 (ii), p. 29) is convergent. A direct comparison of nth terms is unsuccessful (for $1/n^2 \geqslant 1/n(n+1)$), but on the other hand $1/(n+1)^2 \leqslant 1/n(n+1)$, for all n (because $(n+1)^2 = (n+1)(n+1) \geqslant n(n+1)$), and thus by part (i) of the Comparison Test, the series $\sum_{n=1}^{\infty} 1/(n+1)^2 = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$ is convergent. Our series $\sum_{n=1}^{\infty} 1/n^2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ is obtained from this adding the single extra term 1, and so it converges also (Rule 8).

This does not, of course, tell us what the sum S of $\sum_{n=1}^{\infty} 1/n^2$ is, but only that it exists. We can however notice that

$$S = \sum_{n=1}^{\infty} 1/n^2 = 1 + \sum_{n=1}^{\infty} 1/(n+1)^2 \leqslant 1 + \sum_{n=1}^{\infty} 1/n(n+1) = 1 + 1 = 2.$$

See also § 1, Chapter 3 (p. 60).

Example 4. $\sum_{n=1}^{\infty} 1/n^s$ *is convergent, if* $s \geqslant 2$. For if $s \geqslant 2$, then $n^s \geqslant n^2$ for all n, hence $1/n^s \leqslant 1/n^2$, for all n. By the Comparison

Test, since $\Sigma 1/n^2$ is convergent, then $\Sigma 1/n^s$ must converge as well.

Example 5. $\sum_{n=0}^{\infty} 1/(3^n+1)$. It is clear that $1/(3^n+1) \leqslant 1/3^n = (\frac{1}{3})^n$, for all n. Since the geometric series $\sum_{n=0}^{\infty} (\frac{1}{3})^n$ converges, so does $\sum_{n=0}^{\infty} 1/(3^n+1)$.

The following modification of the comparison test is often easier to use than the first form, although it is less general.

Test 3. *Comparison Test (second form). If* $\sum_{n=1}^{\infty} p_n, \sum_{n=1}^{\infty} q_n$ *are two series of positive terms, and if* $p_n/q_n \to$ *a non-zero (finite) limit* C *as* $n \to \infty$, *then either* Σp_n *and* Σq_n *both converge, or both diverge.*

By the definition of a limit (p. 5), we know that, given any positive h, there exists an integer N_h such that p_n/q_n lies between $C-h$ and $C+h$, if $n>N_h$. Take for h any number less than C (e.g. $\frac{1}{2}C$), and write $N_h=N$, for short. Suppose that Σq_n is *convergent*. We have $p_n/q_n<C+h$, hence $p_n<(C+h)q_n$, for $n=N+1$, $N+2, \ldots$. Now $\sum_{n=N+1}^{\infty} q_n$ is convergent (Rule 8), therefore $\sum_{n=N+1}^{\infty} (C+h)q_n$ is convergent, and so, by the Comparison Test (1st form), $\sum_{n=N+1}^{\infty} p_n$ is also convergent. It follows (by Rule 8(ii)) that $\sum_{n=1}^{\infty} p_n$ itself is convergent. Similarly, if Σq_n is divergent, we can prove that Σp_n is divergent as well, by comparing it with $\Sigma(C-h)q_n$.

Example 6. This test makes the convergence of $\Sigma 1/n^2$ much easier to prove. Take $p_n=1/n^2$, and $q_n=1/n(n+1)$. Then

$p_n/q_n=n(n+1)/n^2=1+\frac{1}{n}\to 1$ as $n\to\infty$. Since Σq_n converges, so does Σp_n.

Series Σp_n, where $p_n=(c_hn^h+\ldots+c_1n+c_0)/(d_kn^k+\ldots+d_1n+d_0)$ is a rational function of n, are easily handled by Test 3. We have only to take $q_n=n^{h-k}$; then $p_n/q_n\to\frac{c_h}{d_k}$ (see p. 12).

Example 7. $\sum_{n=1}^{\infty} (n^3-1)/(2n^4+n+1)$. Take $q_n=n^{3-4}=n^{-1}$, which we know gives the divergent series $\Sigma n^{-1}=\Sigma 1/n$. Then $p_n/q_n=n(n^3-1)/(2n^4+n+1)\to\frac{1}{2}$ as $n\to\infty$. Therefore Σp_n diverges also.

Example 8. $\sum_{n=1}^{\infty} (3n^2+n+1)/(n^5+1)$. Take $q_n=n^{2-5}=1/n^3$. From Example 4 above, Σq_n converges. We find that $p_n/q_n=(3n^5+n^4+n^3)/n^5+1)\to 3$ as $n\to\infty$. Hence by Test 3, Σp_n is convergent.

It will be noticed that Test 3 could not be used, for example for the series $\sum_{n=1}^{\infty}(4-n)/(n^3+1)$, in which all the terms after the fourth are *negative*. However it is possible to modify Test 3 by allowing *one* of the two series involved to have negative terms, as follows:

Test 3′. If Σu_n is any series, and if Σq_n is a series of positive terms, and if $u_n/q_n\to$ a non-zero limit C (which might be negative) as $n\to\infty$, then either Σu_n and Σq_n both converge, or both diverge.

The proof remains unchanged if $C>0$. If $C<0$, we simply consider $\Sigma(-u_n)$ instead of Σu_n; this does not affect the convergence. Using Test 3′ we may discuss *any* series whose nth term is a rational function of n, even if its terms are not all positive. For the 'known' series Σq_n can always be taken to be of the form Σn^s, which has only positive terms. Thus if $u_n=(4-n)/(n^3+1)$, take $q_n=n^{1-3}=1/n^2$. Then Σq_n converges, $u_n/q_n\to -1$ as $n\to\infty$, and therefore Σu_n also converges.

8. THE RATIO TEST

The next test depends, essentially, on a comparison of the given series with a geometric series. We shall describe it first for series of positive terms, and give later (p. 47) a version suitable for series which contain negative terms.

Test 4. *Ratio Test (positive terms). Let* $\sum_{n=1}^{\infty} p_n$ *be a series of positive terms, and suppose that* $p_{n+1}/p_n \to$ *a limit L as* $n \to \infty$. *Then (i) If* $L < 1$, *the series* Σp_n *is convergent, and (ii) If* $L > 1$, *the series is divergent. If* $L = 1$, *this test fails, and the question of the convergence of* Σp_n *must be investigated by some other method.*

(i) Suppose $L < 1$. By the definition of the limit (p. 5), it must be possible, given any h, to find an integer N_h such that p_{n+1}/p_n lies between $L-h$ and $L+h$, for every $n > N_h$. Take $h = \frac{1}{2}(1-L)$, so that $L+h = \frac{1}{2}(1+L) < 1$, and write $N_h = N$, for short. We have $p_{N+1}/p_N < L+h$, $p_{N+2}/p_{N+1} < L+h$, etc. Thus

$$p_{N+n} = \frac{p_{N+n}}{p_{N+n-1}} \cdots \frac{p_{N+3}}{p_{N+2}} \cdot \frac{p_{N+2}}{p_{N+1}} \cdot p_{N+1} < (L+h)^{n-1} p_{N+1},$$

for all n. The series

$$\sum_{n=1}^{\infty} p_{N+1}(L+h)^{n-1} = p_{N+1}\{1 + (L+h) + (L+h)^2 + \ldots\}$$

is convergent, because $L+h < 1$. Therefore by the Comparison Test (1st form), $\sum_{n=1}^{\infty} p_{N+n} = p_{N+1} + p_{N+2} + \ldots$ is also convergent. By Rule 8, the full series $\sum_{n=1}^{\infty} p_n$ is itself convergent.

(ii) Suppose that $L > 1$, then we can find N such that $p_{n+1}/p_n > L-h$ for all $n > N$; we take h to be $\frac{1}{2}(L-1)$ this time, so that $L-h = \frac{1}{2}(L+1) > 1$. Now $p_{n+1}/p_n > 1$, for $n > N$, implies that $p_{n+1} > p_n$ for $n > N$, i.e. (p_n) is an *increasing* sequence. Since the p_n are all positive, it is impossible for them to tend to zero. Therefore Σp_n diverges, by Test 1.

Example 1. $\sum_{n=0}^{\infty} 1/n!$ is convergent. ($n! = 1.2.3 \ldots (n-1).n$

if n is a positive integer, and $0! = 1$, by definition.) Take $p_n = 1/n!$, then $p_{n+1}/p_n = n!/(n+1)! = 1/n+1 \to 0$ as $n \to \infty$. Since this limit $L = 0$ is less than 1, the series converges. (Note: p_n is actually the $(n+1)$th term of this series, since we start with a term p_0. This makes no difference, for the limit of (p_{n+1}/p_n) is the same as that of (p_{n+2}/p_{n+1})).

Example 2. $\sum_{n=1}^{\infty} n^2 x^n$ $(x > 0)$. Here $p_{n+1}/p_n = x(n+1)^2/n^2$ $= x\left(1+\frac{1}{n}\right)^2 \to x$, as $n \to \infty$. Therefore $\Sigma n^2 x^n$ converges if $x < 1$. If $x = 1$, the ratio test fails. However, for $x = 1$ the series becomes Σn^2, whose nth term tends to $+\infty$, and which therefore diverges (Test 1).

Example 3. $\sum_{n=1}^{\infty} x^n/n^2$ $(x > 0)$. Here $p_{n+1}/p_n = xn^2/(n+1)^2$ $= x/\left(1+\frac{1}{n}\right)^2 \to x$ as before. Therefore the series converges if $x < 1$, and diverges if $x > 1$. When $x = 1$, the ratio test fails. But in this case the series becomes $\Sigma 1/n^2$ which converges, as we have seen (p. 35).

9. THE INTEGRAL TEST

Suppose that it is possible to express the nth term of a series $\sum_{n=1}^{\infty} p_n$ in the form $p_n = f(n)$, where $f(x)$ is a continuous function defined for *all* $x \geqslant 1$ (and not just for integral values $x = n$), and satisfying the following conditions:

(*a*) $f(x) > 0$, for $x \geqslant 1$, and

(*b*) $f(x)$ is *decreasing* as x increases from 1.

Example 1. The series $\sum_{n=1}^{\infty} 1/n^s$ $(s > 0)$ is in this class. For $p_n = f(n)$, where $f(x)$ is the function $1/x^s$. Conditions (*a*) and (*b*) are satisfied.

The graph $y=f(x)$ is a curve of the form suggested in Fig. 12. Let $F(t)=\int_1^t f(x)dx$; this is the area under the curve between

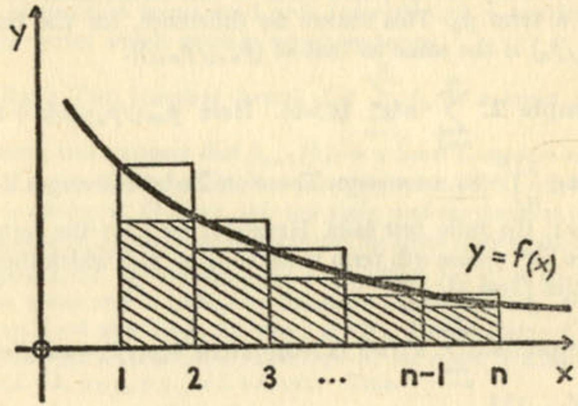

FIG. 12

$x=1$ and $x=t$. Let $s_n=p_1+p_2+\ldots+p_n$, the nth partial sum of the series $\sum_{n=1}^{\infty} p_n$. The area of the shaded part of Fig. 12 is $p_2+p_3+\ldots+p_n=s_n-p_1$ (for the width of each shaded rectangle is 1, and their heights are $f(2)=p_2$, $f(3)=p_3$, etc.). This area is less than the area of the curvilinear region $F(n)$, and $F(n)$ is, in turn, less than the sum of the larger rectangles in Fig. 12. Their heights are $p_1, p_2, \ldots, p_{n-1}$ respectively. Thus

$$s_n-p_1<F(n)<s_{n-1}.$$

The sequence $(F(n))$, like the sequence (s_n), is an *increasing* sequence. It is possible that $F(n)$ tends to a finite limit A, as $n\to\infty$. This limit A would represent the total area under the curve $y=f(x)$, to the right of the line $x=1$.

