INTRODUCTION

TO

GENERAL TOPOLOGY

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AUTHOR'S PREFACE TO THE POLISH EDITION

The theorems of any geometry (*e.g.* Euclidean) follow, as is well known, from a number of axioms, *i.e.* hypotheses about the space considered, and from accepted definitions. A given theorem may be a consequence of some of the axioms and may not require all of them. Such a theorem will be true not only in the space defined by all the axioms, but also in more general spaces.

It will, therefore, be of importance to introduce axioms gradually and to deduce from them as many conclusions as possible.

We thus arrive at the concept of an abstract space (Fréchet), *i.e.* a set of elements whose nature is immaterial, and as to which we assume only certain conditions given by the axioms defining that space. Theorems obtained for a given abstract space are true for each set of elements which satisfies the axioms of that space; however, the set may also satisfy other axioms. Herein lies the practical advantage of the study of abstract spaces. For, with a suitable choice of axioms for such a space, the theorems obtained for that space may be applied to different branches of mathematics, *e.g.* to various types of geometry, to the theory of functions, and to others.

This book is divided into seven chapters. In the first chapter we consider an abstract space, which is a set K of any elements, and the only assumption we make is that certain parts of the space considered are called open sets, and that open sets have certain simple properties specified in hypothesis I (page 1). From these simple suppositions, after having introduced the corresponding definitions, we deduce a series of conclusions. In each of the following chapters we add new axioms (*i.e.* introduce new conditions to be satisfied by the open sets of the space considered) and we deduce a new series of theorems.

The axiomatic development based on the concept of an open set (as a basic concept) seemed to us simpler and more intuitive than other axiomatic treatments which will be mentioned.

Such an axiomatic treatment of the theory of point sets, apart from its logical simplicity, has also an advantage in that it supplies excellent material for exercise in abstract thinking and logical argument in the deduction of theorems from stated suppositions alone; *i.e.* in proving the theorems by drawing logical conclusions only and without any appeal to intuition, which is so apt to mislead one in the theory of sets.

The contents of this book are divided in such a manner that theorems are proved in those chapters in which the axioms necessary for the proof are introduced. The theorems of a given chapter will not in general be true in spaces which satisfy axioms of preceding chapters; an exercise very useful to the reader would be to prove, by suitably constructed examples of abstract spaces, that a given theorem is not true in a space which does not satisfy all the axioms of the chapter considered. It must be noted, however, that a strict adherence to the principle that each theorem be placed so that it is not true in spaces satisfying only the conditions of preceding chapters, is not always possible; for, it may happen that we are not able to decide whether a given theorem follows already from a given set of axioms, or else such a decision may be very complicated. The consideration of such a theorem in a given chapter is, therefore, not advisable from the pedagogical point of view, especially if its proof in a later chapter (*i.e.* under an increased number of axioms) is much simpler.

WACLAW SIERPIŃSKI

Warsaw, February, 1928.

TRANSLATOR'S PREFACE

In view of the achievements of Professor W. Sierpiński in the field of the theory of sets it was thought desirable to make his book on the *Theory of Aggregates* accessible to English-speaking students of Mathematics. Since, however, the first volume[1] on "Transfinite Numbers" has been already translated into French by the author himself, only the second volume is being offered in English translation, especially since its contents are for the most part independent of those of the first volume. Moreover, in order to enable the reader to follow the English translation, I have outlined in an appendix[2] some of the ideas and results supplied by the first volume and made use of in this.

I had hoped that Professor Sierpiński would himself write a brief note on transfinite numbers, but certain arrangements with his French publishers stood in the way. It is to be hoped that the reader not already acquainted with the subject may find the appendix useful.

I should like to take this opportunity to express my gratitude to Dean A. T. DeLury and Professor E. F. Burton for their kind interest in the publication of this book, and to Professor S. Beatty and Professor W. J. Webber whose encouragement and generous help made this translation possible.

My sincere thanks are also due to the staff of the University of Toronto Press for their untiring co-operation.

C. C. KRIEGER

Toronto, September, 1933.

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[1] *Leçons sur les nombres transfinis*, Borel series, Paris, 1928.

[2] I have been guided by the author's *Introduction to the Theory of Sets and Topology*, Lwów, 1930 (in Polish).

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CHAPTER I

CLASSES IN WHICH OPEN SETS ARE DEFINED

1. Let K be a set of any elements. We shall suppose that certain subsets of K are called *open*, where the convention determining which of the subsets of K shall be called open is quite arbitrary apart from satisfying the following axioms:

(I) $\begin{cases} (i) & \textit{A null set is an open set,} \\ (ii) & \textit{The set } K \textit{ is an open set,} \\ (iii) & \textit{The sum of any aggregate of open sets} \subset K \textit{ is an open set.} \end{cases}$

In this chapter we shall deduce results which follow solely from the axioms stated above and from the definitions to be given below. When we speak of sets in this chapter, we shall always mean sets which are subsets of the same fundamental set K.

2. We call p a *limit element* of the set $E \subset K$ if every open set containing p (whether p belongs to E or not) contains at least one element of E different from p.

It is easily seen that if p is a limit element of E, then p is also a limit element of the set $E-(p)$ (where (p) is the set consisting of the element p only). The set of all limit elements of the set E is called the *derived set* of E and is denoted by E'. (If the set E has no limit element, then its derived set is a null set, *i.e.* $E'=0$; in particular, the derived set of a null set is a null set.)

From the definitions of a limit element and a derived set we get immediately

Theorem 1. *If $E_1 \subset E$, then $E_1' \subset E'$.*

3. A set containing all its limit elements is said to be *closed*.

It follows at once from the definition of a derived set that *for a set E to be closed, it is necessary and sufficient that $E' \subset E$.*

If E be a given set, the set $K-E$ will be called the *complement* of the set E (with respect to the fundamental set K) and will be denoted by CE.

It follows at once from the definition of a closed set that, *if the set E is closed and p is not an element of the set E, there exists an open set $U \subset CE$ and containing p.* In fact, if there were no such set U, every open set containing p would contain at least one element of E different from p. (since $p \bar{\epsilon} E$); this would imply that p is a limit element of E and so, since E is closed, an element of E, contrary to the hypothesis.

Theorem 2. *In order that a set E be closed, it is necessary and sufficient that its complement CE be open.*

Proof. Let E be a set whose complement $U=CE$ is open. If E were not closed, there would be an element p such that $p \epsilon E'$ but $p \bar{\epsilon} E$. Since $p \bar{\epsilon} E$, therefore, $p \epsilon CE$, *i.e.* $p \epsilon U$; from the fact that U is open and $p \epsilon E'$, U would have to contain at least one element of the set E, which is impossible, since $U=CE$. The set E is, therefore, closed, and so the condition of the theorem is seen to be sufficient.

Let now E be a closed set. Denote by U the sum of all open sets contained in CE; evidently, $U \subset CE$, where by (I) the set U is open.

It will be shown that $U=CE$; since $U \subset CE$, it will be sufficient to prove that $U \supset CE$, *i.e.* every element of the set CE belongs to U. Let p be a given element of the set CE; we have, therefore, $p \bar{\epsilon} E$. Since the set E is closed, there exists an open set G such that $p \epsilon G$ and $G \subset CE$. According to the definition of the set U, since $G \subset CE$, we have $G \subset U$, and since $p \epsilon G$, it follows that $p \epsilon U$.

We have thus proved that $U=CE$, and hence CE is an open set. The condition of Theorem 2 is, therefore, necessary. From Theorem 2 and hypothesis (I) it follows at once that *the set K is closed*, and that *a null set is closed*.

Theorem 3. *The product of any number of closed sets is a closed set.*

Proof. Let $P=\Pi E$ denote the given product of closed sets E.

From the well-known formula of De Morgan, we shall have $CP = \sum CE$, where the sum extends over all sets E which are factors of the product $\prod E$. The sets CE are open by Theorem 2; the set $CP = \sum CE$ will be open by (I), and so by Theorem 2 the set P will be closed.

Theorem 4. *If E be a closed set, then every set containing E' and contained in E is closed.*

Proof. If a set E is closed, we have $E' \subset E$. If, further, a set T satisfies the conditions $E' \subset T \subset E$, then by Theorem 1, $T' \subset E'$, and since $E' \subset T$, it follows that $T' \subset T$, which proves that T is closed.

In particular, *the derived set of a closed set is closed.*

4. **Theorem 5.** *The set $E+E'$ is closed for every set E.*

Proof. Let E be any set, q a limit element of the set $T = E+E'$; it will be sufficient to show that q is an element of the set T, in other words, that if $q \bar{\epsilon} E$, then $q \epsilon E'$. Let us suppose, therefore, that $q \bar{\epsilon} E$, and let U be any open set containing q.

From the definition of a limit element, since $q \epsilon T'$, there is in U an element p of the set T, different from q. If $p \bar{\epsilon} E$, since $p \epsilon T = E+E'$, we have $p \epsilon E'$; thus, there is in U (which is open and contains p) an element of E different from q, since $q \bar{\epsilon} E$. U being any open set containing q, it follows that $q \epsilon E'$.

We note, however, that it does not follow from (I) and the definitions of limit element and derived set that the derived set of every set is closed.

The set $E+E'$ is called the *enclosure* of the set E and is denoted by $\bar{E}$. By Theorem 5, the enclosure of any set is closed. Let now F be any closed set containing E. We have $E \subset F$, and, therefore, by Theorem 1, $E' \subset F'$; since $F' \subset F$, F being closed, we get also $E' \subset F$. Hence $\bar{E} = E+E' \subset F$. *The enclosure of the set E is, therefore, a subset of every closed set containing E.* In other words, it is the smallest closed set containing E, or the product of all closed sets containing E.

Furthermore, it is easily seen, that if

$$E_1 \subset E, \text{ then } \bar{E}_1 \subset \bar{E}, \tag{1}$$

for by Theorem 1, $E_1 \subset E$ implies $E_1' \subset E'$, whence

$$\overline{E}_1 = E_1 + E_1' \subset E + E' = \overline{E}.$$

A closed set is evidently identical with its enclosure, and, therefore,

$$\overline{\overline{E}} = \overline{E}, \tag{2}$$

i.e. the enclosure of the enclosure of a given set is itself the enclosure of the set.

5. The sum of all open sets contained in a given set E is called the *interior* of E and is denoted by $I(E)$. (In particular, the set $I(E)$ may be a null set.) The interior of a set E is, therefore, the largest open set contained in E. It is obvious that, if $E_1 \subset E$, then $I(E_1) \subset I(E)$. We shall prove that

$$I(E) = E . C((CE)'). \tag{3}$$

Let $T = E . C((CE)')$. Then by the Theorem of De Morgan, $CT = CE + (CE)'$, which shows that CT is the enclosure of CE and so, by Theorem 5, is a closed set. It follows that T is open, and hence from the definitions of T and $I(E)$, since $T \subset E$, $T \subset I(E)$.

On the other hand, since $I(E)$ is open and contained in E, the set $CI(E)$ is closed and contains CE, and, therefore, contains $\overline{CE} = CT$, since $\overline{CE}$, as the enclosure of CE, is by § 4 a subset of every closed set containing CE.

We have, therefore, $CI(E) \supset CT$, which gives $I(E) \subset T$, and, since we had above $T \subset I(E)$, it follows that $I(E) = T$; this proves (3). Since $CT = \overline{CE}$, we have proved also that

$$I(E) = C(\overline{CE}). \tag{4}$$

Every element belonging to the interior $I(E)$ of a set E, is called an *interior element* of the set E. Hence, in order that a given element of E be an interior element of E, it is necessary and sufficient (by (3)) that it be not a limit element of the complement of E.

Every element belonging to the set $I(CE)$ is called an *exterior element* of E.[1] The set of all exterior elements of E is denoted by $E^*(E)$; from (3) and (4) we shall have

[1]The exterior elements of E are, therefore, those elements that do not belong to E and are not limit elements of E.

(5) $$E^*(E) = I(CE) = CE \, . \, C(E') = C(\overline{E}).$$

Those elements which are neither interior elements nor exterior elements of E constitute the *frontier set* of E, which is denoted by $F(E)$; we have then from (4) and (5)

$$F(E) = C[I(E) + E^*(E)] = C[I(E)] \, . \, C[E^*(E)] = \overline{CE} \, . \, \overline{E}.$$

From (4)

(6) $$F(E) = \overline{E} \, . \, \overline{CE} = \overline{E} - I(E).$$

Since $\overline{E} \, . \, \overline{CE} = (E+E') \, . \, [CE + (CE)'] = E \, . \, (CE)' + E' \, . \, CE$ and since $E \, . \, CE = 0$ and $E' \, . \, (CE)' = E' \, . \, (CE)' \, . \, [E + CE] \subset E \, . \, (CE)' + E' \, . \, CE$, it follows from (6) that

(7) $$F(E) = E \, . \, (CE)' + E' \, . \, CE.$$

In virtue of Theorem 3 and the fact that the enclosure of a set is closed, the relation (6) provides the information that *the frontier of a set is a closed set.*

Also it follows at once from (6) that a given set and its complement have the same frontier.

The set

(8) $$B(E) = E \, . \, F(E) = E \, . \, \overline{CE} = E \, . \, (CE)' = E - I(E)$$

is called the *border* of the set E and its elements, the *border elements.* The border elements of E are those elements of E which are not interior elements of E. A set consisting of border elements only is called a *border set.* Hence in order that a given set E be a border set, it is necessary and sufficient, by (8), that

$$E = B(E),$$

which may be written in either of the equivalent forms

$$E \subset (CE)', \; I(E) = 0,$$

the latter indicating that E does not contain interior elements. It follows at once that *a subset of a border set is a border set.*

Since, by (8), the border of a set E does not contain any interior elements of E, and certainly not its own interior elements, *the border of a border set is a border set.* We have, therefore, for every set E

(9) $$BB(E) = B(E).$$

From the definition of the interior $I(E)$ it follows at once, that in order that E be open it is necessary and sufficient that

$$E = I(E).$$

It follows from this, that for every set E

(10) $$II(E) = I(E).$$

We have, from (5), $CE^*(E) = \overline{E}$, and hence $E^*E^*(E) = ICE^*(E) = I(\overline{E})$, which, by (1), gives (when $E_1 \subset E_2$) $E^*E^*(E_1) \subset E^*E^*(E_2)$.[2]
Since $\overline{E} \supset E$, we have

$$I(\overline{E}) \supset I(E),$$

and, therefore,

$$E^*E^*(E) = I(\overline{E}) \supset I(E),$$

and, hence, from (10),

$$E^*E^*E^*E^*(E) = E^*E^*I(\overline{E}) \supset II(\overline{E}) = I(\overline{E}) = E^*E^*(E),$$

while

$$E^*E^*(E) = I(\overline{E}) \subset \overline{E},$$

and so

$$E^*E^*E^*E^*(E) \subset E^*E^*(\overline{E}) = I(\overline{\overline{E}}) = I(\overline{E}) = E^*E^*(E).$$

Therefore,

$$E^*E^*E^*E^*(E) = E^*E^*(E).$$[3]

As to the frontier of a set, it follows from (6) and (8) that for a closed set E,

(11) $$F(E) = \overline{E} - I(E) = E - I(E) = B(E);$$

the frontier of a closed set is, therefore, always a border set (a property which need not be true for non-closed sets); we have, therefore, for a closed E

(12) $$IF(E) = 0.$$

Since $F(E)$ is always closed, we may substitute $F(E)$ for E in (11); by (12), this gives, for a closed E,

$$FF(E) = F(E).$$

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[2]Again from (5), we have $E^*(E_1) \supset E^*(E_2)$, for $E_1 \subset E_2$.

[3]See M. Zarycki, *Fund. Math.*, vol. IX, p. 6.

from which we get at once, by replacing E by the closed set $F(E)$,

$$FFF(E) = FF(E) \tag{13}$$

for all sets E, a result also obtained by Zarycki.

6. A set, every element of which is a limit element of the set, is said to be *dense-in-itself*. According to this definition, a null set is to be considered dense-in-itself. It follows at once from the definition of the derived set, that, *for a given set E to be dense-in-itself, it is necessary and sufficient that $E \subset E'$.*

Theorem 6. *If E be a set dense-in-itself, then every set containing E and contained in E' is dense-in-itself.*

Proof. If E is dense-in-itself, we have $E \subset E'$. If further, the set T is such that $E \subset T \subset E'$, then, by Theorem 1, we have $E' \subset T'$, and so we get $T \subset T'$.

In particular, *the derived set of a set dense-in-itself is dense-in-itself.* Furthermore, it follows that *the enclosure of a set dense-in-itself is dense-in-itself*, for $E \subset E'$ implies $\overline{E} = E + E' = E'$.

Theorem 7. *The sum of any number of sets dense-in-themselves is dense-in-itself.*

Proof. Let $S = \sum E$ denote the sum of sets dense-in-themselves. We have then $E \subset E'$ for every set E of the sum $S = \sum E$. On the other hand, since $E \subset S$, we have, from Theorem 1, $E' \subset S'$, and, therefore, $E \subset S'$ for every E of the sum S; thus $S \subset S'$.

Given any set E, denote by N the sum of all sets dense-in-themselves and contained in E; by Theorem 7, the set N will be dense-in-itself. N will be, as is easily seen, the largest subset of E dense-in-itself, *i.e.* the subset containing every subset of E dense-in-itself. The set N is called the *nucleus* of E. A set, whose nucleus is a null set (*i.e.* a set not containing any subset dense-in-itself other than the null set) is said to be *scattered.*

Theorem 8. *Every set E can be represented in the form*

$$E = N + R,$$

where N is the nucleus of E, and R is scattered or null.

In fact, if N be the nucleus of E, we have $N \subset E$, and we may, therefore, write $E-N=R$, where $R \subset E$, and where $E=N+R$. If R were not a scattered set, it would contain a subset dense-in-itself, which according to the definition of the nucleus, would have to be contained in N; this is impossible, since $N.R=0$. R is, therefore, scattered, which proves the theorem.

A set which is dense-in-itself and closed, is called *perfect.* Obviously, *in order that a set* E *be perfect, it is necessary and sufficient that* $E=E'$. (In particular, the null set will be considered to be a perfect set.)

7. The set E_1 is said to be *closed in the set* E, *if*

$$E_1'.E \subset E_1, \tag{14}$$

i.e. if the set E_1 contains all those of its limit elements which belong to E.

If $E_1 \subset E$, then (14) is equivalent to the equality

$$E_1=(E_1+E_1').E,$$

for (14) gives $E_1'.E \subset E_1.E$, from which $(E_1+E_1').E=E_1.E+E_1'.E=E_1.E=E_1$. On the other hand, if $E_1=(E_1+E_1').E$, (14) follows.

Hence if $E_1 \subset E$ is closed in E, we have $E_1=\overline{E}_1.E$. Therefore, a set contained in E and closed in E, is the product of E and a closed set. Conversely, if $E_1=F.E$, where F is a closed set, then E_1 is closed in E, for from $E_1 \subset F$, F being closed, we have (by (1)) $\overline{E}_1 \subset F$, and so $E_1=F.E$ gives $E_1=\overline{E}_1.E$.

A set contained in a closed set, and closed in that set, is itself closed. This follows immediately from the relation $E_1=\overline{E}_1.E$ and Theorem 3.

A closed set is evidently closed in every set (since the relation $E_1' \subset E_1$ implies (14) for every set E).

If a set E_1 is closed in a set E, then it is closed in every subset of E (for if $E_1'.E \subset E_1$ and $E_2 \subset E$, then $E_1'.E_2 \subset E_1'.E \subset E_1$), but it may not be closed in a set $T \supset E$.

Theorem 9. *The product of any number of sets closed in the set* E_0 *is a set closed in* E_0.

Proof. Let $P=\Pi E$ denote the given product of the sets E closed in E_0. We have then $P\subset E$ for every factor E of the product, whence by Theorem 1, $P'\subset E'$ and, therefore, also $P'.E_0 \subset E'.E_0$. But $E'.E_0\subset E$ for every E of P (since the sets E are closed in E_0); we have then $P'.E_0\subset E$ for every E of P and, therefore, $P'.E_0\subset P$, which proves that P is closed in E_0.

Theorem 10. *The nucleus of any set is closed in that set.*

Proof. Let E_1 denote the nucleus of the set E (§ 6). Since the nucleus E_1 is dense-in-itself and contained in E, we have $E_1\subset E_1'.E$, and since obviously $E_1'.E\subset E_1'$ we have, by Theorem 6, that the set $E_1'.E$ is dense-in-itself and is, therefore, contained in E_1, the nucleus of E (as a subset of E dense-in-itself). We have then $E_1'.E\subset E_1$, and so E_1 is closed in the set E.

It follows from the above that the nucleus of a closed set is closed and dense-in-itself and therefore perfect.

A set $E_1\subset E$ is called *perfect in the set E* if $E_1'.E=E_1$. Hence, in order that a set contained in E be perfect in E, it is necessary and sufficient that the set be closed in E and dense-in-itself.

A set, perfect in a closed set, is evidently perfect.

8. Two sets A and B are said to be *separated*, if

(15) $$A\neq 0,\ B\neq 0,\ A.B=A.B'=A'.B=0.$$

Theorem 11. *If A and B be separated sets, and A_1 and B_1 be sets such that*

(16) $$A_1\neq 0,\ B_1\neq 0,\ A_1\subset A,\ B_1\subset B,$$

then the sets A_1 and B_1 are also separated.

Proof. From (15) and (16), we have $A_1.B_1\subset A.B=0$, whence $A_1.B_1=0$; since $A_1\subset A$, $B_1\subset B$, it follows, by Theorem 1, that $A_1'\subset A'$, $B_1'\subset B'$, and so from (15) and (16), $A_1.B_1'\subset A.B'=0$ and $A_1'.B_1\subset A'.B=0$, which gives $A_1.B_1'=A_1'.B_1=0$. We have thus proved, that the sets A_1 and B_1 are separated.

A set E which cannot be expressed as the sum of two separated sets, is said to be *connected* (Hausdorff).[4]

[4]According to Lennes, a set of points is connected if at least one of any two complementary subsets contains a limit point of points in the other set (*American Jour. of Maths.*, vol. XXXIII (1911), p. 303).