Example 2. In the case $f(x)=1/x^2$, $F(n)=\int_1^n dx/x^2=1-\frac{1}{n}$. As $n\to\infty$, this $\to 1$. Therefore the area A in this case is 1.

On the other hand, if we take $f(x)=1/x$, we get $F(n)=\int_1^n dx/x$ $=\log_e n$, which $\to+\infty$ as $n\to\infty$. This means that the area under the curve $y=1/x$ to the right of $x=1$ is infinite.

When this area A is finite, we have $s_n-p_1<F(n)<A$, for all n, so that $s_n<p_1+A$ or all n, i.e. the sequence (s_n) is *bounded above*. Since it is also increasing, it must tend to a limit S, and that means $\sum_{n=1}^{\infty} p_n$ is *convergent*. Incidentally it follows by taking limits of the terms in the inequality $s_n-p_1<F(n)<s_{n-1}$ (see above) that $S-p_1\leqslant A\leqslant S$, i.e. S lies between A and $A+p_1$.

If, on the other hand, $F(n)\to+\infty$ as $n\to\infty$, we see from the inequality $s_{n-1}>F(n)$, or equivalently $s_n>F(n+1)$, that $s_n\to+\infty$ also. In this case, therefore, $\sum_{n=1}^{\infty} p_n$ *diverges*.

We have now established the following

Test 5. (*Integral Test*). *If the nth term p_n of the series $\sum_{n=1}^{\infty} p_n$ can be expressed in the form $p_n=f(n)$, where $f(x)$ is a continuous function of x satisfying conditions (a) and (b) (p. 39) then*

(i) $\sum_{n=1}^{\infty} p_n$ converges if $\int_1^n f(x)dx$ tends to a finite limit A as $n\to\infty$, and the sum S of the series lies between A and $A+p_1$, while

(ii) $\sum_{n=1}^{\infty} p_n$ diverges if $\int_1^n f(x)dx$ tends to $+\infty$ as $n\to\infty$.

Example 3. Consider the two functions $1/x^2$ and $1/x$ (Example 2, above). Since $\int_1^n dx/x^2\to 1$ as $n\to\infty$, we deduce that $\sum_{n=1}^{\infty} 1/n^2$ converges and that its sum S lies between 1 and $1+p_1=2$. On the other hand, $\int_1^n dx/x\to+\infty$ as $n\to\infty$, and therefore $\sum_{n=1}^{\infty} 1/n$ diverges. (We have of course already proved these facts by independent methods.)

Example 4. More generally, take the series $\sum_{n=1}^{\infty} 1/n^s$ $(s>0)$. Put $f(x)=1/x^s$, which satisfies conditions (*a*) and (*b*). The integral $\int_1^n dx/x^s$ has the value $\frac{1}{s-1}\left\{1-\frac{1}{n^{s-1}}\right\}$ (we are assuming here that $s\neq 1$; the case $s=1$ has been dealt with in the previous example). Now $1/n^{s-1}\to 0$ as $n\to\infty$, if $s-1>0$ (see p. 7), while if $s-1<0$, then $1/n^{s-1}\to+\infty$. Therefore $\Sigma 1/n^s$ converges if $s>1$, and diverges if $s<1$.

To summarize our knowledge of this type of series (see also Example 2, p. 35)

$$\sum_{n=1}^{\infty} 1/n^s \textit{ converges if } s>1, \textit{ and diverges if } s\leq 1.$$

Example 5. $\sum_{n=1}^{\infty} \sqrt{n^3+2n^2+n}/\sqrt[4]{1+7n^{11}}$. Let p_n be the nth term of this series. We can use the principle employed in Examples 6, 7, 8 (p. 37). The quotient of the leading terms of the numerator and denominator is $\sqrt{n^3}/\sqrt[4]{7n^{11}}=\frac{1}{\sqrt[4]{7}}n^{-5/4}$. Therefore if $q_n=n^{-5/4}=1/n^{5/4}$, we have $p_n/q_n\to 1/\sqrt[4]{7}$ as $n\to\infty$. The series Σq_n is convergent, by what we have just proved, and hence Σp_n is also convergent, by Test 3.

Example 6. *Euler's constant.* Sometimes the method used in proving the integral test can give useful information even about *divergent* series. For example, consider the sequence (a_n) whose nth term is $a_n=s_n-F(n)$. From the inequality $s_{n-1}>F(n)$ (p. 40) we see that $a_n=s_{n-1}+p_n-F(n)>p_n>0$, for all n; i.e. (a_n) is *bounded below*. We shall prove next that (a_n) *is a decreasing sequence.* For $a_n-a_{n+1}=(s_n-s_{n+1})-(F(n)-F(n+1))$ $=-p_{n+1}+$(the area under $y=f(x)$ between $x=n$ and $x=n+1$). In Fig. 13, the shaded rectangle has area p_{n+1}, and it is clear

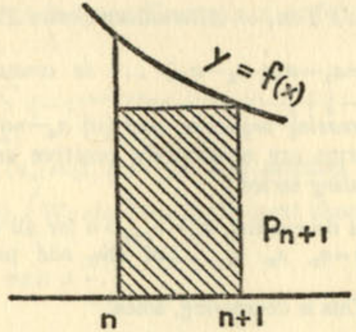

FIG. 13

that this is less than the corresponding area under the curve. Therefore $a_n-a_{n+1}>0$, i.e. $a_n>a_{n+1}$, for all n. It follows from the Fundamental Theorem that (a_n) tends to a finite limit.

In the case $f(x)=1/x$, this limit has the value $0{\cdot}5772\ldots$ and is denoted by γ (Euler's constant). Thus

$$(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+1/n)-\log_e n\to\gamma=0{\cdot}5772\ldots, \text{ as } n\to\infty$$

This means that, although we can give no simple formula for the finite sum $1+\frac{1}{2}+\frac{1}{3}+\ldots+1/n$, it is approximately equal to $\log_e n+0{\cdot}5772\ldots$, for large values of n.

10. SERIES WITH POSITIVE AND NEGATIVE TERMS. LEIBNIZ'S TEST

We have dealt so far mainly with series Σp_n, whose terms p_n are all positive. For such a series, the sequence (s_n) of partial sums is an *increasing* sequence. If, however, we take a series such as $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}(-1)^{n+1}/n$, which has both positive and negative terms, then the sequence (s_n) will not be increasing. The partial sums of the series mentioned start off $s_1=1$, $s_2=0{\cdot}5$, $s_3=0{\cdot}833\ldots$, $s_4=0{\cdot}5833\ldots$, $s_5=0{\cdot}7833\ldots$ In fact, this series is convergent, as we can prove with the help of the following test.

Test 6. (*Leibniz's Test, or Alternating Series Test.*) *The series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$ *is convergent provided* (*i*) (a_n) *is a decreasing sequence, and* (*ii*) $a_n \to 0$ *as* $n \to \infty$. (A series whose terms are alternately positive and negative is called an *alternating* series.)

Because (a_n) is decreasing, $a_n - a_{n+1} > 0$ for all n. Consider the sequence $(s_{2n-1}) = s_1, s_3, s_5, \dots$ of the *odd* partial sums of $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. This is decreasing, since

$$s_{2n+1} - s_{2n-1} = -a_{2n} + a_{2n+1} < 0,$$

for all n, and bounded below, because

$$s_{2n-1} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2n-3} - a_{2n-2}) + a_{2n-1} > 0$$

for all n. Therefore (s_{2n-1}) tends to a limit, S', say. Similarly, the sequence $(s_{2n}) = s_2, s_4, s_6, \dots$ is increasing, since

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} > 0,$$

for all n, and also bounded above, for

$$s_{2n} = a_1 - (a_2 - a_3) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n} < a_1,$$

for all n. Therefore (s_{2n}) tends to a limit, say S''. Now

$$s_{2n-1} - s_{2n} = a_{2n},$$

and (a_{2n}), being a subsequence of (a_n), tends to 0. Thus

$$S' = \lim s_{2n-1} = \lim (s_{2n} + a_{2n}) = S'' + 0,$$

i.e. $S' = S''$. Denote by S the common value $S' = S''$. Then both the even and the odd terms of the sequence $(s_n) = s_1, s_2, s_3, s_4, \dots$ tend to the same limit S, and so (s_n) itself tends to this limit. Therefore the series converges.

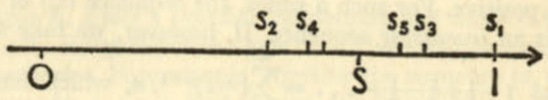

FIG. 14

Example 1. The series $\sum_{n=1}^{\infty}(-1)^{n+1}/n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent. For this is the case $a_n = 1/n$, which certainly satisfies the two conditions (i) and (ii) of Leibniz's test. (We shall

prove in the next chapter (Example 1, p. 67) that the sum is $\log_e 2 = 0{\cdot}6931\dots$.)

Example 2. $\sum_{n=1}^{\infty}(-1)^{n+1}/(2n-1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ is convergent, since $(a_n) = (1/(2n-1))$ also satisfies both the conditions (i) and (ii). (We shall see in the next chapter (Example 3, p. 68) that its sum is $\frac{\pi}{4}$.)

II. ABSOLUTE CONVERGENCE

It is convenient at this point to recall the following definition: if x is any number, then $|x|$ (the 'modulus' or 'absolute value' of x) is the numerical value of x, disregarding its sign. For instance, $|-3| = 3$, $|4| = 4$, $|0| = 0$.

This can be expressed more formally by saying $|x| = x$, if $x \geq 0$, while $|x| = -x$, if $x \leq 0$.

Example 1. If R is a positive number, $|x| < R$ means the same as $-R < x < R$. Similarly $|x - a| < R$ means that $a - R < x < a + R$.

Example 2. $|xy| = |x|\,|y|$, for any two numbers x and y. This generalizes to n factors, $|x_1 x_2 \dots x_n| = |x_1||x_2| \dots |x_n|$. In particular, putting $x_1 = x_2 = \dots = x_n = x$, we have $|x^n| = |x|^n$.

Example 3. It is sometimes useful to notice that $|x + y| \leq |x| + |y|$, for any x and y. In fact, $|x + y| = |x| + |y|$ if x and y both have the same sign, but if they have opposite signs (and neither is zero) then $|x + y| < |x| + |y|$. The reader should check this by trying a few cases. In general $|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|$, for any n numbers $x_1, x_2, \dots, x_n$.

Suppose now that $\sum_{n=1}^{\infty} u_n$ is a series, some of whose terms u_n are negative. We can make the *series of absolute values* $\sum_{n=1}^{\infty} |u_n|$; the terms of this series are all positive or zero. Now it is possible that Σu_n is convergent, but that $\Sigma|u_n|$ is divergent. An example of this is provided by the series $1-\frac{1}{2}+\frac{1}{3}-\ldots$, which was shown to be convergent, by Leibniz's test. However the series of absolute values $1+\frac{1}{2}+\frac{1}{3}+\ldots$ is divergent (p. 29).

On the other hand, the following statement is true: *If the series* $\sum_{n=1}^{\infty}|u_n|$ *is convergent, then the series* $\sum_{n=1}^{\infty}u_n$ *is also convergent.* A series Σu_n, for which $\Sigma|u_n|$ is convergent, is called an *absolutely convergent* series. Our statement can then be phrased as follows: *Every absolutely convergent series is convergent.* The proof is given below. Sometimes a series which, like $1-\frac{1}{2}+\frac{1}{3}-\ldots$, is convergent, but *not* absolutely convergent, is called a 'conditionally' convergent series.

Suppose that $\sum_{n=1}^{\infty}u_n$ is a series, such that $\sum_{n=1}^{\infty}|u_n|$ is convergent. Write $a_n=\frac{1}{2}(u_n+|u_n|)$, $b_n=\frac{1}{2}(|u_n|-u_n)$. If $u_n>0$, then

$$a_n=\tfrac{1}{2}(u_n+u_n)=u_n,$$

while $b_n=\frac{1}{2}(u_n-u_n)=0$. If $u_n<0$, then $a_n=\frac{1}{2}(u_n-u_n)=0$, and

$$b_n=\tfrac{1}{2}(-u_n-u_n)=-u_n.$$

So in either case a_n and b_n are $\geqslant 0$. Further, $|u_n|=a_n+b_n$ and $u_n=a_n-b_n$. Since $\Sigma|u_n|$ is convergent, and $a_n\leqslant a_n+b_n=|u_n|$, it follows from the comparison test (first form) that Σa_n is convergent. Similarly Σb_n is convergent. Hence $\Sigma u_n=\Sigma(a_n-b_n)$ is also convergent.

Example 1. $\sum_{n=1}^{\infty}(-1)^{n+1}/n^2=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ is absolutely convergent, since the series of absolute values is $\Sigma 1/n^2$, which is convergent. Hence $\sum_{n=1}^{\infty}(-1)^{n+1}/n^2$ is convergent. (This could also have been proved using Leibniz's test.)

Example 2. $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ is convergent (for any fixed value of x). For let $u_n=(\sin nx)/n^2$, then $|u_n|=|\sin nx|/n^2$.

Now $|\sin nx|\leqslant 1$, for any values of n and x, hence $|u_n|\leqslant 1/n^2$, for all n. Hence $\Sigma|u_n|$ converges, by the comparison test. Therefore Σu_n is convergent as well. We have proved in fact that $\Sigma(\sin nx)/n^2$ is *absolutely* convergent.

Example 3. If a series $\sum_{n=1}^{\infty}u_n$ is absolutely convergent, and if $\sum_{n=1}^{\infty}|u_n|$ has sum T, then the sum S of $\sum_{n=1}^{\infty}u_n$ has absolute value less than or equal to T. For let A, B be the sums of the two series Σa_n, Σb_n used in the proof above. The sum T of $\Sigma\,|u_n|=\Sigma(a_n+b_n)$ is $A+B$, and the sum S of $\Sigma u_n=\Sigma(a_n-b_n)$ is $A-B$. Now A and B are both positive (or zero), hence $|S|=|A-B|=|A+(-B)|\leqslant A+|-B|=A+B=T$.

Test 7. *Ratio Test (general form). Let* $\sum_{n=1}^{\infty}u_n$ *be a series none of whose terms* u_n *is zero, and suppose that* $|u_{n+1}/u_n|\to$ *a limit* L *as* $n\to\infty$. *Then (i) If* $L<1$, *then* Σu_n *is absolutely convergent (and hence is convergent), while (ii) If* $L>1$, *then* Σu_n *is divergent. If* $L=1$, *the test gives no information about the convergence of* Σu_n.

If $L<1$, we have by Test 4 that $\Sigma|u_n|$ is convergent. Hence Σu_n is absolutely convergent. If $L>1$, we must argue rather differently, for the fact that $\Sigma|u_n|$ is divergent does *not* imply that Σu_n is divergent (take e.g. $u_n=(-1)^{n+1}/n$). However, if $L>1$, we can apply the argument on p. 38 (with $p_n=|u_n|$) to show that $|u_n|\to+\infty$ as $n\to\infty$. Therefore u_n cannot tend to zero, and so Σu_n is divergent by Test 1.

Some examples of the use of this test are given in the next section; its application raises no new difficulties.

12. POWER SERIES

A series of the form $\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+\ldots$, involving the powers of a variable x, is called a 'power series'. The constants $c_0, c_1, c_2, \ldots$ are called the *coefficients* of the series; they can be positive, negative or zero.

Example 1. $\sum_{n=0}^{\infty} x^n = 1+x+x^2+\ldots$ is a power series. So is $\sum_{n=0}^{\infty} x^n/n! = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$

Let $\sum_{n=0}^{\infty} c_n x^n$ be a given power series. A number R such that the series is absolutely convergent when $|x|<R$, and divergent if $|x|>R$, is called the *radius of convergence* of $\sum_{n=0}^{\infty} c_n x^n$. For example, the radius of convergence of the geometric series $1+x+x^2+\ldots$ is 1, for it converges if $|x|<1$, and diverges if $|x|>1$ (p. 28). In a great many cases the radius of convergence can be found by using the ratio test, as in the examples which follow.

Example 2. Consider $\sum_{n=0}^{\infty} x^n/n!$. To apply the ratio test, take $u_n = x^n/n!$ (we must remember that this will be negative, if $x<0$ and n is odd). Then $|u_{n+1}/u_n| = |x|/(n+1) \to 0$ as $n\to\infty$. Therefore the series converges absolutely, for all x. In such a case we say the radius of convergence is infinite. (This series is called the *exponential series*, and its sum, written $\exp(x)$ or e^x, is a function of x called the exponential function. See also Example 2, p. 52.)

Example 3. $\sum_{n=0}^{\infty} n!x^n = 1+x+2!x^2+3!x^3+\ldots$ Here $|u_{n+1}/u_n| = (n+1)|x|$, which tends to $+\infty$ as $n\to\infty$, unless $x=0$. Although this case '$L=+\infty$' has not been covered in our statement of the ratio test 7, it is easy to adapt the argument which was used for the case $L>1$ to prove that a series for which

$\lim |u_{n+1}/u_n| = +\infty$ is divergent. In particular, $\sum_{n=1}^{\infty} n!x^n$ is divergent for all x, except $x=0$, and therefore $R=0$ for this series. Naturally a power series with zero radius of convergence has little interest for practical purposes.

Example 4. *The logarithmic series.* $\sum_{n=1}^{\infty} (-1)^{n+1}x^n/n = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ $|u_{n+1}/u_n| = |-xn/n+1| = |x|(n/n+1) \to |x|$, as $n\to\infty$. Therefore the series converges if $|x|<1$, and diverges if $|x|>1$. Thus the radius of convergence is 1. (We shall see (p. 67) that the sum of this series is $\log_e(1+x)$.)

Example 5. *The binomial series.* By putting $y=1$ in the binomial theorem (p. 25) we obtain the finite series $(1+x)^a = \sum_{n=0}^{a}\binom{a}{n}x^n$. Now the 'binomial coefficient' $\binom{a}{n}$ can be defined, even when a is not a positive integer, by the usual formulae $\binom{a}{n} = \frac{a(a-1)\ldots(a-n+1)}{1.2.\ldots n}$ (if $n>0$), $\binom{a}{0}=1$. However, in this case the series $\sum_{n=0}^{\infty}\binom{a}{n}x^n$ will be *infinite*, because the coefficients $\binom{a}{n}$ are all non-zero, *unless a is a positive integer or zero*. This infinite series which we get by using a value of a which is not a positive integer or zero is called the *binomial series*. To investigate its convergence, we observe that $|u_{n+1}/u_n| = \left|\binom{a}{n+1}x \Big/ \binom{a}{n}\right| = \left|\frac{a-n}{n+1}\right||x| \to |x|$, as $n\to\infty$. Therefore *the binomial series* $\sum_{n=0}^{\infty}\binom{a}{n}x^n = 1+ax+\frac{a(a-1)}{1.2}x^2+\ldots$ *converges absolutely if* $|x|<1$, *and diverges if* $|x|>1$; in other words its radius of convergence is 1. (The sum of this series is in fact $(1+x)^a$, for any value of a. See p. 71.)

Example 6. *The sine and cosine series.* The two series

$$\text{(i)} \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)! = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \text{ and}$$

$$\text{(ii)} \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n+1}/(2n+1)! = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

are closely related to the exponential series. For (i) we find $|u_{n+1}/u_n| = x^2/(2n+1)(2n+2) \to 0$ as $n \to \infty$, and again, for (ii), $|u_{n+1}/u_n| = x^2/2n(2n+1) \to 0$ as $n \to \infty$. Therefore each of these series converges absolutely for all values of x, i.e. they have infinite radius of convergence. (It is shown in text-books on the calculus that the sum of the series (i) is $\cos x$, and the sum of (ii) is $\sin x$; in each case x being in radian measure.)

Example 7. It is possible to prove that every power series $\sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence R (which may be infinite). However it may not be possible to find R by the ratio test. For example, the series $\sum_{n=0}^{\infty} c_n x^n = 1 + \left(\frac{x}{2}\right) + x^2 + \left(\frac{x}{2}\right)^3 + x^4 + \left(\frac{x}{2}\right)^5 + \ldots$ would give $|u_{n+1}/u_n| = (\frac{1}{2})^{n+1}|x|$, if n is even, and $|u_{n+1}/u_n| = 2^n|x|$, if n is odd. It is clear that $|u_{n+1}/u_n|$ does not tend to a limit as $n \to \infty$, and therefore the ratio test cannot be applied. On the other hand, the series converges absolutely if $|x| < 1$, by the comparison test (Test 2) (for $|u_n| \leqslant |x|^n$, all n, and $\Sigma|x|^n$ is convergent if $|x| < 1$), while if $|x| > 1$, the nth term u_n does not tend to zero and therefore $\Sigma c_n x^n$ diverges (Test 1). Thus this power series has radius of convergence 1.

Interval of Convergence. If R is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$, then the interval $-R < x < R$ is called the *interval of convergence* of the series. The values $x = R$ and $x = -R$ are called the *end-points* of the interval of convergence. In general the ratio test gives us no information about the convergence of $\Sigma c_n x^n$ at these end-points, and there is no universal

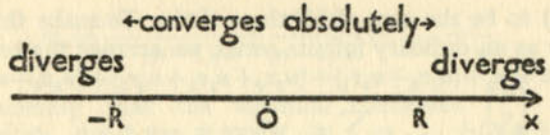

FIG. 15

rule; some power series converge at one end-point only, some at both, and some at neither.

Example 8. The end-points of the interval of convergence of the logarithmic series (Example 4) are $x=1$ and $x=-1$. At $x=1$, the series becomes $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, which is convergent (p. 44). At $x=-1$ we get $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots$, that is, the harmonic series with the sign changed throughout. This is divergent.

13. MULTIPLICATION OF SERIES

Let $\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \ldots$ and $\sum_{n=1}^{\infty} v_n = v_0 + v_1 + v_2 + \ldots$ be two infinite series. If we try to multiply these together we get an infinite number of products of the form $u_i v_j$, which we can arrange in an 'infinite square' as below.

u_0v_0	u_0v_1	u_0v_2	u_0v_3	u_0v_4	. . .
u_1v_0	u_1v_1	u_1v_2	u_1v_3	u_1v	. . .
u_2v_0	u_2v_1	u_2v_2	u_2v_3	u_2v	. . .
u_3v_0	u_3v_1	u_3v_2	u_3v_3	u_3v_4	. . .
u_4v_0	u_4v_1	u_4v_2	u_4v_3	u_4v_4	. . .
.	.	.	.	.	
.	.	.	.	.	

FIG. 16

We should expect the product $(u_0+u_1+u_2+\ldots)(v_0+v_1+v_2$

$+\dots$) to be the sum of all these terms. To make this sum appear as an ordinary infinite series, we arrange the terms as follows: $u_0v_0+(u_1v_0+u_0v_1)+(u_2v_0+u_1v_1+u_0v_2)+(u_3v_0+u_2v_1+u_1v_2+u_0v_3)+\dots=\sum_{n=0}^{\infty} w_n$, where $w_n=u_nv_0+u_{n-1}v_1+u_{n-2}v_2+\dots+u_0v_n$. The terms $w_0, w_1, w_2, \dots$ of the 'product series' Σw_n are the sums of the terms on the successive 'diagonals' in Fig. 16.

The Multiplication Theorem. *If $\sum_{n=0}^{\infty} u_n$, $\sum_{n=0}^{\infty} v_n$ are both absolutely convergent series, with sums S, T respectively, then the product series $\sum_{n=0}^{\infty} w_n$ defined above is absolutely convergent, and its sum is ST.*

Example 1. The series $\sum_{n=0}^{\infty} x^n$, $\sum_{n=0}^{\infty} y^n$ are both absolutely convergent if $|x|$ and $|y|$ are less than 1, and their sums are $1/1-x$ and $1/1-y$, respectively. The product series is $\sum_{n=0}^{\infty} w_n$, where $w_n=x^ny^0+x^{n-1}y^1+x^{n-2}y^2+\dots+x^0y^n$. Thus $1/(1-x)(1-y)=1+(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\dots$ The special case $x=y$ gives $1/(1-x)^2=1+2x+3x^2+4x^3+\dots=\sum_{n=1}^{\infty} nx^{n-1}$.