Theorem 12. *In order that a set E be connected it is necessary and sufficient that it be not expressible as the sum of two mutually exclusive, non-null sets closed in E.*

Proof. Suppose the set E to be the sum of two mutually exclusive, non-null sets A, B, closed in E. We have then $E=A+B$, where

$$A\neq 0,\ B\neq 0,\ A.B=0,\ A'.E\subset A,\ B'.E\subset B. \tag{17}$$

From $E=A+B$, we have $A=A.E$, $B=B.E$, and from (17), $A.B'=A.E.B'\subset A.B=0$, *i.e.* $A.B'=0$;
similarly,

$$A'.B=A'.E.B\subset A.B=0,\ \textit{i.e.}\ A'.B=0.$$

We obtain thus (15), which proves that the sets A and B are separated. The set $E=A+B$ is, therefore, not connected. We have thus proved that the condition of our theorem is necessary.

Suppose now that the set E is not connected. It can, therefore, be represented as a sum of two separated sets A and B which satisfy (15). From $E=A+B$ and (15), we get

$$A'.E=A'.(A+B)=A'.A+A'.B=A'.A\subset A,$$

whence $A'.E\subset A$; this proves that A is closed in E. Similarly, it may be shown, that B is closed in E. The set E is, therefore, by (15), the sum of two mutually exclusive, non-null sets closed in E. The condition of the theorem is, therefore, sufficient.

If E be closed, sets closed in E are obviously also closed (§ 7) and conversely; we thus get as an immediate deduction from Theorem 12,

Theorem 13. *In order that a closed set be connected, it is necessary and sufficient that it shall not be the sum of two closed, mutually exclusive, non-null sets.*[5]

We now proceed to prove

Theorem 14. *A connected set which is contained in the sum of two separated sets is contained in one of these sets.*

[5]An analogous theorem follows, as is easily seen, also for open sets.

Proof. Let us suppose that E is connected and contained in the sum of the two separated sets A and B. We have then (15) and $E \subset (A+B)$, whence

$$E = E.(A+B) = E.A + E.B.$$

Put

$$E.A = A_1,\ E.B = B_1;$$

then

$$A_1 \subset A,\ B_1 \subset B,$$

and

$$A_1.B_1 = E.A.B = 0$$

(since, from (15), $A.B=0$). If we had $A_1 \neq 0$, $B_1 \neq 0$, the sets A_1 and B_1 would satisfy all conditions of Theorem 11 and so would be separated, which is impossible, since

$$E = E.A + E.B = A_1 + B_1$$

and E is connected. Hence, either $A_1 = 0$, or $B_1 = 0$, and, therefore, either

$$E = B_1 = E.B \subset B,$$

or

$$E = A_1 = E.A \subset A.$$

Theorem 15. *If E be a connected set, then every set containing E and contained in $\overline{E}$ is connected.*

Proof. Let us suppose that E is connected, and let T denote a set such that $E \subset T \subset \overline{E}$. If T were not connected, we could write $T = A+B$, where the sets A and B are separated. Now E is connected, and $E \subset T = A+B$; therefore, by Theorem 14, it is contained in one of the sets A and B, say $E \subset A$. Hence, by Theorem 1, $E' \subset A'$, and, therefore, $E'.B \subset A'.B$. But $A'.B=0$, A and B being separated; we thus have $E'.B=0$.

Similarly, $E.B=0$, since $E \subset A$, and $A.B=0$. The supposition regarding T requires, then, that

$$B = (A+B).B = T.B \subset \overline{E}.B = (E+E').B = E.B + E'.B = 0,$$

which is impossible. The set T must, therefore, be connected. In particular, if we put $T = \overline{E}$, it follows from the above theorem that *the enclosure of a connected set is a connected set.*

Theorem 16. *A connected set which contains elements of each of two complementary sets contains at least one element of their frontier.*

Proof. Let S be a connected set containing elements of each of two complementary sets E and T. Put $A=E.S$, $B=T.S$; evidently, $S=A+B$, and $A\neq 0$, $B\neq 0$, $A.B=0$. Since S is connected, the sets A and B cannot be separated, and so all conditions of (15) cannot be satisfied; it follows that $A.B'+A'.B\neq 0$. But, from $A=E.S\subset E, B=T.S\subset T$, and Theorem 1, we have $A'\subset E', B'\subset T'$, and, therefore, $A.B'+A'.B\subset E.S.T'+T.S.E'$; thus

$$S.(E.T'+E'.T)=E.S.T'+T.S.E'\supset A.B'+A'.B\neq 0,$$

i.e. $S.(E.T'+E'.T)\neq 0$, which establishes the theorem, since from $T=CE$ and (7), the set $E.T'+E'.T$ is the frontier of the set E.

Theorem 17 (Hausdorff). *If every two elements of a set E belong to some connected subset of the set E, then the set E is connected.*

Proof. Suppose that the set E is not connected, *i.e.* $E=A+B$, where A and B are separated. Let a denote any element of A, b any element of B, E_1 any subset of E containing the elements a, b. Putting

$$A_1=A.E_1,\ B_1=B.E_1,$$

we shall have $A_1\neq 0$, $B_1\neq 0$, since $a\in A.E$, $b\in B.E$, and obviously $A_1\subset A$, $B_1\subset B$; by Theorem 11, the sets A_1, B_1 will be separated. The set E_1 is, therefore, not connected. We have thus proved that no subset of E, containing the elements a and b is connected, contrary to the hypothesis of our theorem. The set E must, therefore, be connected.

LEMMA (Hausdorff). *The sum of two connected sets having an element in common is connected.*

Proof. Let us suppose that $E=E_1+E_2$, where E_1 and E_2 are connected sets containing a common element a, and let us assume that E is not connected, *i.e.* $E=A+B$, where A and B are separated. Since $a\in E=A+B$, a belongs to one of the sets A, B, say to A. On the other hand, since $B\neq 0$, and $B\subset E=E_1+E_2$, at least one of the sets $B.E_1$, $B.E_2$, the former say, is not a null set. Putting $A_1=A.E_1$, $B_1=B.E_1$, then $A_1\neq 0$ (since $a\in A.E_1=A_1$), $B_1\neq 0$, and

$A_1 \subset A$, $B_1 \subset B$; by Theorem 11, then, the sets A_1 and B_1 are separated, and, therefore, the set $E_1 = E.E_1 = (A+B).E_1 = A.E_1 + B.E_1 = A_1 + B_1$ cannot be connected, contrary to hypothesis. The set E must, therefore, be connected.

Theorem 18 (Hausdorff). *The sum of any aggregate of connected sets, every pair of which has an element in common, is a connected set.*

Proof. Let S denote the sum of the connected sets E, every pair of which has at least one element in common, and let a and b be any two elements of S. There are, therefore, two terms E_1 and E_2 of the sum S, such that $a \epsilon E_1$, $b \epsilon E_2$, where we can have $E_1 \neq E_2$ or $E_1 = E_2$. In any case, $E_1.E_2 \neq 0$ by hypothesis, and, since E_1 and E_2 are connected, the set $E_1 + E_2$ is connected, by the above lemma. Finally, since this set contains the elements a and b, we have proved that the set S satisfies the conditions of Theorem 17, and is, therefore, connected.

Let E be a given set and a one of its elements. There are connected sets contained in E and containing the element a, *e.g.*, the set consisting of the single element a. Denote by $C(a)$ the sum of all connected sets containing a and contained in E; by Theorem 18, this will be a connected set. The set $C(a)$, *i.e.* the greatest connected subset of E containing the element a, is called the *component* of E corresponding to the element a. In a particular case, the component $C(a)$ may reduce to the element a itself. It follows at once from the definition of components, and from Theorem 18, that components corresponding to two different elements of E are either identical, or have no elements in common.

A connected set S is said to be *locally connected* at a point p,[6] if for every open set U containing p there is a connected subset T of S, contained in U and containing p, and such that $p \bar{\epsilon} (S-T)'$. A connected set, which is locally connected at every one of its points, is said to be uniformly connected.

9. Let P and Q be given sets. Suppose that each element of the set P is correlated with some element of the set Q (where the

[6]According to the terminology of Mazurkiewicz. Hahn employs here the term: *zusammenhangend im kleinen*.

same element of Q may be correlated with different elements of P, and where there may be elements of Q which are not correlated with any of the elements of P). Each such correlation is said to determine a *mapping* (single-valued) of the set P on the set Q, or to define a single-valued *function* of the elements of the set P; thus if q denotes an element of the set Q, which is correlated with an element p of the set P, we write

$$q=f(p)$$

and call q the *transform* of the element p.

The set P^* of all the elements $f(p)$, correlated with the elements p of the set P is called the *transform of the set P* (obtained by means of the function f) and is denoted by $f(P)$. If, further, p_1 and p_2, different elements of the set P, are correlated always with different elements $f(p_1)$ and $f(p_2)$, we say that the function f establishes a *one-to-one* or (1, 1) correspondence between the elements of the sets P and P^*; corresponding to each element q of the set P^* there exists, then, one and only one element $p=\phi(q)$ of the set P, such that $\phi(q)=p$, and the function ϕ establishes a mapping *inverse* to that which is established by the function f, namely a mapping of the set P^* on the set P, and we have $\phi(P^*)=P$. We then say that the function f has an *inverse function* (or, that it is *biuniform* in the set P).

In particular, the sets P and P^* may be identical; we then have a mapping of the set P on itself.

If f be a single-valued function defined for the elements of the set P, and if E be any subset of P, then the set of all elements $f(p)$, corresponding to the elements p of the set E, will be denoted by $f(E)$.

It is easily seen that for every (single-valued) function f, defined in the set P, we have

$$f(E_1+E_2)=f(E_1)+f(E_2), \text{ for } E_1\subset P,\ E_2\subset P,$$

and generally, for every sum $S=\sum E$ of sets $E\subset P$,

$$f(\sum E)=\sum f(E),$$

where the summation extends over all the sets E which form the sum S. Hence *the transform of a sum is the sum of the transforms.*

As regards the transform of a difference, we can merely state that in general

$$f(E_1 - E_2) \supset f(E_1) - f(E_2), \text{ for } E_1 \subset P,\ E_2 \subset P$$

i.e. the transform of a difference contains the difference of the transforms.

If $E_1 \subset E_2 \subset P$, we have evidently $f(E_1) \subset f(E_2)$, *i.e. the transform of a subset of a set is a subset of the transform of the set.* From this it follows at once that for every product $\prod E$ of given sets $E \subset P$, we have

$$f(\prod E) \subset \prod f(E) \tag{18}$$

(where the product on the right hand side extends over all the factors E of the product $\prod$), *i.e. the transform of a product is contained in the product of the transforms.*

If, however, the function f establishes a (1, 1) mapping (of the elements of P) then, for every product $\prod E$ of sets $E \subset P$, we have

$$f(\prod E) = \prod f(E). \tag{19}$$

In fact, let us suppose that the function f establishes a (1, 1) mapping of the elements of the set P, and let ϕ be its inverse function, defined in the set $P^* = f(P)$. Let further $f(E) = E^*$ for every factor E of the product $\prod E$; since $E \subset P$, we shall have $E^* \subset P^*$ and, therefore, from the result just established, that the transform of a product is contained in the product of transforms,

$$\phi(\prod E^*) \subset \prod \phi(E^*), \tag{20}$$

where the product extends over all the factors E of the product $\prod E$. It follows at once from (20) that

$$f[\phi(\prod E^*)] \subset f[\prod \phi(E^*)],$$

whence

$$\prod f(E) \subset f(\prod E);$$

on account of (18) this gives (19).

Similarly, it can be easily shown, that, if the function f establishes a (1, 1) mapping of the elements of the set P, and if $E_1 \subset P$, $E_2 \subset P$, then

$$f(E_1 - E_2) = f(E_1) - f(E_2).$$

Let $f(p)$ denote a single-valued function (not necessarily biuniform), defined in the set P. For each subset T of the set $f(P)$ denote by $g(T)$ the set of all those elements p of the set P for which $f(p) \in T$. It can easily be proved, that

$$g(\Sigma T) = \Sigma g(T)$$

for every sum ΣT of sets contained in $f(P)$; also that

$$g(T_1 - T_2) = g(T_1) - g(T_2), \text{ for } T_1 \subset f(P),\ T_2 \subset f(P),$$

and

$$g(\Pi T) = \Pi g(T)$$

for every product ΠT of sets contained in $f(P)$.

In the case, where the function f is biuniform in the set P and ϕ denotes the inverse function of f, we have, as is easily seen, $g(T) = \phi(T)$, for $T \subset f(P)$.

It may be noted that, in accordance with Lebesgue's notation, the set $g(T)$ may be denoted by $E\,[f(p) \subset T]$.

10. Let E be a given set, f a function (single-valued), defined for the elements of the set E. We shall assume that the set E is composed of elements of some class K, which satisfies hypothesis (I), and that the values of the function f are elements of the same class K, or of another class K_1, provided that the latter satisfies also hypothesis (I) (when K_1 is substituted for K). In fact, for the discussion of §§ 10-14 it would be sufficient to assume that open sets which do not necessarily satisfy hypothesis (I) are defined in the classes K and K_1.

A function f is said to be *continuous on the set E at an element p_0* of that set, if, for every open set V such that $f(p_0) \in V$, there exists an open set U such that $p_0 \in U$, and such that the condition

$$p \in U.E$$

implies

$$f(p) \in V$$

(or, what amounts to the same thing, $f(p) \in V.T$, where $T = f(E)$, since $f(p) \in T$, for $p \in E$).[7]

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[7]We note that the continuity of a function could be also defined as a so-called limit-continuity (Limesstetigket, see *e.g.* H. Tietze, *Über Analysis Situs*, Hamburg, 1923, p. 2).

The infinite sequence $p_1, p_2, \ldots$ of the elements of the class K is said to have for its limit the element p (written, $\lim_{n \to \infty} p_n = p$), if, for every open set U

If the function f is continuous in the set E at every element of that set, the set $T=f(E)$ is said to be a *continuous transform* of the set E, obtained by means of the function f. It is easily seen, that if $p_0 \epsilon E_1 \subset E$ and if the function f is continuous in the set E at the element p_0, then f is continuous also in the set E_1 at the element p_0.

Theorem 19. *If the function f be continuous in the set E at the element p_0 of that set, and the function g be continuous in the set $T=f(E)$ at the element $q_0=f(p_0)$ of that set, then the function $\phi(p)=g[f(p)]$ is continuous in the set E at the element p_0.*

Proof. Let W be an open set such that

(21) $$\phi(p_0) \epsilon W.$$

Since $\phi(p_0)=g[f(p_0)]=g(q_0)$, (21) gives

$$g(q_0) \epsilon W;$$

since the function g is continuous in the set T at the element q_0, there exists an open set V, such that

(22) $$q_0 \epsilon V$$

and

(23) $$g(q) \epsilon W, \text{ whenever } q \epsilon V.T.$$

From (22), since $q_0=f(p_0)$, we have $f(p_0) \epsilon V$; since the function f is continuous in the set E at the element p_0, there exists an open set U, such that

(24) $$p_0 \epsilon U,$$

and

(25) $$f(p) \epsilon V.T, \text{ whenever } p \epsilon U.E.$$

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containing p, there exists an integer μ, such that $p_n \epsilon U$, whenever $n>\mu$. A function $f(p)$ is said to possess *limit-continuity* in the set E at elements p of that set, if, for every infinite sequence $p_1, p_2, \ldots$ of elements of E for which $\lim_{n\to\infty} p_n=p$, we have also $\lim_{n\to\infty} f(p_n)=f(p)$. Clearly a function which is continuous at a given element in the sense described in the text, has also limit-continuity at that element. It could, however, be shown, that the converse is not necessarily true (unless special assumptions be made with regard to the class K; *cf.* § 31 and § 33, Theorem 40).

From (25) and (23), we have

$$g[f(p)] \epsilon W, \text{ whenever } p \epsilon U.E,$$

or

(26) $$\phi(p) \epsilon W, \text{ whenever } p \epsilon U.E.$$

We have, therefore, proved that for every open set W, for which (21) holds, there exists an open set U, for which (24) and (26) are satisfied; the continuity of the function ϕ in the set E at the element p_0, is thus established.

In particular, if f be continuous in the whole set E and if g be continuous in the whole set $T=f(E)$, then the function $\phi(p)=g[f(p)]$ is continuous in the whole set E. In brief, *a continuous transform of a continuous transform of a given set is a continuous transform of that set.*

Theorem 20. *In order that a function f, defined in a set E, be continuous in that set, it is necessary and sufficient that the transform of a limit element of any subset of E and belonging to E be an element or a limit element of the transform of that subset of E.* In other words, *it is necessary and sufficient, that*

(27) $$f(E.E_1') \subset f(E_1)+[f(E_1)]', \textit{ whenever } E_1 \subset \text{E}.$$

Proof. Let f be a function defined and continuous in the set E, let E_1 be a subset of E, and p_0 an element of the set $E.E_1'$, and suppose that $f(p_0) \bar{\epsilon} f(E_1)$. Let further V denote an open set such that $f(p_0) \epsilon V$. It follows, from the definition of continuity of the function f in the set E at the element p_0, that there exists an open set U, such that $p_0 \epsilon U$ and

(28) $$f(p) \epsilon V, \text{ whenever } p \epsilon U.E.$$

But, since $p_0 \epsilon E.E_1'$, we have $p_0 \epsilon E_1'$; since $p_0 \epsilon U$ and U is open, there exists an element $p \epsilon U.E_1$, from which it results that $f(p) \epsilon f(E_1)$, and, from (28), $f(p) \epsilon V$; since $f(p_0) \bar{\epsilon} f(E_1)$, we have $f(p) \neq f(p_0)$. Hence, in every open set containing $f(p_0)$, there exists an element of $f(E_1)$ different from $f(p_0)$, which proves that $f(p_0) \epsilon [f(E_1)]'$. We have thus proved that the condition of Theorem 20 is necessary.

Suppose now, that f is a function defined in the set E, and assume that f is not continuous in E at an element p_0 of that set. It follows from the definition of continuity that there exists an open set V, such that $f(p_0) \in V$, and that in every open set U containing p_0, there is an element p of the set E, such that $f(p) \bar{\in} V$. Denote by E_1 the set of all the elements p of the set E for which $f(p) \bar{\in} V$. Then in every open set U containing p_0 there is an element p of the set E_1, which element is different from p_0, since $f(p_0) \in V$, but $f(p) \bar{\in} V$ from the definition of E_1. Therefore, $p_0 \in E_1'$, and so $p_0 \in E.E_1'$ (since $p_0 \in E$). But from the definition of the set E_1 it follows that $V.f(E_1)=0$, and, therefore, $V.[f(E_1)]'=0$ (since V is open); since $f(p_0) \in V$, we have $f(p_0) \bar{\in} f(E_1)$, and $f(p_0) \bar{\in} [f(E_1)]'$, but since $p_0 \in E.E_1'$, we have $f(p_0) \in f(E.E_1')$. Hence the set $E_1 \subset E$ does not satisfy condition (27). Therefore, if the function f is not continuous in the (whole) set E the relation (27) is not true. It follows, therefore, that (27) is a sufficient condition for the continuity of the function f in the set E.

Theorem 20 is thus proved.

Let now f be a function biuniform and continuous in E. Let further, E_1 denote a subset of E, and p_0 an element of $E.E_1'$, and V any open set containing $f(p_0)$. Since f is continuous there exists an open set U, containing p_0, for which (28) holds. But from $p_0 \in E.E_1'$ we have $p_0 \in E_1'$; since $p_0 \in U$ and U is open, there exists an element $p \in U.E_1$, different from p_0, whence $f(p) \in f(E_1)$ and $f(p) \neq f(p_0)$ (since f is biuniform in E). We conclude then (since V is any open set containing $f(p_0)$) that $f(p_0) \in [f(E_1)]'$. We have thus proved, that

(29) $$f(E.E_1') \subset [f(E_1)]', \text{ whenever } E_1 \subset E.$$

If, on the other hand, a function f defined in E satisfies (29) it certainly satisfies (27) and, therefore, by Theorem 20, it is a continuous function in E. We have then

Theorem 20a. *In order that a function f, defined in the set E and biuniform in that set, be continuous in E, it is necessary and sufficient that*

$$f(E.E_1') \subset [f(E_1)]', \text{ for } E_1 \subset E.$$

We may also deduce from Theorem 20, the following

COROLLARY. *If f be continuous in E, and T_1 be a set closed in the set $T=f(E)$, then the set E_1 of all the elements p of E for which $f(p) \epsilon T_1$ is closed in E.*

Proof. Let p_0 denote a limit element of E_1 which belongs to E. We have then $p_0 \epsilon E.E_1'$ and so, by Theorem 20, either $f(p_0) \epsilon f(E_1) = T_1$ or $f(p_0) \epsilon T_1'$. In the first case, $p_0 \epsilon E_1$ by the definition of E_1, while in the second case (since $p_0 \epsilon E$), $f(p_0) \epsilon T_1'.T$, and, therefore, $f(p_0) \epsilon T_1$, this set being closed in T (§ 7), and so again $p_0 \epsilon E_1$. Consequently, in either case, the relation $p_0 \epsilon E.E_1'$ implies that $p_0 \epsilon E_1$; this proves that $E.E_1' \subset E_1$, *i.e.* that E_1 is closed in E.

11. Theorem 21. *A continuous transform of a connected set is connected.*

Proof. Let E be a connected set, $T=f(E)$ its continuous transform; suppose that T is not connected.

There is, then, a division $T=A_1+B_1$, such that

(30) $$A_1 \neq 0,\ B_1 \neq 0,\ A_1.B_1=A_1.B_1'=A_1'.B_1=0.$$

Denote by A the set of all elements p of E for which $f(p) \epsilon A_1$, and by B the set of all those elements p of E for which $f(p) \epsilon B_1$. Since $A_1 \neq 0$, $B_1 \neq 0$, $A_1+B_1=T=f(E)$, $A_1.B_1=0$, we have, evidently, $A \neq 0$, $B \neq 0$, $A.B=0$, $A+B=E$; since E is connected, we cannot have simultaneously $A.B'=A'.B=0$. Suppose $A.B' \neq 0$. There exists, then, an element $p_0 \epsilon A.B'$. Let V be any given open set such that $f(p_0) \epsilon V$. Since $p_0 \epsilon A.B' \subset A \subset E$, and since f is continuous in E, there exists an open set U, such that $p_0 \epsilon U$ and

(31) $$f(p) \epsilon V, \text{ for } p \epsilon U.E.$$

Since $p_0 \epsilon A.B' \subset B'$, and from the definition of a derived set there exists an element $p_1 \neq p_0$, such that $p_1 \epsilon U.B \subset U.E$, and so, by (31), $f(p_1) \epsilon V$. But, since $p_1 \epsilon U.B \subset B$ and $f(B)=B_1$, we have $f(p_1) \epsilon B_1$; but $p_0 \epsilon A.B' \subset A$, from which it follows that $f(p_0) \epsilon f(A)=A_1$; the relation $A_1.B_1=0$ gives, therefore, $f(p_1) \neq f(p_0)$. Hence, every open set V, containing $f(p_0)$, contains at least one element $f(p_1)$ of B_1 different from $f(p_0)$, and so $f(p_0) \epsilon B_1'$. Since $f(p_0) \epsilon A_1$, it results that $A_1.B_1' \neq 0$, contrary to (30).