Example 2. Let $\exp(x)$ stand for the sum of the series $\sum_{n=0}^{\infty} x^n/n!$ (see p. 48). We have seen that this series converges absolutely for all x. Therefore $\exp(x)\ \exp(y)=\left(\sum_{n=0}^{\infty} x^n/n!\right)\left(\sum_{n=0}^{\infty} y^n/n!\right)=\sum_{n=0}^{\infty} w_n$, where

$$w_n=\frac{x^n}{n!}+\frac{x^{n-1}y}{(n-1)!1!}+\frac{x^{n-2}y^2}{(n-2)!2!}+\dots+\frac{y^n}{n!}.$$

Remembering that the binomial coefficient $\binom{n}{r}$ equals $n!/r!(n-r)!$, we have $w_n=\frac{1}{n!}(x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\dots+y^n)=\frac{1}{n!}(x+y)^n$. Therefore $\exp(x)\ \exp(y)=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!}=\exp(x+y)$. From this fundamental 'functional equation' $\exp(x)\exp(y)=\exp(x+y)$ it is possible to deduce the properties of this very important function $\exp(x)$ (cf. P. J. Hilton, *Differential Calculus*, in this series).

To prove the Multiplication Theorem, consider the nth partial sums s_n, t_n, q_n of the three series Σu_n, Σv_n, Σw_n, respectively. For example, $q_5=w_0+w_1+w_2+w_3+w_4$ is the sum of all the terms in the 'triangle' above the diagonal line in Fig. 16. This triangle can be divided into a square and two smaller triangles, as shown. The sum of the terms in the square works out to be

$$(u_0+u_1+u_2)(v_0+v_1+v_2)=s_3t_3.$$

Therefore $q_5=s_3t_3+A+B$, where $A=u_0v_3+u_0v_4+u_1v_3$, and

$$B=u_3v_0+u_4v_0+u_3v_1.$$

We know that $\Sigma|u_n|$ and $\Sigma|v_n|$ are convergent, because we have assumed that Σu_n and Σv_n are *absolutely* convergent. Suppose that S^* and T^* are the sums of $\Sigma|u_n|$, $\Sigma|v_n|$ respectively, and let $L_n=|u_n|+|u_{n+1}|+\dots$ be the remainder after n terms of $\Sigma|u_n|$, while $M_n=|v_n|+|v_{n+1}|+\dots$ is the corresponding remainder of $\Sigma|v_n|$. Now (see Example 3, p. 45) $|A|<|u_0||v_3|+|u_0||v_4|+|u_1||v_3|<(|u_0|+|u_1|)(|v_3|+|v_4|)<S^*M_3$, and $|B|<|u_3||v_0|+|u_4||v_0|+|u_3||v_1|<(|v_0|+|v_1|)(|u_3|+|u_4|)<T^*L_3$. Therefore $|q_5-s_3t_3|=|A+B|<|A|+|B|<S^*M_3+T^*L_3$. In general, we find in exactly the same way that $|q_n-s_mt_m|<S^*M_m+T^*L_m$, where $m=\frac{1}{2}n$, if n is even, and $m=\frac{1}{2}(n-1)$, if n is odd.

Now the remainders L_m, M_m tend to 0 as $m\to\infty$ (see Example 4, p. 32). Therefore $|q_n-s_mt_m|\to 0$ as $n\to\infty$ (for naturally $m\to\infty$ as $n\to\infty$), i.e. $q_n-s_mt_m\to 0$ as $n\to\infty$. However, by definition $s_m\to S$ and $t_m\to T$ as $m\to\infty$, and hence $q_n=s_mt_m+(q_n-s_mt_m)\to$

$ST+0=ST$ as $n\to\infty$. This proves that Σw_n is convergent and that its sum if ST, as required.

To prove that Σw_n is *absolutely* convergent, we must show that $\Sigma|w_n|$ is convergent. Now we may apply what we have just proved, to the product of the series $\Sigma|u_n|$, $\Sigma|v_n|$; the resulting product series ΣW_n is convergent, and $W_n=|u_n||v_0|+|u_{n-1}||v_1|+\ldots+|u_0||v_n|$. But $|w_n|=|u_nv_0+u_{n-1}v_1+\ldots+u_0v_n|\leqslant W_n$, for all n. Therefore $\Sigma|w_n|$ is convergent, by the comparison test.

14. NOTES ON THE USE OF THE CONVERGENCE TESTS

The first thing we want to know about a series Σu_n is whether it is convergent or not. The following notes are intended to give some advice on choosing the convergence test most suitable to the series on hand.

(1) As a rough guide, the *ratio test* should be tried whenever u_n contains either (i) factors with n in the exponent, e.g. $u_n=nx^{n+1}$ or $u_n=(2^n+n)/(5^n+4n^3)$ or (ii) factors containing factorials of functions of n, e.g. $1/n!$, $n^2/(2n+1)!$. Case (i) includes *all power series*. (This is not to say that the ratio test will work in all these cases, but it is worth trying first.) Only the general form (Test 7) need be remembered, because the earlier version (Test 4) comes to exactly the same thing, when the series Σu_n has only positive terms.

Warning. It is essential to find the *limit* L of $|u_{n+1}/u_n|$ first, and then see whether $L<1$, >1, or $=1$. It is no use saying '$|u_{n+1}/u_n|<1$ for all n'—this is *not* the same as '$L=\lim|u_{n+1}/u_n|$ is <1'. For example, the series $\sum_{n=1}^{\infty}1/n$ is divergent, yet $u_{n+1}/u_n=n/n+1<1$ for all n. In fact $L=\lim n/n+1=1$, so that the ratio test cannot be applied in this case.

(2) If u_n is a rational function of n (p. 37) the ratio test is *always* useless (it will give $L=1$), but, in compensation, the comparison test (Test 3, or its modified form Test 3′) will always be successful. The same applies to any series Σp_n where p_n is a quotient of *roots* of polynomials, e.g. $p_n=\sqrt[3]{1+n+n^2}/\sqrt{1+n^4+9n^7}$. Such series 'behave like' the series whose nth

term is the quotient of the leading terms in the numerator and denominator of p_n; this quotient is (apart from a constant factor) a power n^s, for some fixed s. So we take Σq_n to be Σn^s (this is a 'known' series; we found that Σn^s converges if $s<-1$, and diverges if $s\geqslant-1$, see p. 42). For example in the case mentioned above, the quotient of the leading terms is $\sqrt[3]{n^2}/\sqrt{9n^7}=\frac{1}{3}n^{-17/6}$. Take $q_n=n^{-17/6}$. We can verify that $p_n/q_n\to\frac{1}{3}$ as $n\to\infty$, and therefore, since Σq_n converges, so does Σp_n (Test 3). See also Examples 7, 8 p. 37, and Example 5, p. 42.

(3) Although the ratio test will very often determine the interval of convergence of a power series $\sum_{n=0}^{\infty}c_nx^n$, it will not be sufficiently sensitive to determine whether the series is convergent at the end-points. Often *Leibniz's test* can decide this question at one of the end-points.

(4) We have only used the *integral test* to investigate the series $\Sigma 1/n^s$ (see also Exercise 14 at the end of this chapter). However the idea of comparing an infinite sum with an integral is one which is greatly exploited in the more advanced theory of series. See also Example 6, p. 42, and Example 2, p. 59.

EXERCISES ON CHAPTER II

1. Show that $1+3+5+\ldots+(2N-1)=N^2$.
2. Find $\sum_{n=1}^{N}1/n(n+1)(n+2)$.
3. Prove that $\sin nx\sin\frac{1}{2}x=\frac{1}{2}\cos(n-\frac{1}{2})x-\frac{1}{2}\cos(n+\frac{1}{2})x$. Hence calculate $\sum_{n=1}^{N}\sin nx$.
4. Prove that $\sum_{n=1}^{\infty}1/n(n+1)(n+2)$ converges, and find its sum. (See Exercise 2.)

5. Calculate $s_N = \sum_{n=1}^{N} nx^{n-1}$. (Differentiate the formula $1+x+x^2+\ldots+x^N=(1-x^{N+1})/(1-x)$ with respect to x.)

6. Using Exercise 5, prove that $\sum_{n=1}^{\infty} nx^{n-1}$ converges if $-1<x<1$, and find its sum. (Alternative methods are given in Example 1, p. 52, and Example 2, p. 71.)

7. Express as fractions (i) $0\cdot\dot{1}4\dot{8}$, (ii) $0\cdot\dot{0}\dot{1}$, (iii) $0\cdot72\dot{1}\dot{2}$.

8. Calculate the first 6 partial sums of (i) $\sum_{n=0}^{\infty}(0\cdot2)^n$, (ii) $\sum_{n=0}^{\infty}(-0\cdot2)^n$.

   Find also the sums of these series, using the formula $S=1/1-x$.

9. Find whether the following series are convergent or divergent (all sums are from $n=1$ to ∞). (i) Σn. (ii) $\Sigma 1/(2n+1)$. (iii) $\Sigma 2^n/(n^5+1)$. (iv) $\Sigma(n-1)/(n^2+n-3)$. (v) $\Sigma\left\{\left(\frac{3}{4}\right)^n+\frac{1}{n^2}\right\}$.
   (vi) $\Sigma(n+2^n)/(n2^n)$. (vii) $\Sigma(2n^2+1)/(2n^5-1)$.
   (viii) $\Sigma(n+1)/(n+2)2^n$. (ix) $\Sigma(2^n+n)/(3^n+n^2)$.
   (x) $\Sigma\sqrt[3]{1+n+n^2}/\sqrt{1+n^4+n^7}$. (xi) $\Sigma n(n+1)/\sqrt{n^2(n+2)}$.
   (xii) $\Sigma\{(1-2n)/\sqrt[3]{1+8n^4}\}^4$. (xiii) $\Sigma(n!)^2/(2n)!$.
   (xiv) $\Sigma(1+2^n)/(1+n2^n)$. (xv) $\Sigma(-1)^{n+1}/n^3$.
   (xvi) $\Sigma(-1)^{n+1}/\sqrt{n}$. (xvii) $\Sigma(-1)^{n+1}\sqrt{n}$.
   (xviii) $\Sigma(-1)^{n+1}n!/n^n$. (xix) $\Sigma\{1-(-2)^n\}/\{1+2^n\}$.
   (xx) $\Sigma\{1-(-2)^n\}/\{1+3^n\}$. (xxi) $\Sigma\{(-9)^n+n\}/\{n!+1\}$.
   (xxii) $\Sigma(-n)^{-n}$.

10. Find the radius of convergence of each of the following power series. (i) $\sum_{n=1}^{\infty}x^n/n$. (ii) $\sum_{n=0}^{\infty}(-1)^n x^n/(n+1)$. (iii) $\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$.

    (iv) $\sum_{n=0}^{\infty}x^{n^2}$. (v) $\sum_{n=0}^{\infty}(n+1)x^n/(2^n+n)$. (vi) $\sum_{n=}^{\infty}(2n)!x^n/(n!)^2$.

    (vii) $\sum_{n=0}^{\infty}(n^2+2^n)x^n$. (viii) $\sum_{n=0}^{\infty}x^n/(n!+1)$.

11. Investigate the convergence of $\sum_{n=1}^{\infty} n^s x^n$, for all values of s and x.

12. By multiplying the series for $1/1-x$ and $1/(1-x)^2$ (see Example 1, p. 52), find a power series for $1/(1-x)^3$. What is the radius of convergence of your series?