Similarly, it can be shown that the assumption $A'.B \neq 0$ also leads to a contradiction. The set T must, therefore, be connected.

It is easily seen that the set of all real numbers can be considered as the class K which satisfies hypothesis (I), if, apart from the null set, open sets are defined to mean sets U of real numbers having the following properties: that if $x_0 \epsilon U$ there exists two numbers a and b such that $a<x_0<b$, and that every number x such that $a<x<b$ belongs to U.

A connected set of real numbers has, evidently, the property that, if two numbers a and b belong to it, so does every number contained between a and b. It follows that a connected set of real numbers must be an interval, closed or otherwise, finite or infinite. In fact, suppose that $a \epsilon E$, $b \epsilon E$, $a<c<b$, and $c \bar{\epsilon} E$; if we denote by U_1 and U_2 the set of all real numbers $< c$, and $> c$ respectively, then putting $E_1=E.U_1$, $E_2=E.U_2$ (since U_1 and U_2 are open sets), we shall have $E=E_1+E_2$, *i.e.* E is expressible as a sum of two separated sets.

If a function whose values are real numbers be defined in a set contained in the class K, we say that this function is a *real function* defined in that set.

From Theorem 21 and the properties of connected sets of real numbers deduced above the following result may be deduced at once.

Corollary. *If a real continuous function f be defined in a connected set E, contained in the class K, and if y_1 and y_2 are any two values which the function takes at the elements of E, then f takes, in E, every value intermediate between y_1 and y_2.*

This theorem is a generalization of a well-known result in Analysis, concerning a similar property of real functions of a real variable which are continuous in a given interval.

It is easily seen that the converse of this theorem is also true. *If every real function, defined and continuous in E, takes in the set E every value between two assumed values, then E is connected.* In fact, if E were the sum of two separated sets A and B, then the function $f(p)$ equal to zero in A and equal to unity in B, would be obviously continuous in the whole set E, but would not take any value between 0 and 1.

12. If the function f establishes a (1, 1) mapping of the set E on the set U, and if f is continuous in the whole set E, and if ϕ, the inverse function of f, is continuous in the whole set U, then U is said to be a biuniform and bicontinuous transform of E. Evidently, the set E is then also a biuniform and bicontinuous transform of U.

Two sets E and U, each of which is a biuniform and bicontinuous transform of the other, are said to be *homeomorphic*, or, in symbols, $E\,h\,U$;[8] if we wish to express that the function f transforms the set E into U in a (1, 1) and bicontinuous manner, we write $E\,h_f\,U$.

Evidently, if

$$E\,h_f\,U,\ E_1 \subset E,\ \text{and}\ f(E_1) = U_1,$$

then

$$E_1\,h_f\,U_1;$$

hence, if two sets are homeomorphic, any two corresponding subsets of these sets are homeomorphic.

It follows, from Theorem 19, that if

$$E\,h_f\,U,\ \text{and}\ U\,h_g\,T,$$

and if we put

$$\phi(p) = g[f(p)]\ \text{in}\ E,\ \text{then}$$

$$E\,h_\phi\,T;$$

the relation of homeomorphism is, therefore, *transitive*.

Theorem 22. *The necessary and sufficient condition for two sets to be homeomorphic is the existence of a (1, 1) correspondence between the elements of the sets such that the transform of a limit element (belonging to the set) of any subset of either set, is a limit element of the transform of that subset.*

Proof. Suppose, $E\,h_f\,U$, and let p_0 be an element of E, and also a limit element of E_1, a subset of E, *i.e.* $p_0 \in E.E_1'$, where $E_1 \subset E$. The function f, which establishes a homeomorphic mapping of E on U, is continuous and biuniform in E; by Theorem 20a, the relation (29) (§ 10) is satisfied, and so $f(p_0) \in [f(E_1)]'$,

[8] It is evident, that in such a case $U\,h\,E$; the relation of homeomorphism is therefore, *symmetrical*.

since $p_0 \epsilon E.E'$, $E_1 \subset E$, which proves that $f(p_0)$ is a limit element of the transform of E_1. The condition of the theorem is, therefore, necessary.

Let now E and T be two sets such that between their elements a (1, 1) correspondence f can be established satisfying the condition of Theorem 22. Since the conditions are symmetrical with respect to E and T, it will be obviously sufficient to prove that f is continuous in E. From the conditions of Theorem 22, we get at once (29); since f is biuniform in E, it must accordingly be continuous in E, by Theorem 20a. The condition of Theorem 22 is, therefore, sufficient.

Theorem 22 is, therefore, proved.

COROLLARY 1. *In order that two sets be homeomorphic, it is necessary and sufficient that there exist such a (1, 1) correspondence between their elements, that the transform of a subset of either set closed in the set is closed in the transform of that set.*

Proof. Suppose $E\, h_f\, U$, and let E_1 denote a subset of E which is not closed in E. There exists, therefore, an element a, such that $a \epsilon E.E_1'$, but $a \bar{\epsilon} E_1$. Since f is biuniform, $f(a) \bar{\epsilon} f(E_1)$, and since $a \epsilon E.E_1' \subset E_1'$, then, by Theorem 22, $f(a) \epsilon [f(E_1)]'$, and, therefore, also $f(a) \epsilon f(E).[f(E_1)]'$. Therefore, $f(E_1)$ is not closed in $f(E)$ (since $f(a) \bar{\epsilon} f(E_1)$). We have thus proved that the transform of a subset of E not closed in E, is a subset of $f(E)$, which is not closed in $f(E)$. Hence the condition of Corollary 1 is necessary.

Suppose now that a (1, 1) correspondence $U=f(E)$ satisfying the condition of the corollary can be established between the elements of the sets E and U, and let a be an element of E but not a limit element of E_1, a subset of E. Then $f(a) \bar{\epsilon} [f(E_1)]'$. For let $E_2 = E_1-(a)$, $E_3=\bar{E}_2.E$; E_3 is, obviously, closed in E (as a product of E and the closed set $\bar{E}_2$), and does not contain a, since $\bar{E}_2=E_2+E_2'$ and $a \bar{\epsilon} E_2=E_1-(a)$, and $a \bar{\epsilon} E_2'$ (since $a \bar{\epsilon} E_1'$). By the condition of the corollary $f(E_3)$ is closed in $f(E)$, and, hence, $f(a) \bar{\epsilon} [f(E_3)]'$; for $f(a) \bar{\epsilon} f(E_3)$ (since $a \bar{\epsilon} E_3$, and f is biuniform). But $E_3 = \bar{E}_2.E \supset E_2 \supset E_1-(a)$, and hence, $f(E_3) \supset f(E_1-(a))$, and $f(E_3) \supset f(E_1) - f(a)$; if then $f(a) \epsilon [f(E_1)]'$, we should have $f(a) \epsilon [f(E_3)]'$, which is not the case. We have, therefore, $f(a) \bar{\epsilon} [f(E_1)]'$.

We have thus proved that the transform of an element of E

which is not a limit element of $E_1 \subset E$ is an element of $U=f(E)$, which is not a limit element of the transform $f(E_1)$ of E_1. Hence, as is easily seen, the condition of Theorem 22 is satisfied, and so $E\, h_f\, U$. The condition of the corollary is thus seen to be sufficient, and so Corollary 1, to Theorem 22, is proved.

From Corollary 1, we get at once

COROLLARY 2. *The necessary and sufficient condition for two classes K_1 and K_2 to be homeomorphic is that there exist a (1, 1) correspondence between their elements such that the transform of a closed set of each class is always a closed set of the other.*

Passing to complements, from Corollary 2 we obtain immediately

COROLLARY 3. *The necessary and sufficient condition for two classes K_1 and K_2 to be homeomorphic is that there exist a (1, 1) correspondence between their elements such that the transform of an open set of each class is always an open set of the other.*

The following corollary may also be easily deduced from Theorem 22.

COROLLARY 4.[9] *The necessary and sufficient condition for a biuniform function to establish a homeomorphic mapping of the set E on the set $U=f(E)$, is that it should satisfy the condition*

$$f(E.E_1') = f(E).[f(E_1)]',$$

for every $E_1 \subset E$.

COROLLARY 5. *The necessary and sufficient condition for two classes K and K_1 to be homeomorphic is the existence of a (1, 1) correspondence between their elements such that derived sets of corresponding sets are corresponding sets, i.e., if $U=f(E)$, then $U'=f(E')$.*

If a set P is homeomorphic with a certain subset of a set Q and Q is homeomorphic with a certain subset of P, then P and Q are said to have the same *dimensional type* (type de dimensions, Fréchet; Homöie, Mahlo). We denote this by $dP=dQ$. Obviously, if $dP=dQ$, then also $dQ=dP$, and, if $dP=dQ$ and $dQ=dR$, then $dP=dR$. Homeomorphic sets have evidently the same dimensional type, but the converse is not necessarily true. If P is

[9] S. Saks, *Fund. Math.*, vol. V, p. 291.

homeomorphic with a certain subset of Q, but Q is not homeomorphic with any subset of P, we say that P has a smaller dimensional type than Q, and we write $dP<dQ$ (or $dQ>dP$). It is easily seen that if $dP<dQ$, and $dQ<dR$, then $dP<dR$.

13. A property of a set E, which is possessed by every set homeomorphic with E, is said to be a *topological property* of E. The purpose of Topology is the investigation of topological properties of sets, *i.e.* of properties invariant under biuniform and bicontinuous transformation.

We shall give a few examples of topological properties.

It follows from Theorem 21, that the *connectivity of a set is a topological property* (since by Theorem 21 the connectivity of a set is invariant under any continuous transformation of the set).

Density-in-itself of a set is a topological property. We shall even show that density-in-itself is invariant under every biuniform and merely continuous transformation.

Let E denote a given set dense-in-itself, and f a function biuniform and continuous in E. Let q_0 be any element of $U=f(E)$. It is required to show that q_0 is a limit element of U.

Let V denote any given open set, such that $q_0 \epsilon V$. Since $q_0 \epsilon U$, there exists an element p_0 of E such that $f(p_0)=q_0$.

It follows from the definition of continuity of the function (§ 10), and from $f(p_0) \epsilon V$, that there exists an open set W, such that $p_0 \epsilon W$, and the condition $p \epsilon W.E$ implies $f(p) \epsilon V$.

Again, since E is dense-in-itself, the open set W containing the element p of E contains an element p_1 of E different from p. From $p_1 \neq p$ and the properties of the function f, we have $q_1=f(p_1)\neq f(p)$. On the other hand, since $p_0 \epsilon W.E$ we have, by the definition of W, $f(p_1) \epsilon V$; also $p_1 \epsilon E$ implies that $f(p_1) \epsilon U$, and, therefore, there exists in the set V an element $q_1=f(p_1)$ of U different from q_0. Hence q_0 is a limit element of U.

From the fact that density-in-itself of a set is a topological property it follows at once that *the property of being a scattered set is a topological one* (§ 6). (The property of being scattered is not, however, invariant under a (1, 1) and merely continuous transformation; for a continuous and biuniform transform of a scattered set may be dense-in-itself; *e.g.*, the set of rational numbers may be a continuous and biuniform transform of the set of natural numbers). It is easily seen that a homeomorphic mapping transforms the nucleus of a set into the nucleus of the transform.

A set which does not contain any of its limit elements is said to be *isolated*. (Hence in order that a set E be isolated it is necessary and sufficient that $E.E'=0$.) It has been shown previously that a biuniform and continuous transformation maps a limit element of a set into a limit element of the transform. It follows at once, that *the property of being isolated is a topological property*.

14. An element q (belonging to E or not) such that every open set containing q contains a non-countable number of elements of E is said to be an *element of condensation* of the set E. This definition will be referred to as the Lindelöf definition.

Fréchet calls an element of condensation of a set E an element q (belonging to E or not) which is a limit element of every set obtained after a finite or countable set of elements have been removed from E.

It is easily seen that the definitions of Lindelöf and Fréchet are equivalent. In fact, if q be an element of condensation of E according to Lindelöf's definition and if P be any finite or countable set and U any open set containing q, then U contains a non-countable set of elements of E, and, therefore, also of the set $E-P$, and so q is a limit element of $E-P$, *i.e.* an element of condensation of E according to the definition of Fréchet.

On the other hand, if q is not an element of condensation of E according to Lindelöf's definition, there exists an open set U containing q, such that $P=U.E$ is finite or countable. But $U.(E-P)=U.E-U.E=0$; hence, q is not a limit element of the set $E-P$, and, therefore, not an element of condensation of E according to the definition of Fréchet.

The definitions of an element of condensation given by Lindelöf and Fréchet are thus seen to be equivalent.

We proceed to prove next that *the set of all elements of condensation of a given set* (belonging to that set or not) *is closed*. In fact, let E denote a given set and Q the set of all the elements of condensation of E (belonging to E or not). Let p be any limit element of Q, and U any open set containing p. Then U contains at least one element q of Q. But, since $q \epsilon U$, it follows from the definition of Q, that U contains a non-countable set of elements of E. Since U is any open set containing p, it follows that p is an element of condensation of E, and so an element of Q. Hence, Q contains every one of its limit elements and is, therefore, closed.

CHAPTER II

FRÉCHET'S CLASSES (H)

15. We shall introduce now two new axioms concerning the class K:

(II)
- (*iv*) *If p and q are two different elements of the class K, there exists an open set containing p, but not containing q;*
- (*v*) *The product of two open sets is an open set.*

We shall deduce now a series of results from hypotheses (I) and (II).

Theorem 23. *The derived set of a sum of two sets is the sum of the derived sets of these sets.*

Proof. Let E_1 and E_2 be two given sets (of elements of the class K). We have evidently from Theorem 1,

$$E_1' + E_2' \subset (E_1 + E_2)'; \tag{1}$$

it will, therefore, be sufficient to show that

$$E_1' + E_2' \supset (E_1 + E_2)'. \tag{2}$$

Suppose that p is an element of K such that $p \,\bar{\epsilon}\, E_1'$ and $p \,\bar{\epsilon}\, E_2'$. There exist, accordingly, open sets U_1 and U_2, such that $p \epsilon U_1$, $p \epsilon U_2$, $U_1.E_1 - (p) = 0$, $U_2.E_2 - (p) = 0$. From axiom (v), the set $U = U_1.U_2$ is open, where evidently $p \epsilon U$ and $U.(E_1 + E_2) - (p) = U.E_1 + U.E_2 - (p) \subset U_1.E_1 + U_2.E_2 - (p) = 0$, whence $p \,\bar{\epsilon}\, (E_1 + E_2)'$. The assumption that $p \,\bar{\epsilon}\, E_1'$, and $p \,\bar{\epsilon}\, E_2'$, implies that $p \,\bar{\epsilon}\, (E_1 + E_2)'$; therefore, (2) is proved. (1) and (2) give

$$(E_1 + E_2)' = E_1' + E_2'.$$

This result may be extended by induction to any finite number of sets.

From Theorem 23 and the expression $\overline{E}=E+E'$ for the enclosure of a set (§ 4) we get

$$\overline{E_1+E_2}=\overline{E}_1+\overline{E}_2,$$

i.e. the enclosure of the sum of two sets is the sum of the enclosures of these sets.

Theorem 24. *The derived set of a finite set is a null set.*

By Theorem 23, generalized to a finite number of terms, it will be sufficient to prove that the derived set of a set, consisting of one element only, is a null set. Let E, then, be a set consisting of one element p only. It follows from axiom (*iv*) that no element q is a limit element of E, whence $E'=0$.

Corollary. *If p be a limit element of E, then every open set containing p contains an infinity of different elements of E.*

In fact, suppose that some open set U containing p contains a finite number of elements of E. Put $E_1=E.U$, $E_2=E-U$. The derived set of E_1 is, by Theorem 24, a null set; hence $p\bar{\epsilon}E_1'$; but also $p\bar{\epsilon}E_2'$, since the open set U containing p does not contain any element of E_2. It follows, therefore, from Theorem 23, that p does not belong to $(E_1+E_2)'=E'$, *i.e.* p is not a limit element of E. This proves the corollary.

As a further corollary to Theorems 23 and 24 we note that *the derived set of a set does not change when any finite number of elements is removed from the set.* For if E_1 is a finite set contained in E, then, letting $E-E_1=E_2$, we shall have $E=E_1+E_2$, whence, by Theorem 23, $E'=E_1'+E_2'$; since E_1 is finite, $E_1'=0$ by Theorem 24, and, therefore, $E'=E_2'$.

Theorem 25. *The derived set of every set is a closed set.*

Proof. Let E denote a given set. It is required to show that $(E')'\subset E'$, *i.e.* if q is a limit element of E' then it is an element of E'. Suppose, then, that q is a limit element of E' and let U be any open set containing q.

Since q is a limit element of E', there exists in U an element $p\,\epsilon\,E'$. Since p is a limit element of E, the open set U, containing p, contains an infinity of different elements of E (Theorem 24, Corollary) and so an infinity of elements of E different from q. Hence every open set U, which contains q, contains an infinity of elements of E different from q, and so $q \in E'$.

It follows easily by induction from axiom (v) that *the product of any finite number of open sets is open.* From which, passing to complements (see proof of Theorem 3) we get at once

Theorem 26. *The sum of a finite number of closed sets is closed.*

We may note that it could be easily proved that axiom (iv) is equivalent to the theorem, that the derived set of a set consisting of one element is a null set.

16. A finite set or a set, every infinite subset of which has a derived set different from zero, is said to be *compact.* Obviously *a subset of a compact set is compact.*

It is easily seen that *the sum of a finite number of compact sets is compact.* (For if $E=E_1+E_2+ \ldots +E_n$, and if T be an infinite subset of E, then one at least of the sets $T.E_1$, $T.E_2, \ldots, T.E_n$ is infinite and so will have a derived set different from zero, if the sets $E_1, E_2, \ldots, E_n$ are compact; by Theorem 1, this derived set will be a subset of the derived set of T, whence $T' \neq 0$.)

The sum of a countable set of compact sets is said to be *semi-compact.* It is easily seen that a subset of a semi-compact set is semi-compact and that the sum of a countable set of semi-compact sets is semi-compact.

Theorem 27 (Cantor). *If*

$$E_1 \supset E_2 \supset E_3 \ldots$$

be an infinite decreasing sequence of closed, compact, non-null sets, then the product of these sets is not a null set.

Proof. From each of the sets E_n select a single element p_n. Let P_1 denote the set consisting of all the different terms of the sequence p_n ($n=1, 2, 3, \ldots$). If P_1 were finite, then at least one of its elements would occur in the sequence p_n an infinite number of times and would be, therefore, as is easily seen, a common element of all the sets E_n ($n=1, 2, 3, \ldots$).

We may, therefore, suppose that P_1 is infinite. As an infinite subset of the compact set E_1, P_1 will have a derived set P_1', which is not a null set. Denote by P_n the set consisting of all the different terms of the sequence $p_n, p_{n+1}, p_{n+2}, \ldots$; this set is obtained by removing a finite number of elements from P_1; hence (§ 15) $P_n' = P_1'$, for $n = 1, 2, \ldots$

But, since $p_k \in E_k \supset E_{k+1}$, we have $P_n \subset E_n$, whence

$$P_n' \subset E_n' \subset E_n,$$

since E_n is closed; therefore, since $P_n' = P_1'$,

$$E_n \supset P_1' \neq 0, \text{ for } n = 1, 2, 3, \ldots;$$

this establishes Theorem 27.

17. Theorem 28 (Borel). *Let E be a closed and compact set. If*

$$O_1, O_2, O_3, \ldots$$

be an infinite sequence of open sets, such that

$$E \subset O_1 + O_2 + O_3 + \ldots, \tag{3}$$

then, there exists a finite number n, such that

$$E \subset O_1 + O_2 + \ldots + O_n. \tag{4}$$

(In other words, a closed and compact set, which can be covered by a countable set of open sets, can be covered by a finite number of these sets.)

Proof. Put

$$O_1 + O_2 + \ldots + O_n = S_n,$$

$$K - S_n = F_n,$$

$$E . F_n = E_n,$$

for $n = 1, 2, 3, \ldots$

The sets S_n are open (as sums of open sets by (*iii*)); hence, their complements F_n are closed, and, therefore, E_n is closed (as a product of closed sets, Theorem 3), and compact (as a subset of the compact set E). If the sets E_n were not null sets for $n=1, 2, 3, \ldots$, then, by Cantor's theorem (Theorem 27, § 16), their product P would not be a null set, and we would have

$$P \subset E_n \subset F_n = K - S_n,$$

whence

$$P.S_n = 0,$$

and (since $O_n \subset S_n$) certainly

$$P.O_n = 0, \text{ for } n = 1, 2, 3, \ldots ;$$

from this and from (3)

$$P.E = 0,$$

which is impossible, since $P \neq 0$, and $P \subset E_n \subset E$.

There exists, therefore, an integer n, such that

$$E_n = 0,$$

i.e. $E.F_n = 0$, and so, since $F_n = K - S_n$ and $E \subset K$, we get

$$E \subset S_n,$$

i.e. the relation (4).

Borel's theorem can be stated more generally as follows:

If E be closed and compact and everyone of its elements be interior[1] *to at least one of the infinite sequence of sets* $P_1, P_2, P_3, \ldots$, *there exists a positive integer n, such that every element of E is interior to at least one of the sets*

$$P_1, P_2, \ldots, P_n.$$

To prove the above it would be sufficient to denote by O_k the interior of P_k (for $k=1, 2, 3, \ldots$) (§ 5) and to apply Theorem 28, which is justifiable, since the interior of each set is open.

18. For all sets A and B contained in the class K, satisfying hypotheses (I) (§ 1) and (II) (§ 15), we have, as we have seen (§§ 4 and 15),

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[1] *I.e.* an interior element.