13. Show that $\dfrac{\log_e(1-x)}{x-1}=x+(1+\frac{1}{2})x^2+(1+\frac{1}{2}+\frac{1}{3})x^3+\ldots$, if $|x|<1$.

14. Prove that $\sum_{n=1}^{\infty}1/(n+1)\log(n+1)$ is divergent. (Integral test.)

CHAPTER THREE

Further Techniques and Results

1. NUMERICAL CALCULATION OF THE SUM OF A SERIES

After a series $\sum_{n=1}^{\infty} u_n$ has been proved convergent, there still remains the problem of calculating the sum S. We know that the *partial* sum s_N of the first N terms of the series is an approximation to S, for S is, by definition, the limit of s_N as $N \to \infty$. However, different convergent series may differ very much in their 'rapidity of convergence', we may find in one case that s_5 differs from S by less than 0·0001, for example (so that we get to within 0·0001 of the true sum S by adding up only the first five terms of the series), while to obtain the same accuracy in another case might require thousands of terms. It is essential to have some way of estimating by how much each partial sum s_N differs from S; then we can tell in advance how many terms of the series must be added up, to get an answer within a prescribed amount of the full sum S.

We have seen (p. 32) that $S = s_N + R_N$, where R_N is the sum of the 'remainder after N terms' $\sum_{n=N+1}^{\infty} u_n = u_{N+1} + u_{N+2} + \ldots$.

If we could calculate R_N exactly, then of course we could find S exactly. But usually we can only 'estimate' R_N, which means that we prove (for a given N), that R_N lies between two numbers a and b. This tells us that $S = s_N + R_N$ lies between $s_N + a$ and $s_N + b$. If a and b are very close together, we obtain in this way a good estimate for S.

The three examples which follow illustrate some of the ways in which this estimation of R_N can be carried out. It is also interesting to notice how the three series differ in rapidity of convergence.

Example 1. The number e is defined to be exp(1), i.e. the sum of the series $\sum_{n=0}^{\infty} 1/n! = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$ The remainder after N terms is

$$R_N = \frac{1}{N!} + \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \ldots = \frac{1}{N!}\left\{1 + \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \ldots\right\} < \frac{1}{N!}\left\{1 + \frac{1}{(N+1)} + \frac{1}{(N+1)^2} + \ldots\right\}.$$

The series in the brackets is geometric, and its sum is $1/\left(1 - \frac{1}{N+1}\right) = \frac{N+1}{N}$. It is obvious that R_N is positive, and so we have the estimate $0 < R_N < \frac{N+1}{N.N!}$. For example, $0 < R_8 < \frac{9}{8.8!} = \frac{1}{7168} < 0{\cdot}0002$. The partial sum $s_8 = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{7!}$ works out to be 2·71825 . . ., and therefore $e = s_8 + R_8$ lies between $s_8 + 0$ and $s_8 + 0{\cdot}0002$, i.e. e lies between 2·7182 . . . and 2·7184 . . . This *proves* that the value of e correct to four significant figures is 2·718, and we could obviously get any higher degree of accuracy required, by taking N large enough.

Example 2. If a series $\sum_{n=1}^{\infty} p_n$ can be proved convergent by the integral test, it is always possible to derive an estimate for R_N by comparing this with an integral. Suppose that $p_n = f(n)$, where $f(x)$ is the function described on p. 39. Let A_N denote the area under the curve $y = f(x)$ to the right of the line $x = N$; this area is finite, since we are assuming that $\int_1^n f(x)dx \to$ a finite limit A, as $n \to \infty$ (p. 41). In fact, A_N is the limit of $\int_N^n f(x)dx$, as $n \to \infty$. By considering the sum of the shaded rectangles in Fig. 17, we see that $R_N = p_{N+1} + p_{N+2} + p_{N+3} + \ldots$ is less than

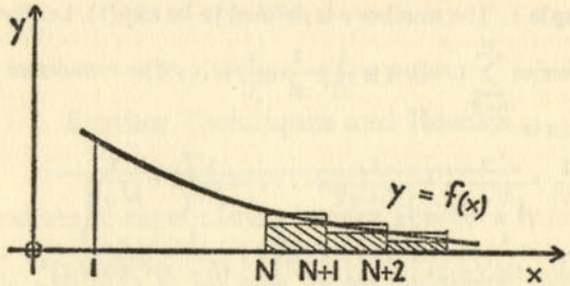

FIG. 17

A_N, while the sum $p_N+p_{N+1}+p_{N+2}+\dots=p_N+R_N$ of the larger rectangles is greater than A_N. From these two remarks we get the estimate $A_N-p_N<R_N<A_N$.

As an example, take the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Here $A_N=\lim_{n\to\infty}\int_N^n dx/x^2$ $=\lim_{n\to\infty}\left(\frac{1}{N}-\frac{1}{n}\right)=\frac{1}{N}$, and $p_N=\frac{1}{N^2}$. Therefore $\frac{1}{N}-\frac{1}{N^2}<R_N<\frac{1}{N}$, and consequently $S=s_N+R_N$ lies between $s_N+\frac{1}{N}-\frac{1}{N^2}$ and $s_N+\frac{1}{N}$. The difference between these two numbers is $1/N^2$; thus if we want a value which is guaranteed to be within 0·01 of the true sum S, we may take $N=10$. Now $s_{10}=1+\frac{1}{2^2}+\dots+\frac{1}{10^2}$ $=1\cdot549\dots$, therefore S lies between $s_{10}+0\cdot1-0\cdot01=1\cdot639\dots$ and $s_{10}+0\cdot1=1\cdot649\dots$ (The value correct to three decimal places is 1·645, but we should have to take more than 30 terms to obtain this accuracy, by this method. A much better estimate for R_N can be found by approximating to the curve in Fig. 17 by trapezoidal figures, instead of by rectangles.)

Example 3. Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+\dots$ be an alternating series, such that (a_n) is decreasing and tends to zero

as $n\to\infty$. It follows from Leibniz's test (p. 44) that the series is convergent. It is also clear from the discussion on p. 44, and from Fig. 14, that $S<s_N$, for all odd N, and $S>s_N$, for all even N. For every N, S lies between s_{N-1} and s_N, and consequently $|R_N|<|s_N-s_{N-1}|=a_N$. Thus $-a_N<R_N<0$ if N is odd, and $0<R_N<a_N$, if N is even, i.e. the Nth partial sum s_N is within a_N of S, for every N. For example, the Nth partial sum of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ is within $1/N$ of the sum of the series. This estimate compares unfavourably with those of the last two examples; to be sure of a value within 0·01 of the true sum, for example, we should have to add together the first 100 terms.

2. ESTIMATING THE REMAINDER OF A POWER SERIES

Many important functions such as $\exp(x)$, $\sin x$, $\cos x$, $(1+x)^a$, $\log_e(1+x)$ can be expressed as the sums of power series (see § 12, ch. II). The general problem of 'expanding' a *given* function $f(x)$ in a power series (that is, of finding a power series $\sum_{n=0}^{\infty}c_nx^n$ such that $f(x)$ is the sum of this series for all values of x inside the interval of convergence) is dealt with in books on the theory of functions. We shall adopt a different point of view, in that we shall suppose here that the power series $\sum_{n=0}^{\infty}c_nx^n$ is given, and see what information can be derived concerning its sum $f(x)$.

For a number x within the interval of convergence, the series $\sum_{n=0}^{\infty}c_nx^n$ is absolutely convergent (by definition of the interval of convergence). The remainder after N terms is $R_N(x)=c_Nx^N$ $+c_{N+1}x^{N+1}+c_{N+2}x^{N+2}+\dots$ This might be either positive or negative, but we consider its absolute value $|R_N(x)|$, and we know that $|R_N(x)|\leqslant|c_Nx|+|c_{N+1}x^{N+1}|+|c_{N+2}x^{N+2}|+\dots$ (see Example 3, p. 45). It is important to have an 'estimate' for $R_N(x)$ for two reasons: (i) for numerical computation—very often a power series provides the most convenient way of calculating the values of a function $f(x)$, and (ii) for theoretical purposes; for example in deriving a new series by integration (see § 3). The technique for estimating the remainder of a power series is to compare the remainder series with a suitable geometric series. The following examples illustrate this.

Example 1. *Geometric series.* This is one of the rare cases where the remainder can be calculated exactly. For $R_N(x)=x^N+x^{N+1}+\ldots=x^N(1+x+x^2+\ldots)=x^N/1-x$, provided that $|x|<1$. In fact this also follows from the formula $s_N=1+x+x^2+\ldots+x^{N-1}=(1-x^N)/(1-x)=(1/1-x)-x^N/(1-x)$ (see Example 3, p. 25). This gives the identity (which holds for all x except $x=1$)

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^{N-1}+\frac{x^N}{1-x}.$$

Example 2. *Exponential series.* The remainder after N terms of the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is $R_N(x)=\frac{x^N}{N!}+\frac{x^{N+1}}{(N+1)!}+\frac{x^{N+2}}{(N+2)!}+\ldots$

Therefore

$$|R_N(x)|\leqslant\frac{|x|^N}{N!}+\frac{|x|^{N+1}}{(N+1)!}+\frac{|x|^{N+2}}{(N+2)!}+\ldots$$

$$=\frac{|x|^N}{N!}\left\{1+\frac{|x|}{N+1}+\frac{|x|^2}{(N+1)(N+2)}+\ldots\right\}$$

$$<\frac{|x|^N}{N!}\left\{1+\frac{|x|}{N+1}+\frac{|x|^2}{(N+1)^2}+\ldots\right\}.$$

The series in the last brackets is geometric, and provided that $|x|<N+1$, it converges and has the sum $1/\left(1-\frac{|x|}{N+1}\right)$. Therefore we have the estimate

$$|R_N(x)|<\frac{|x|^N}{N!}\Big/\left\{1-\frac{|x|}{N+1}\right\}.$$

(If $x>0$, we can see at once that $R_N(x)>0$. So in this case we have a rather better estimate $0<R_N(x)<\frac{x^N}{N!}\Big/\left\{1-\frac{x}{N+1}\right\}$.)

Suppose, for example, that we want to calculate $\exp(-\frac{1}{2})$ to 4 decimal places. Our estimate gives $|R_6(-\frac{1}{2})|<\frac{(\frac{1}{2})^6}{6!}\Big/\left\{1-\frac{\frac{1}{2}}{7}\right\}$, which a rough calculation shows to be less than 0·00003. On the other hand $s_6=1-\frac{1}{2}+(\frac{1}{2})^2/2!-(\frac{1}{2})^3/3!+(\frac{1}{2})^4/4!-(\frac{1}{2})^5/5!=0·60651\ldots$ Therefore the full sum is between $s_6-0·00003=0·60648$ and $s_6+0·00003=0·60654$. It follows that $\exp(-\frac{1}{2})=0·6065$, correct to 4 decimal places.

Example 3. *Binomial series.* The remainder after N terms of the series $\sum_{n=0}^{\infty}\binom{a}{n}x^n$ is $R_N(x)=\binom{a}{N}x^N+\binom{a}{N+1}x^{N+1}+\binom{a}{N+2}x^{N+2}+\ldots$ Noticing that $\binom{a}{N+1}=\frac{a-N}{N+1}\binom{a}{N}$, $\binom{a}{N+2}=\frac{a-N}{N+1}\frac{a-N-1}{N+2}\binom{a}{N}$, etc., we have

$$|R_N(x)|<\left|\binom{a}{N}x^N\right|\left\{1+\left|\frac{a-N}{N+1}\right||x|+\left|\frac{a-N}{N+1}\right|\left|\frac{a-N-1}{N+2}\right||x|^2+\ldots\right\}.$$

We shall consider first the case $a\geqslant-1$. If this is so, then $|a-N|\leqslant N+1$, $|a-N-1|\leqslant N+2$, etc., and therefore $|R_N(x)|\leqslant\left|\binom{a}{N}\right||x|^N\{1+|x|+|x|^2+\ldots\}$. Thus for $|x|<1$, and if $a\geqslant-1$, we have the estimate

$$|R_N(x)|\leqslant\left|\binom{a}{N}\right||x|^N/(1-|x|).$$

If $a<-1$, then $\left|\frac{a-N}{N+1}\right|>1$. However we find in this case that $\left|\frac{a-N}{N+1}\right|>\left|\frac{a-N-1}{N+2}\right|>\left|\frac{a-N-2}{N+3}\right|>\ldots$, and thus $|R_N(x)|<\left|\binom{a}{N}\right||x|^N\left\{1+\left|\frac{a-N}{N+1}\right||x|+\left|\frac{a-N}{N+1}\right|^2|x|^2+\ldots\right\}$. Therefore if $|x|\left|\frac{a-N}{N+1}\right|<1$, we have that $|R_N(x)|<\left|\binom{a}{N}\right||x|^N/\left\{1-\left|\frac{a-N}{N+1}\right||x|\right\}$.