1) $\overline{A+B}=\overline{A}+\overline{B}$,

2) $A\subset\overline{A}$,

3) $\overline{0}=0$,

4) $\overline{\overline{A}}=\overline{A}$.

In his work on the operation $\overline{A}$ in topology,[2] Kuratowski assumes the relations written above as axioms defining the symbol $\overline{A}$. He assumes, namely, that to each subset A of a given class K can be attached a certain set $\overline{A}$, satisfying the axioms 1)-4) but otherwise perfectly arbitrary, and he investigates what are the conclusions which follow from these four axioms (and corresponding definitions).

Theorems, obtained by Kuratowski in this manner will be true in every class K which satisfies hypotheses (I) and (II). There are, however, theorems which can be deduced from hypotheses (I) and (II), but which cannot be deduced from axioms 1)-4); such, for instance, is the theorem that $\overline{(a)}=(a)$ for every element of the class K.

19. Let K denote a given class satisfying hypotheses (I) and (II), and let a be any element of that class. We shall understand by a neighbourhood of an element a any open set containing a. It is easily seen, that neighbourhoods, thus defined, have the following four properties:

(α) *To every element a* (of the class considered) *corresponds at least one of its neighbourhoods; every neighbourhood of a contains a.*

(β) *If V_1 and V_2 are two neighbourhoods of a, there exists a neighbourhood V of a such that $V\subset V_1.V_2$.*

(γ) *For every pair of different elements there exists a neighbourhood of either not containing the other.*

(δ) *For every element b contained in a neighbourhood V of an element a, there exists a neighbourhood W of b, such that $W\subset V$.*

Property (α) follows directly from our definition of a neighbourhood and from the fact that K is an open set containing every element $a\in K$ (*i.e.* K is a neighbourhood of every element of the class K).

[2] *Fund. Math.*, vol. III, pp. 182 *et seq.*

Property (β) follows immediately from our definition of a neighbourhood and from axiom (*v*) (§ 15). Similarly, the property (γ) follows from the definition of a neighbourhood and from axiom (*iv*) (§ 15).

Finally, property (δ) follows directly from our definition of a neighbourhood. For, if $b \in V$, and if V is a neighbourhood of the element a, then V is open, and since $b \in V$, V is a neighbourhood of the element b.

Properties (α), (β), (γ), and (δ) are, therefore, proved.

Suppose now that with each element a of any given class K there is associated a certain set of subsets of K, which are called *neighbourhoods* of a; this association may be quite arbitrary, apart from having the subsets satisfy the conditions (α), (β), (γ), and (δ). Classes, in which neighbourhoods are so defined, are called by Fréchet classes (H).[3]

We have thus proved that if, in a class K satisfying hypotheses (I) and (II), we define the neighbourhood of an element a to be an open set containing a, then the class K will be a class (H).

Let K denote a given class (H), E a given set contained in K, and let a be a given element of the class K. Fréchet calls a a *limit element* of E, if every neighbourhood of a contains at least one element of E different from a. The set of all limit elements of E is called by Fréchet the *derived* set of the set E; a set containing its derived set is called a *closed* set, and the complement of a closed set an *open* set.

It is thus seen that, for the class K, satisfying hypotheses (I) and (II), in which neighbourhoods are open sets, these definitions are in accordance with those accepted in §§ 2 and 3.

Suppose, now, that K_1 and K_2 are two classes (H), consisting of the same elements, in which, however, neighbourhoods may be defined in a different way (*e.g.*, a neighbourhood V of a certain element a belonging to K_1 may not be a neighbourhood of a when considered as belonging to K_2). If, however, for every set E contained in K_1 and K_2 a limit element of $E \subset K_1$ is also a limit element of $E \subset K_2$, and conversely, then the classes K_1 and K_2 are described by Fréchet as *equivalent*. Derived sets of the same set

[3] *Annales de l'École Normale*, vol. XXXVIII (1921), p. 366.

contained in two equivalent classes are obviously identical, and every set, which is closed (open) when considered as a subset of one of these classes, is also closed (open) when considered as a subset of the other.

It is easily seen that for two classes (H), K_1 and K_2 (consisting of the same elements) to be equivalent, it is necessary and sufficient that, for every neighbourhood V_1, of any element a of the class K_1, there should exist a neighbourhood V_2 of a, considered in K_2, such that $V_2 \subset V_1$, and conversely.

Let K be any given class (H). We shall show that an equivalent class (H) will be obtained if we consider the open sets of K to be the neighbourhoods of elements of K.

We shall investigate first which sets of the class K are open (according to the above definition of Fréchet).

It will be shown that every open set of the class K is identical with the sum of an aggregate of neighbourhoods of elements of the class K.

In fact, let U denote the sum of a set of neighbourhoods of any elements of K. If $b \epsilon U$, then it follows from the definition of the set U that there exists a neighbourhood V of a certain element a, such that $b \epsilon V \subset U$. But by (δ) there exists a neighbourhood W of b, such that $W \subset V$, and hence $W \subset U$. Put $K - U = F$; hence, $W.F = 0$, and so, since W is a neighbourhood of b, b cannot be a limit element of F. Hence, U does not contain any of the limit elements of $F = K - U$; therefore, every limit element of F belongs to F, *i.e.* F is closed, and so $U = K - F$ is open.

On the other hand, let U denote an open set of the class K. The set $F = K - U$ is, therefore, closed. If, then, $a \epsilon U$, a cannot be a limit element of F; there exists, then, a neighbourhood V of a such that $V.F = 0$, and hence $V \subset U$. Denote by S the sum of all neighbourhoods V of elements a, belonging to U, such that $V \subset U$; obviously $S = U$.

We have thus proved that for a set consisting of elements of a class K to be open, it is necessary and sufficient that it should be the sum of a class of neighbourhoods of elements belonging to K.[4]

[4]In particular, every neighbourhood of the class (H) is an open set.

Denote now by K_1 a class consisting of the same elements as the class K in which open sets containing a are considered to be neighbourhoods of the element a. We shall show that the class K_1, so defined, is a class (H) equivalent to the class K.

That the neighbourhoods in the class K_1, so defined, possess property (α) follows from the definitions of neighbourhoods of the class K_1, and from the fact that every neighbourhood in the class K is an open set (with respect to K).

Let now W_1 and W_2 be two neighbourhoods of an element a in the class K_1 and hence two open sets of K containing a. W_1 is, therefore, as previously proved, the sum of a certain set of neighbourhoods of K; since $a \epsilon W_1$, there exists a neighbourhood V of a certain element b in K, such that $a \epsilon V \subset W_1$. But, since $a \epsilon V$, from the property (δ) of neighbourhoods of the class K, it follows that there exists a neighbourhood V_1 of a in K, such that $V_1 \subset V \subset W_1$. Similarly, since $a \epsilon W_2$, we conclude that there exists a neighbourhood V_2 of a in K such that $V_2 \subset W_2$; but, by the property (β) of neighbourhoods in the class K, there exists a neighbourhood W of a in K, such that $W \subset V_1 . V_2$, and hence $W \subset W_1 . W_2$; consequently, in the class K_1, W is a neighbourhood of a, contained in $W_1 . W_2$: *i.e.* the neighbourhoods of the class K_1 satisfy (β). The property (γ) of the neighbourhoods of K_1 follows at once from the corresponding property of the neighbourhoods of K and from the fact that every neighbourhood of an element in K, being an open set in K, is at the same time a neighbourhood of that element in K_1.

Finally, the property (δ) of the neighbourhoods of K_1 follows directly from their definition.

Hence the neighbourhoods of the class K_1 possess the properties (α), (β), (γ), and (δ), and, therefore, K_1 is a class (H) of Fréchet.

Since every neighbourhood of an element in K is also its neighbourhood in K_1, it follows that, if a be a limit element of E in K_1, it will certainly be a limit element of E in K.

Suppose now that a is a limit element of E in K, and let W denote any neighbourhood of a in K_1. Then W is an open set of K and so the sum of a set of neighbourhoods of K. Since $a \epsilon W$, there exists a neighbourhood V of a certain element b of K, such that $a \epsilon V \subset W$. But, from $a \epsilon V$ and property (δ) of neighbourhoods of the class K, it follows that there exists a neighbourhood V_0 of a in K such that $V_0 \subset V$, and so $V_0 \subset W$. Since a is a limit element of E in K, there exists an element p of E, such that $p \neq a$, and $p \in V_0$ and so, since $V_0 \subset W$, certainly $p \in W$. We have thus proved that in every neighbourhood of a in K_1 there exists an element of E different from a, and that, therefore, a is a limit element of E in K_1.

Hence the classes K and K_1 are equivalent.

It follows, from the above, that open sets are the same in the class K as in K_1. Hence the neighbourhoods of elements in K_1 are open sets of K_1 containing these elements.

Furthermore it is easily seen that open sets of a class (H) satisfy hypotheses (I) and (II).

In fact, axiom (*i*) is satisfied, since the set of all elements of the class (H) is evidently a closed set in that class (according to Fréchet's definition).

That axiom (*ii*) is satisfied follows from the property (α) of neighbourhoods, and from the fact that the sum of any set of neighbourhoods of elements belonging to the class (H) is an open set of that class.

That axiom (*iii*) is satisfied follows from the necessary and sufficient condition, given above, for a set of the class (H) to be open in that class.

Axiom (*iv*) follows at once from property (γ) and from the fact that neighbourhoods of a class (H) are open sets of that class.

It remains, therefore, to prove axiom (*v*). Let W_1 and W_2 be two open sets of a given class (H), and let $P = W_1 . W_2$. Let a denote an element of P; hence, $a \in W_1 . W_2$. Since $a \in W_1$, it follows from the property of open sets of the class (H) that there exists a neighbourhood U of some element b, such that $U \subset W_1$. According to property (δ) there exists a neighbourhood V_1 of a, such that $V_1 \subset U$, and so certainly $V_1 \subset W_1$. Similarly, since $a \in W_2$, we conclude that there exists a neighbourhood V_2 of a, such that $V_2 \subset W_2$. But by property (β) there exists a neighbourhood V of a, such that $V \subset V_1 . V_2$, and hence certainly $V \subset W_1 . W_2 = P$.

Now it is evident that P is the sum of all these neighbourhoods (of its elements), contained in P; P is, therefore, an open set.

We have proved, therefore, in this article, that every class K, in which open sets are defined quite arbitrarily apart from their satisfying hypotheses (I) and (II), becomes a class (H), if open sets containing an element a be accepted as neighbourhoods of that element; on the other hand, every class (H) of Fréchet is equivalent to a class K in which neighbourhoods are open sets (and conversely), and where the open sets satisfy hypotheses (I) and (II).

Hausdorff[5] gives the name *topological space* to a class in which neighbourhoods are defined to possess properties (α), (β), (δ), together with the property

(γ_1). *For every pair of different elements* (a, b) *there exists a neighbourhood* V_1 *of the element* a, *and a neighbourhood* V_2 *of* b *such that* $V_1 . V_2 = 0$.

Property (γ) follows evidently from property (γ_1) and, therefore, Hausdorff's topological space is a class (H) of Fréchet. (The converse, however, is not necessarily true, as can be easily shown.)