As an example, let us calculate $\sqrt[3]{2}$, correct to 6 decimal places.

Divide the equation $2\times 8^3=1024=10^3\times 1\cdot024$ by 8^3 and then take cube roots; one finds $\sqrt[3]{2}=\frac{10}{8}\times(1+0\cdot024)^{1/3}$. Now $(1+0\cdot024)^{1/3}$ is the sum of the binomial series $\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}(0\cdot024)^n$. Our estimate above gives $|R_4(0\cdot024)|<\left|\binom{\frac{1}{3}}{4}\right|(0\cdot024)^4/\{1-(0\cdot024)\}$

$$=\left|\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{1.2.3.4}\right|(0\cdot024)^4/0\cdot976=\frac{1.2.5.8}{1.2.3.4}(0\cdot008)^4/0\cdot976$$

$=\frac{4096}{2\cdot928}\times 10^{-11}<2\times 10^{-8}$. The partial sum $s_4=1+\frac{1}{3}(0\cdot024)-\frac{1}{9}(0\cdot024)^2+\frac{5}{81}(0\cdot024)^3=1\cdot00793685\ldots$ Therefore $(1\cdot024)^{1/3}$ lies between $s_4-0\cdot00000002=1\cdot00793683\ldots$ and $s_4+0\cdot00000002=1\cdot00793687\ldots$ It follows that $\sqrt[3]{2}=1\cdot25\times(1\cdot024)^{1/3}$ lies between $1\cdot25992103$ and $1\cdot25992109$. This proves that its value correct to 6 places of decimals is $1\cdot259921$.

3. INTEGRATION OF POWER SERIES

Suppose that $\sum_{n=0}^{\infty} c_n x^n=c_0+c_1x+c_2x^2+\ldots$ is a power series with radius of convergence R, and that $f(x)$ denotes its sum, for $|x|<R$. If $R_N(x)$ is the remainder $c_Nx^N+c_{N+1}x^{N+1}+\ldots$, we have the equation

$$f(x)=c_0+c_1x+c_2x^2+\ldots+c_{N-1}x^{N-1}+R_N(x).$$

Replacing, for convenience, the variable x by t, and then integrating this formula from $t=0$ to $t=x$, we find

$$\int_0^x f(t)\,dt=c_0x+c_1\frac{x^2}{2}+c_2\frac{x^3}{3}+\ldots+c_{N-1}\frac{x^N}{N}+\int_0^x R_N(t)\,dt \quad (1)$$

If it is possible to prove that $\int_0^x R_N(t)\,dt\to 0$ as $N\to\infty$, then we can show that the series $c_0x+c_1\frac{x^2}{2}+c_2\frac{x^3}{3}+\ldots=\sum_{n=0}^{\infty} c_n\frac{x^{n+1}}{n+1}$ is

convergent, and has sum $\int_0^x f(t)\,dt$. For equation (1) shows that $\int_0^x f(t)\,dt=$ the Nth partial sum of this series $+\int_0^x R_N(t)\,dt$. Therefore if $\int_0^x R_N(t)\,dt\to 0$ as $N\to\infty$, it means that the Nth partial sum of $\sum_{n=0}^{\infty} c_n\frac{x^{n+1}}{n+1}$ tends to $\int_0^x f(t)\,dt$ as $N\to\infty$.

The process of getting the series $c_0x+c_1\frac{x^2}{2}+c_2\frac{x^3}{3}+\ldots$ from $c_0+c_1x+c_2x^2+\ldots$ is called 'integrating term by term'. There is a general theorem that

If $\sum_{n=0}^{\infty} c_nx^n$ is a power series with radius of convergence R and sum $f(x)$, then the series $\sum_{n=0}^{\infty} c_n\frac{x^{n+1}}{n+1}$ obtained by integrating it term by term has the same radius of convergence, and its sum is $\int_0^x f(t)\,dt$.

We shall not attempt to prove this result here. However it is possible in many important particular cases to prove quite easily that the 'integrated remainder' $\int_0^x R_N(t)\,dt$ does tend to zero as $N\to\infty$ (provided that $|x|<R$). This, by what we have said above, gives a proof of the theorem just stated, for the particular series on hand.

(i) *The series for* log $(1+x)$.[1] Replace x by $-t$ in the formula at the end of Example 1, p. 62. We shall also replace N by n. This gives

$$\frac{1}{1+t}=1-t+t^2+\ldots+(-1)^{n-1}t^{n-1}+(-1)^n\frac{t^n}{1+t}.$$

[1] Unless the contrary is explicitly indicated, 'log' stands for 'logarithm to the base e'.

Integrate both sides from $t=0$ to $t=x$, giving

$$\int_0^x \frac{dt}{1+t} = \log(1+x)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n L_n(x) \quad (2)$$

where $L_n(x) = \int_0^x t^n dt/(1+t)$. Any attempt to evaluate this integral directly would lead back to the starting point. However we can make an estimate for $L_n(x)$, as follows.

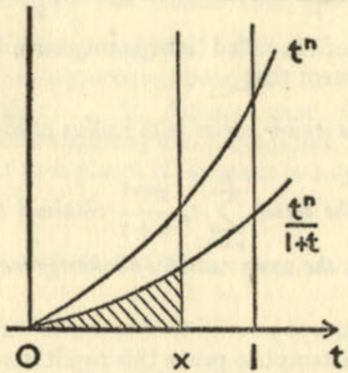

FIG. 18

Take first the case $0 \leqslant x \leqslant 1$. If t is between 0 and x, then $1+t \geqslant 1$, so that $t^n/1+t \leqslant t^n$ (see Fig. 18). Therefore $L_n(x)$ (represented by the shaded area in Fig. 18) is $\leqslant \int_0^x t^n dt = x^{n+1}/n+1$.

Now suppose that $-1 < x < 0$. Introduce a new variable $u=-t$, so that $L_n(x) = \int_0^x t^n dt/1+t$ becomes $(-1)^{n+1}\int_0^y u^n du/1-u$, where $y=-x$ $(=|x|)$. If u is between 0 and y, then $1-u \geqslant 1-y$, and hence $u^n/1-u \leqslant u^n/1-y$. Consequently $\int_0^y u^n du/(1-u) \leqslant \int_0^y u^n du/(1-y) = y^{n+1}/(n+1)(1-y)$. Therefore $|L_n(x)| \leqslant |x|^{n+1}/(n+1)(1-|x|)$, if $-1 < x < 0$.

It is now clear that $L_n(x) \to 0$ as $n \to \infty$, if $-1 < x \leqslant 1$. For $x^{n+1}/(n+1) \to 0$ as $n \to \infty$, if $0 \leqslant x \leqslant 1$, and $|x|^{n+1}/(n+1)(1-|x|) \to 0$, if $-1 < x < 0$. Therefore

(*a*) The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=0}^{\infty} (-1)\frac{x^n}{n}$ converges if $-1 < x \leqslant 1$, and its sum is $\log(1+x)$.

Going back to the original formula (2) we have also

(*b*) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n L_n(x)$,

where $0 \leqslant L_n(x) \leqslant x^{n+1}/(n+1)$, if $0 \leqslant x \leqslant 1$, and $|L_n(x)| < |x|^{n+1}/(n+1)(1-|x|)$ if $-1 < x < 0$.

This second statement gives estimates for the remainder of the logarithmic series, which can be used when calculating the sum for particular values of x.

Example 1. Put $x=1$, and we have $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log(1+1) = \log_e 2$. The case $x=-1$ gives the divergent series $-1 - \frac{1}{2} - \frac{1}{3} - \dots$

(ii) *The series for* $\tan^{-1}x$. Starting again with the formula at the end of Example 1, p. 62, we replace x this time by $-t^2$. The resulting identity

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \dots + (-1)^{n-1}t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}$$

is integrated from $t=0$ to $t=x$. This gives

$$\int_0^x \frac{dt}{1+t^2} = \tan^{-1}x$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + (-1)^n T_n(x), \quad (3)$$

where $T_n(x) = \int_0^x t^{2n} dt/1+t^2$.

$T_n(x)$ is the area under the curve $y = t^{2n}/1+t^2$, between $t=0$ and $t=x$. Because this curve is symmetrical about the y-axis, $T_n(-x) = T_n(x)$, for any x, so it is enough to examine the case $x \geqslant 0$. Suppose in fact that $0 \leqslant x \leqslant 1$. For any value of t, $1+t^2 \geqslant 1$, therefore $t^{2n}/1+t^2 \leqslant t^{2n}$. It follows that $T_n(x)=\int_0^x t^{2n}dt/(1+t^2)$ $\leqslant \int_0^x t^{2n}dt = x^{2n+1}/(2n+1)$. This tends to zero as $n \to \infty$, since $0 \leqslant x \leqslant 1$. By the symmetry which was mentioned above, it follows also that $T_n(x) \to 0$ as $n \to \infty$, for $-1 \leqslant x \leqslant 0$. Therefore

(*a*) The series $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots=\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$ converges when $-1 \leqslant x \leqslant 1$, and its sum is $\tan^{-1}x$ ('Gregory's Series').[1]

Going back to the equation (3), we have also the formula which exhibits the remainder, namely

(*b*) $\tan^{-1}x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots+(-1)^{n-1}\dfrac{x^{2n-1}}{2n-1}+(-1)^nT_n(x)$,

where $0 \leqslant T_n(x) \leqslant x^{2n+1}/(2n+1)$, if $x \geqslant 0$, and $0 \geqslant T_n(x) \geqslant -|x|^{2n+1}/(2n+1)$ if $x \leqslant 0$.

Example 2. *Leibniz's series.* Put $x=1$, and we get $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\dfrac{\pi}{4}$. In theory, this provides a means for calculating π. However this series, although convergent, is so slowly convergent that the amount of work required to secure even a modest number of decimal places of π would be prohibitive.

Example 3. A more rapidly convergent series for π is

$$\frac{\pi}{6}=\tan^{-1}(1/\sqrt{3})=\frac{1}{\sqrt{3}}\left\{1-\frac{1}{3\cdot3}+\frac{1}{5\cdot3^2}-\frac{1}{7\cdot3^3}+\ldots\right\}.$$

From (*b*), the remainder after n terms is $(-1)^nT_n(1/\sqrt{3})$, and $0 \leqslant T_n(1/\sqrt{3}) \leqslant (1/\sqrt{3})^{2n+1}/(2n+1) = 1/(2n+1)3^n\sqrt{3}$. For example, $T_7(1/\sqrt{3}) \leqslant \frac{1}{15}/3^7\sqrt{3} = 1/32085\sqrt{3} < 1/50{,}000$ $=0\cdot00002$. This means that $\dfrac{\pi}{6}$ lies between s_7 and $s_7-0\cdot00002$ (we must remember that the remainder is $(-1)^nT_n(1/\sqrt{3})$, i.e.

[1] The value of $\tan^{-1}x$ referred to here is the 'principal value', defined as follows: $\theta=\tan^{-1}x$ is the angle between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ such that $\tan\theta=x$. (Radian measure must be used.)

for $n=7$, it is $-T_7(1/\sqrt{3})$). The sum s_7 of the first 7 terms of the series works out to be $0\cdot523612\ldots$, and therefore $0\cdot523592\ldots$ $\leqslant \dfrac{\pi}{6} \leqslant 0\cdot523612\ldots$, i.e. $3\cdot14155\ldots \leqslant \pi \leqslant 3\cdot14167\ldots$

Example 4. The integral $\int e^{-t^2}dt$ is one which cannot be expressed in terms of the 'elementary' functions (log, exp, sin, cos). However it is possible to work out the value of $\int_0^x e^{-t^2}dt$, for any x, by integrating the series $\sum\limits_{n=0}^{\infty}\dfrac{(-t^2)^n}{n!}=1-t^2+t^4/2!$ $-t^6/3!+\ldots$, which has sum $\exp(-t^2)$ or e^{-t^2}. We find that

$$\int_0^x e^{-t^2}dt = x-\frac{x^3}{3}+\frac{x^5}{5\cdot2!}-\frac{x^7}{7\cdot3!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1).n!}.$$

It can be verified, using the ratio test, that this series converges for all x. It has been used to make tables of values of $\int_0^x e^{-t^2}dt$, which are used extensively by statisticians.

Example 5. The binomial series for $a=-\frac{1}{2}$ is $\sum\limits_{n=0}^{\infty}\dbinom{-\frac{1}{2}}{n}x^n$, and its sum is $(1+x)^{-\frac{1}{2}}$, if $|x|<1$ (see Example 3, p. 71). Replace x by $-t^2$; this gives $(1-t^2)^{-\frac{1}{2}}=1/\sqrt{(1-t^2)}=1+\frac{1}{2}t^2+$ $\dfrac{(-\frac{1}{2})(-\frac{3}{2})}{1.\ \ 2}(-t^2)^2+\dfrac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1.\ \ 2.\ \ 3}(-t^2)^3+\ldots$, the coefficient of t^{2n} being $\dfrac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}.\dfrac{1}{2^n}$. Multiply numerator and denominator by $2.4.6\ldots(2n)=n!2^n$, and this becomes $(2n)!/(n!)^2 2^{2n}$. Integrating the series from $t=0$ to $t=x$, we find

$$\int_0^x dt/\sqrt{(1-t^2)}=\sin^{-1}x=\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2 2^{2n}}\frac{x^{2n+1}}{(2n+1)}\ (\text{for } |x|<1).$$

This series can also be used to compute π; for example putting $x=1/2$ we have $\sin^{-1}\left(\dfrac{1}{2}\right)=\dfrac{\pi}{6}=\sum\limits_{n=0}^{\infty}\dfrac{(2n)!}{(n!)^2(2n+1)2^{4n+1}}$.

4. DIFFERENTIATION OF POWER SERIES

If the power series $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$ is *differentiated* 'term by term', the new series $c_1 + 2c_2 x + 3c_3 x^2 + \dots = \sum_{n=0}^{\infty} nc_n x^{n-1}$ results. There is the following general theorem, analogous to the theorem on integration:

If $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence R and sum $f(x)$, then the series $\sum_{n=0}^{\infty} nc_n x^{n-1}$ obtained by differentiating it term by term has the same radius of convergence, and its sum is $f'(x) = \frac{d}{dx} f(x)$.

We shall not give the proof of this result. It is important however to notice that a proof is necessary; the theorem is by no means obvious, as might appear at first sight. There are cases where term by term differentiation of a series (other than a *power* series) destroys its convergence.

Example 1. We saw (Example 2, p. 46) that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent, for all x. Differentiating term by term, we get $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$, which is divergent for $x=0$. Differentiating again gives $\sum_{n=1}^{\infty} (-\sin nx)$. When x is an integral multiple of π radians, all the terms of this series are zero, but for all other values of x, it is *divergent*, because the nth term $u_n = -\sin nx$ does not tend to zero as $n \to \infty$ (see Example 6, p. 15).

Even for a power series, we cannot be sure that the differentiated series will be convergent at the *end-points* of the interval of convergence, even if the original series was. For example,

the logarithmic series $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ converges if $-1 < x \leq 1$. Differentiating this gives the geometric series $1 - x + x^2 - \dots$ (this is not surprising, since we originally obtained the logarithmic series by integrating this geometric series!), which does not converge for $x=1$.

Example 2. The series $1 + x + x^2 + x^3 + \dots$ has sum $1/(1-x)$ and radius of convergence 1. Therefore the series $1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} nx^{n-1}$ has sum $\frac{d}{dx}(1/(1-x)) = 1/(1-x)^2$, for $|x| < 1$. We have already proved this in a different way (Example 1, p. 52).

Example 3. *The Binomial Theorem for arbitrary exponent.* Write $f(x)$ for the sum of the binomial series $\sum_{n=0}^{\infty} \binom{a}{n} x^n$ $(|x| < 1)$. Differentiating, we get

$$f'(x) = \sum_{n=0}^{\infty} n\binom{a}{n} x^{n-1} = \binom{a}{1} + 2\binom{a}{2}x + 3\binom{a}{3}x^2 + \dots, \text{ hence}$$

$$xf'(x) \qquad = \binom{a}{1}x + 2\binom{a}{2}x^2 + \dots$$

Adding, we have $(1+x)f'(x) = a + \sum_{n=1}^{\infty} \left\{ (n+1)\binom{a}{n+1} + n\binom{a}{n} \right\} x^n$. Now $\binom{a}{n+1} = \binom{a}{n}\frac{a-n}{n+1}$, and so $(n+1)\binom{a}{n+1} + n\binom{a}{n} = \binom{a}{n}\{a - n + n\} = a\binom{a}{n}$. Therefore $(1+x)f'(x) = a\sum_{n=0}^{\infty} \binom{a}{n} x^n = af(x)$, i.e. $f'(x)/f(x) = a/(1+x)$, or $\frac{d}{dx}\{\log f(x)\} = a/(1+x)$. Integrating both sides, we find that $\log f(x) = a \log(1+x) + k$, where k is some constant. By putting $x=0$ (which reduces the binomial series to the term 1) we get $k=0$, therefore $\log f(x) = a \log(1+x)$, that is, $f(x) = (1+x)^a$.

Example 4. The exponential series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ reproduces itself on differentiation. This establishes the well-known property of the exponential function, that $\frac{d}{dx}(\exp(x))=\exp(x)$.

5. CAUCHY'S CONVERGENCE PRINCIPLE

If $\sum_{n=1}^{\infty} u_n$ is a convergent series, then the partial sum $s_n=u_1+u_2+\ldots+u_n$ tends to a limit S, the sum of the series, as $n\to\infty$. By the definition of a limit (p. 5), this means that, if any positive number h is given, it is possible to find an integer N_h such that s_n is between $S-h$ and $S+h$, for all $n>N_h$; or, equivalently, that $|S-s_n|(=|s_n-S|)<h$, for all $n>N_h$. Therefore if m, n are two integers both $>N_h$, then $|s_m-s_n|=|(S-s_n)+(s_m-S)|\leq|S-s_n|+|s_m-S|<h+h=2h$. If, in addition, we suppose that $m>n$, then s_m-s_n is the sum $u_{n+1}+u_{n+2}+\ldots+u_m$, a sort of 'excerpt' from the series.

It is convenient to make slight changes of notation: put $k=2h$, and write M_k for N_h, then we can say:

If $\sum_{n=1}^{\infty} u_n$ is a convergent series, then for any positive number k, however small, we can find an integer M_k such that $|u_{n+1}+u_{n+2}+\ldots+u_m|<k$, for any m, n which are both $>M_k$.

The converse of this theorem is of fundamental theoretical importance; it was discovered by A. Cauchy (1789–1857).

Cauchy's Convergence Principle. *Suppose that $\sum_{n=1}^{\infty} u_n$ is any series, with the property that, given any positive number k, however small, it is possible to find an integer M_k such that*

$$|u_{n+1}+u_{n+2}+\ldots+u_m|<k,$$

whenever m, n are both $>M_k$. Then $\sum_{n=1}^{\infty} u_n$ is convergent.

This is not normally used as a *test* of convergence, in the way that for example the ratio and comparison tests are used, because it is usually difficult to estimate the sum $u_{n+1}+u_{n+2}+\ldots+u_m$. However it is used to derive other, more special tests (such as Dirichlet's test, in the next section) which are easier to apply.

The proof of Cauchy's principle is beyond the scope of this book. However it can be roughly explained as follows: we know that $u_{n+1}+u_{n+2}+\ldots+u_m=s_m-s_n$ lies between $-k$ and $+k$, for all m, $n>M_k$. Therefore all the partial sums s_m after s_n (we can take $n=M_k+1$ if we like, for definiteness) lie between s_n-k and s_n+k. If we take k to be very small in relation to the accuracy required, this means that the partial sums s_m for $m>n$ are 'practically' (e.g. up to a certain number of decimal places) indistinguishable from s_n. Thus s_n would, up to this number of decimal places, represent the full sum. In this sense it would be possible to calculate the sum S to any required degree of accuracy.

6. DIRICHLET'S CONVERGENCE TEST

Test 8 (*Dirichlet's Test*). *Let (a_n), (b_n) be two sequences such that (i) The sums $t_n=b_1+b_2+\ldots+b_n$ are bounded, i.e. there exists a number $H>0$ such that $|t_n|<H$, for all n, (ii) The sequence (a_n) is decreasing, and (iii) $a_n\to 0$ as $n\to\infty$. Then the series $\sum_{n=1}^{\infty} a_nb_n$ converges.*

Proof. Take any positive number k. For any m, $n(m>n)$ write $R_{m,n}=a_{n+1}b_{n+1}+a_{n+2}b_{n+2}+\ldots+a_mb_m$. Our aim is to find an integer M_k such that $|R_{m,n}|<k$, whenever m, $n>M_k$. If we can do this (for an arbitrary positive k), then it will follow from Cauchy's convergence principle that $\sum_{n=1}^{\infty} a_nb_n$ converges.

By (iii), $a_n\to 0$ as $n\to\infty$. We take M_k to be an integer such that $a_n<k/2H$, for all $n>M_k$. Such an integer M_k must exist, by the definition of the statement $a_n\to 0$ (p. 5); the reason for the curious number $k/2H$ will appear shortly.

Remembering that $b_{n+1}=t_{n+1}-t_n$, we have

$R_{m,n} = a_{n+1}(t_{n+1}-t_n) + a_{n+2}(t_{n+2}-t_{n+1}) + \dots + a_{m-1}(t_{m-1}-t_{m-2}) + a_m(t_m - t_{m-1}) = -t_n a_{n+1} + t_{n+1}(a_{n+1}-a_{n+2}) + t_{n+2}(a_{n+2}-a_{n+3}) + \dots + t_{m-1}(a_{m-1}-a_m) + a_m t_m$. Now $a_{n+1}-a_{n+2}, a_{n+2}-a_{n+3}, \dots a_{m-1}-a_m$ are all $\geqslant 0$, because (a_n) is a *decreasing* sequence (condition (ii)). Incidentally, conditions (ii) and (iii) together imply that all the a_n are $\geqslant 0$. Therefore $|R_{m,n}| \leqslant |t_n|a_{n+1} + |t_{n+1}|(a_{n+1}-a_{n+2}) + |t_{n+2}|(a_{n+2}-a_{n+3}) + \dots + |t_{m+1}|(a_{m-1}-a_m) + |t_m|a_m$, and since each $|t_n| \leqslant H$ (by (i)), we have $|R_{m,n}| \leqslant H\{a_{n+1} + (a_{n+1}-a_{n+2}) + (a_{n+2}-a_{n+3}) + \dots + (a_{m-1}-a_m) + a_m\} = 2Ha_{n+1}$. If now $m, n > M_k$, it follows that $a_{n+1} < k/2H$, and therefore

$$|R_{m,n}| \leqslant 2Ha_{n+1} < 2H.\frac{k}{2H} = k.$$

Example 1. Leibniz's test (p. 44) is a special case of Test 8. For if we suppose that (a_n) is a decreasing sequence which tends to zero, and that $b_n = (-1)^{n-1}$, then the conditions (i), (ii) and (iii) are all satisfied. For $t_n = 1 - 1 + 1 - \dots + (-1)^{n-1} = 0$ if n is even, and is 1 if n is odd. Thus $|t_n| \leqslant 1$, for all n. Therefore the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges, whenever (a_n) is a decreasing sequence which tends to zero—and this is just what Leibniz's test states.

Example 2. Suppose that $b_n = \cos nx$, where x is any number which is not of the form $2k\pi$ (k integer). It is easy to verify the trigonometric identity $2\cos nx \sin \frac{1}{2}x = (2\sin\frac{1}{2}x)b_n = \sin(n+\frac{1}{2})x - \sin(n-\frac{1}{2})x$. From this, $(2\sin\frac{1}{2}x)t_n = (2\sin\frac{1}{2}x)(b_1+b_2+\dots+b_n) = -\sin\frac{1}{2}x + \sin(n+\frac{1}{2})x$. Now $|\sin\theta| \leqslant 1$, for any θ, and so we deduce that $|2\sin\frac{1}{2}x||t_n| = |-\sin\frac{1}{2}x + \sin(n+\frac{1}{2})x| \leqslant |-\sin\frac{1}{2}x| + |\sin(n+\frac{1}{2})x| \leqslant 1+1 = 2$, and hence (since $\sin\frac{1}{2}x \neq 0$), $|t_n| \leqslant |\operatorname{cosec}\frac{1}{2}x|$, for all n. Thus t_n is bounded. Consequently *any series of the form* $\sum_{n=0}^{\infty} a_n \cos nx$ *is convergent, if* (a_n) *is a decreasing sequence which tends to zero, except possibly when* x *is an integral multiple of* 2π.

Example 3. We can prove similarly that $|\sin x + \sin 2x + \dots + \sin nx| \leqslant |\operatorname{cosec}\frac{1}{2}x|$, when x is not of the form $2k\pi$ (k integer), by starting with the identity $2\sin nx \sin\frac{1}{2}x = \cos(n-\frac{1}{2})x$

$-\cos(n+\frac{1}{2})x$. Therefore *the series* $\sum_{n=1}^{\infty} a_n \sin nx$ *is convergent, provided that* (a_n) *is a decreasing sequence which tends to zero.* (This holds for any value of x, because when x is an integral multiple of 2π, the terms of the series are all zero.)

Example 4. As an example, we can see that the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^s}, \sum_{n=1}^{\infty} \frac{\sin nx}{n^s}$ are both convergent if $0 < s \leqslant 1$, except, in the case of the first series, when x is an integral multiple of 2π. For $\left(\frac{1}{n^s}\right)$ is clearly a decreasing sequence which tends to zero. (Both series converge if $s > 1$; for in that case they both converge *absolutely*. For e.g. $\left|\frac{\cos nx}{n^s}\right| \leqslant 1/n^s$, all n, and $\Sigma 1/n^s$ converges since $s > 1$, hence $\sum_{n=1}^{\infty} \left|\frac{\cos nx}{n^s}\right|$ converges, by the comparison test. When $s \leqslant 1$ this argument fails, and the more sensitive test is necessary.)

Series of the form $\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$, the coefficients a_n, b_n being given constants, are called *trigonometric series.*

EXERCISES ON CHAPTER III

1. Show that the remainder after N terms of $\sum_{n=1}^{\infty} 1/n^4$ lies between $\frac{1}{3N^3} - \frac{1}{N^4}$ and $\frac{1}{3N^3}$ (see Example 2, p. 60). Hence prove that the sum of this series lies between $1{\cdot}081 \dots$ and $1{\cdot}083 \dots$ (Take $N = 5$.)

2. Show that the remainder after N terms of $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$ lies between $\frac{1}{a}\left(\frac{\pi}{2}-\tan^{-1}\frac{N}{a}\right)$ and $\frac{1}{a}\left(\frac{\pi}{2}-\tan^{-1}\frac{N}{a}\right)-\frac{1}{N^2+a^2} (a>0)$.

Prove that the sum of $\sum_{n=1}^{\infty} 1/(n^2+64)$ lies between $0{\cdot}183 \ldots$ and $0{\cdot}193 \ldots$ (Take $N=6$.)

3. Show that $\exp(0{\cdot}1)$ lies between $1{\cdot}1050$ and $1{\cdot}1052$, using the estimate for the remainder of the exponential series given on p. 62. (Take $N=3$.)

4. (i) Show that $e^x=1+x$, with an error which is less than $0{\cdot}0001$ if $|x|<0{\cdot}01$.

(ii) Show that $\sqrt[5]{1+x}=1+\frac{1}{5}x$, with an error which is less than $0{\cdot}0003$, if $|x|<0{\cdot}05$ (see p. 63).

5. Verify that $10\times 5^5=8^5-1518$, whence $\sqrt[5]{10}=\frac{8}{5}\left(1-\frac{1518}{32768}\right)^{1/5}$.

Use 4 (ii) above to prove that $\sqrt[5]{10}$ lies between $1{\cdot}5846 \ldots$ and $1{\cdot}5856 \ldots$

6. (i) Show that $\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots$ if $-1<x<1$.

(ii) Show that $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$ if $-1<x<1$.

(iii) Find a power series expansion for $\log(1+x+x^2)$, valid when $-1<x<1$.

7. Use the estimate for the remainder of the series for $\tan^{-1} x$ which is given on p. 68 to prove that $\tan^{-1} 0{\cdot}2$ lies between $0{\cdot}197395 \ldots$ and $0{\cdot}197397 \ldots$ (radians). (Take $n=3$.)

8. Find the radius of convergence of the series for $\sin^{-1} x$ (p. 69).

9. Show that $\sum_{n=1}^{\infty}(-1)^n a_n \cos nx$ is convergent, if (a_n) is a decreasing sequence which tends to zero, and if x is not an odd multiple of π. (Replace x by $x+\pi$ in Example 2, p. 74.)

ANSWERS TO EXERCISES

Chapter I:

1. (i) $0{\cdot}8$, $0{\cdot}96$, $0{\cdot}992$, $0{\cdot}9984$, $0{\cdot}99968$. (ii) $1{\cdot}2$, $0{\cdot}96$, $1{\cdot}008$, $0{\cdot}9984$, $1{\cdot}00032$. (iii) 1, $1{\cdot}6667$, $2{\cdot}5$, $3{\cdot}4$, $4{\cdot}6667$. (iv) $0{\cdot}4142$, $0{\cdot}3179$, $0{\cdot}2679$, $0{\cdot}2361$, $0{\cdot}2134$. (v) $1, -2, 3, -4, 5$. (vi) $1, 0, -1, 0, 1$. (vii) $-1, -3, -1, -3, -1$. (viii) $1{\cdot}2$, $1{\cdot}5$, $1{\cdot}75$, $1{\cdot}625$. (ix) 1, $1{\cdot}5$, $1{\cdot}75$, $1{\cdot}875$, $1{\cdot}9375$. (x) $1, 1, 1, 1, 1$.

3. (i) $a_n \to 1$. (ii) $a_n \to 1$. (iii) $a_n \to +\infty$. (iv) $a_n \to 0$. (Write a_n in the form $(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})/(\sqrt{n+1}+\sqrt{n})=((n+1)-n)/\sqrt{n+1}+\sqrt{n}=1/(\sqrt{n+1}+\sqrt{n})$. The denominator $\to +\infty$.) (v) osc. (vi) osc. 4. (i) $n>5$. (ii) $n>13$. 5. (i) $n>73$. (ii) $n>145$.

6. (i) 2. (ii) 2. (iii) 0. (iv) $\frac{1}{3}$. (v) $\frac{1}{3}$. (vi) $-\frac{1}{3}$. (vii) 1. (viii) 0. (ix) $+\infty$ (x) -8. (xi) $+\infty$. (xii) $\frac{1}{2}$. (xiii) 0. (xiv) $-\infty$. (xv) 0. (xvi) $+\infty$.

7. $N>19$. 8. $b<1$. 11. 1.

12. Using the recursive formula $x_{n+1}=x_n-0{\cdot}4(x_n^3+x_n-1)$, and taking $x_1=0{\cdot}6$, one has $x_2=0{\cdot}6736$, $x_3=0{\cdot}68191$, $x_4=0{\cdot}68231$, $x_5=0{\cdot}68234$.

Chapter II:

2. $\frac{1}{4}-\frac{1}{2}\frac{}{(N+1)(N+2)}$ 3. $\frac{1}{2}\operatorname{cosec}\frac{1}{2}x\{\cos\frac{1}{2}x-\cos(N+\frac{1}{2})x\}$

4. $\frac{1}{4}$. 5. $\{1-(N+1)x^N+Nx^{N+1}\}/(1-x)^2$.

6. $1/(1-x)^2$. 7. (i) $\frac{4}{27}$. (ii) $\frac{1}{99}$. (iii) $\frac{119}{165}$.

8. (i) 1, $1{\cdot}2$, $1{\cdot}24$, $1{\cdot}248$, $1{\cdot}2496$, $1{\cdot}24992$. $S=1{\cdot}25$. (ii) 1, $0{\cdot}8$, $0{\cdot}84$, $0{\cdot}832$, $0{\cdot}8336$, $0{\cdot}83328$. $S=\frac{5}{6}=0{\cdot}83333 \ldots$

9. (i) div. (ii) div. (iii) div. (iv) div. (v) conv. (vi) div. (vii) conv. (viii) conv. (ix) conv. (x) conv. (xi) div. (xii) conv. (xiii) conv. (xiv) div. (xv) conv. (xvi) conv. (xvii) div. (xviii) conv. (Test 6). (xix) div. (xx) conv. (xxi) conv. (xxii) conv.

10. (i) 1. (ii) 1. (iii) 3. (iv) 1. (v) 2. (vi) $\frac{1}{4}$. (vii) $\frac{1}{2}$. (viii) ∞.

11. conv. $|x|<1$, div. $|x|>1$, for all s. For $x=1$, conv. $s<-1$, div. $s>-1$. For $x=-1$, conv. $s<0$ (Test 6), div. $s>0$.

12. $1+3x+6x^2+10x^3+\ldots=\sum_{n=0}^{\infty}\frac{1}{2}(n+1)(n+2)x^n$. Radius conv. $=1$.

Chapter III:

6. (iii) $x+\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}-\frac{2x^6}{6}+\ldots=\sum_{n=1}^{\infty}c_n x^n$, where $c_n=\frac{1}{n}$ unless n is a multiple of 3, while $c_n=-\frac{2}{n}$ if n is a multiple of 3. (Express $1+x+x^2$ as $(1-x^3)/(1-x)$.)
1.

Index

LIBRARY OF MATHEMATICS

Edited by W Ledermann 6s net each

Complex Numbers	W Ledermann
Differential Calculus	P J Hilton
Differential Geometry	K L Wardle
Electrical and Mechanical Oscillations	D S Jones
Elementary Differential Equations and Operators	G E H Reuter
Fourier Series	I N Sneddon
Integral Calculus	W Ledermann
Linear Equations	P M Cohn
Numerical Approximation	B R Morton
Partial Derivatives	P J Hilton
Principles of Dynamics	M B Glauert
Probability Theory	A M Arthurs
Sequences and Series	J A Green
Sets and Groups	J A Green
Solid Geometry	P M Cohn
Solutions of Laplace's Equation	D R Bland
Vibrating Strings	D R Bland
Vibrating Systems	R F Chisnell
Multiple Integrals	W Lederman

LIBRARY OF MATHEMATICS

Edited by W. Lederman

This series of short text-books is primarily intended for readers who study mathematics as a tool rather than for its own sake. The aim is to cover the topics which are usually included in courses of mathematics for scientists, engineers and statisticians at Universities and Technical Colleges. Each volume is made as nearly self-contained as possible, with exercises and answers, and a few of these books should provide enough reading material for the non-specialist throughout his mathematical studies. Thus each student will be able to build up his own text-book and adapt his reading closely to the syllabus he has to follow.

Generally, techniques are emphasized more than abstract theories, and the exposition has been kept on an elementary level. When it was not feasible to give a rigorous treatment, the underlying assumptions are fully explained.

'The authors obviously understand the difficulties of undergraduates. Their treatment is more rigorous than what students will have been used to at school, and yet it is remarkably clear.

'All the books contain worked examples in the text and exercises at the ends of the chapters. They will be invaluable to undergraduates. Pupils in their last year at school, too, will find them useful and stimulating. They will learn the university approach to work they have already done, and will gain a foretaste of what awaits them in the future.' – *The Times Educational Supplement*

'It will prove a valuable corpus. A great improvement on many works published in the past with a similar objective.' – *The Times Literary Supplement*

'These are all useful little books, and topics suitable for similar treatment are doubtless under consideration by the editor of the series.' – T. A. A. BROADBENT, *Nature*

A complete list of books in the series appears on the inside back cover.

ROUTLEDGE AND KEGAN PAUL

Sequences and Series J A Green RKP