We shall conclude with the remark that Fréchet also investigates classes in which neighbourhoods are defined to possess only property (α). Such classes are called by Fréchet classes (V).[6]

20. Let K be a given class of any elements, and suppose a law to be established which assigns to each set $E \subset K$ a certain set $E' \subset K$ (E' may be a null set), which is called the *derived* set of the set E. The law of assigning this set may be quite arbitrary, apart from its satisfying the following 3 conditions:

1) The derived set of a set consisting of one element is a null set.
2) The derived set of the sum of two sets is the sum of their derived sets.
3) $E' \supset (E')'$ for every set $E \subset K$.

Furthermore, let every element of E' be called a *limit element* of E; the set E will be called *closed* if $E \supset E'$, and *open* if the set $K - E$ is closed.

It can be easily shown that, if by a *neighbourhood* of an element a (belonging to K) we mean any open set containing a, then the

[5]*Grundzüge der Mengenlehre* (1st edition), Leipzig, 1914, p. 213.

[6]Not all the theorems of chapter I are true for Fréchet's classes (V): *e.g.* Theorem 5 is not true; the set $E + E'$ may not be closed in the class (V).

class K, in which the derived set is defined to satisfy conditions 1), 2), and 3), will become a class (H) of Fréchet.

Conversely, it can be shown that, given any class (H) of Fréchet and if by a limit element of E we mean an element a, such that every neighbourhood of a contains at least one element of E different from a, and if by the derived set E' we mean the set of all limit elements of E, then conditions 1), 2), and 3) will be satisfied.

It may be concluded from the above (as in § 19), that the study of classes satisfying conditions 1), 2), and 3) is equivalent to that of classes (H) of Fréchet[7] (and, therefore, also to that of classes K satisfying hypotheses (I) and (II)).

In conclusion, we note that classes have been investigated, in which derived sets not restricted by any conditions are defined.[8]

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[7]In his work *Esquisse d'une théorie des ensembles abstraits*, Calcutta, 1922, p. 335, Fréchet calls classes (H), classes which satisfy conditions 1), 2), and 3).

[8]See *e.g.* my paper: "La notion de dérivée comme base d'une théorie des ensembles abstraits", *Math. Annalen*, vol. XCVII, p. 321.

CHAPTER III

CLASSES (H) WHICH SATISFY THE AXIOM OF COUNTABILITY

21. We shall subject the class K to another axiom, besides those included in hypotheses (I) and (II), namely:

(III) (*vi*). *There exists an infinite sequence of open sets*

(1) $$W_1, W_2, W_3, \dots$$

such that every open set is a sum of a certain aggregate of sets belonging to the sequence (1).

We shall deduce in this chapter a number of results from hypotheses (I), (II), and (III).

The sets of the sequence (1) shall be called for brevity *rational* sets.

Corollary 1. *If a be a given element of the class K, and U an open set containing a, there exists a rational set W containing a and contained in U.*

In fact, by (*vi*), an open set U is the sum of a certain number of sets of the sequence (1); since $a \in U$, there exists a term of that sum to which a belongs; if W of the sequence (1) is this term, we have $a \in W \subset U$.

Furthermore, it follows immediately from axioms (*vi*) and (*iv*) that *every element of K is the product of all rational sets containing that element.* In fact, even more can be proved, namely:

Corollary 2. *Corresponding to every element a of K there exists an infinite sequence of rational sets $V_1, V_2, V_3, \dots$, the product of which consists of the element a only, and such that for every open set U containing a, all but a finite number of the sets $V_1, V_2, V_3, \dots$ are contained in U.*

In fact, let a be a given element, and let

$$W_{n_1},\ W_{n_2},\ W_{n_3},\ \ldots$$

be all the successive terms of the sequence (1) containing a.

The set $W_{n_1}.W_{n_2}\ \ldots\ .W_{n_k}$ is by (v) an open set and obviously contains a; by Corollary 1, there exists, therefore, a rational set V_k, such that

$$a \in V_k \subset W_{n_1}.W_{n_2}\ \ldots\ .W_{n_k}.$$

(It could be assumed that V_k is the first term of the sequence (1), which satisfies the above condition.) It is easily seen that the sequence V_k is the required one. For, it is clear that every one of the terms of the sequence $V_1, V_2, V_3, \ldots$ contains a. On the other hand, let U be any open set containing a. By Corollary 1, there exists a rational set containing a and contained in U, and hence a set belonging to the sequence $W_{n_k} (k=1, 2, \ldots)$, say the set W_{n_q}. But it follows from the definition of the sets V_k, that $V_k \subset W_{n_q}$, for $k \geq q$; therefore, certainly (since $W_{n_q} \subset U$) $V_k \subset U$, for $k \geq q$. Furthermore, the product $V_1.V_2.V_3 \ldots$ consists of the single element a; for, if b were an element different from a, then by axiom (iv), there exists an open set U containing a but not containing b; hence the terms of the sequence V_k, which for k sufficiently large will be contained in U, will not contain b.

Corollary 2 is, therefore, proved. Note that to the conditions already given we could add the condition that the sequence V_k $(k=1, 2, 3, \ldots)$is decreasing.

From Corollary 2, we get at once

COROLLARY 3. *The set of all elements of the class K has the potency of the continuum at most.*

By Corollary 2, every element a of K can be correlated with an infinite sequence of rational sets $V_1, V_2, V_3, \ldots$ such that $V_1.V_2.V_3 \ldots = (a)$, and different elements of K will be correlated with different sequences. Hence, the set of all elements of K has the same potency as a certain subset of the set of all infinite sequences, whose terms are elements of the countable sequence (1). But the set of all such sequences is known to have the potency of the continuum. Hence Corollary 3 is true.

COROLLARY 4. *The set of open (closed) sets contained in K has the potency of the continuum at most.*

In fact, every open set U determines by (*vi*) a certain subset of the countable set (1), namely, that one which consists of all the sets of the sequence (1) contained in U. At the same time, to different open sets, correspond evidently different subsets of (1). Hence, the potency of the set of all open sets (contained in K) is not greater than the potency of the set of all subsets of a countable set, which, as is known, is that of the continuum. Hence, Corollary 4, with reference to open sets, is true. It follows also immediately for closed sets, since the set of all closed sets of the class K, as complements of open sets, has the same potency as the set of open sets of K.

Note that it would be impossible to deduce from the axioms defining thus far the class K, that the set of all open sets of a class K consisting of an infinite number of elements has the potency of the continuum. This result will be obtained in the next chapter (§ 37), after the class K has been subjected to an additional hypothesis (axiom $(iv)_a$, § 30).

In fact, let K denote a class consisting of a countable number of elements $p_1, p_2, p_3, \ldots$. Let open sets of K (apart from the null set) be sets consisting of all but a finite number of elements of K. It is easily seen that this class satisfies hypotheses (I), (II), and (III) (the open sets may be considered to be the rational sets), but the set of all the open sets (and, therefore, also of the closed sets) is countable.

22. Theorem 29. *The set of all the elements of a given set which are not its elements of condensation is at most countable.*

Proof. Let E be a given set $\subset K$, and E_1 the set of all the elements of E which are not its elements of condensation. For every element a of E_1 there exists, therefore (from the definition of an element of condensation, § 14), an open set U containing a and containing at most a countable set of elements of E. There exists, therefore, by Corollary 1, for every element a of E_1 a rational set W containing a and contained in U and so containing at most a countable set of elements of E. Let S denote the sum of all such terms of the sequence (1), which contain at most a countable set of elements of E. Obviously, $E_1 \subset S$ (since every element of E_1 is contained in at least one of the sets of the sum S). From

$E_1 \subset E$, we shall have $E_1 \subset S.E$ (or, even, as is easily seen, $E_1 = S.E$). But from the definition of the sum S, the set $S.E$ is at most countable; hence, the set E_1 is at most countable.

As an immediate corollary from Theorem 29 we get the following:

Every non-countable set contains a non-countable subset of elements which are elements of condensation of the set. A set, every element of which is an element of condensation of the set, is called a *condensed* set.

Theorem 30. *The set of all elements of a given set, which are its elements of condensation is a condensed set* (if not a null set).

Proof. Retaining the same notation as in the proof of Theorem 29, let $E - E_1 = E_2$; it is required to prove that every element of E_2 is an element of condensation of E_2. Hence, let a be a given element of E_2, and let U be any open set containing a. Since $a \epsilon E_2 = E - E_1$, a is an element of condensation of E; the set $E.U$ is, therefore, non-countable. But $E_2.U = E.U - E_1.U$, and $E_1.U \subset E_1$ is, by Theorem 29, at most countable. The set $E_2.U$ is, therefore, non-countable. Since U is any open set containing a, it follows that a is an element of condensation of E_2. Since a is any element of E_2, Theorem 30 is proved.

A condensed set is evidently dense-in-itself. It follows, therefore, from Theorem 30, that the set of all the elements of a given set which are its elements of condensation is dense-in-itself. Moreover, since every non-countable set contains a non-countable subset of elements which are elements of condensation of the set, it follows, therefore, that

Every non-countable set contains a non-countable subset which is dense-in-itself.

From this and the definition of a scattered set (*i.e.* one not containing any subset non-null, dense-in-itself, § 6) we have at once

Theorem 31. *Every scattered set is at most countable.*

Furthermore, it follows immediately from Theorem 29 that the set of all elements of a given set, which are not its limit elements, is at most countable (since every element of condensation of a set is a limit element of that set). Hence, every isolated set is at most countable. (This last result is otherwise a special case of Theorem 31, since an isolated set (§ 13) is scattered.)

Suppose now that E is closed. By Theorem 8 (§ 6), every set E can be expressed in the form $E=N+R$, where N is the nucleus of E, and R is scattered (or null); we have shown towards the close of § 7, that the nucleus of every set is perfect in that set, and that a set perfect in a closed set is perfect. We shall then have from Theorem 31

Theorem 32 (Cantor-Bendixson). *Every closed set is the sum of a perfect set and a scattered set* (where the latter is at most countable by Theorem 31 and where either may be null).

We note that *the division of a closed set into a perfect set and a scattered set is unique.*

In order to prove this we shall show at first, that *if E be dense-in-itself and U be any open set, then the set $E.U$ is dense-in-itself* (or null).

In fact, suppose that the set $E.U$ is not dense-in-itself; it contains, therefore, an isolated element p. Hence, there exists an open set V such that $p \in V$, and no other element of $E.U$ belongs to V. The set $U.V$ is open by axiom (v), and it contains p, since $p \in E.U$ and $p \in V$, and, obviously, it does not contain another element of E different from p (since such an element would belong to $E.U.V$, contrary to the definition of V). Hence, p is an isolated element of E, contrary to the hypothesis that E is dense-in-itself.

Suppose now that two different divisions $E=P+R$ and $E=P_1+R_1$ of the closed set E are possible, where P and P_1 are perfect and R and R_1 scattered (or null), and where $P.R=0$ and $P_1.R_1=0$. Since the divisions are different, we have $R\neq R_1$. In one at least of the sets R and R_1, in R_1 say, there is an element p not in the other, that is, not in R. Since $p \bar{\epsilon} R$ (and $p \in R_1 \subset P_1+R_1=P+R$) we must have $p \in P$. On the other hand, since $p \in R_1$, and $P_1.R_1=0$, we have $p \bar{\epsilon} P_1$, and since P_1 is perfect and so closed, p is neither an element nor a limit element of P_1. There exists, therefore, an open set U such that $p \in U$ and $P_1.U=0$. Hence, $E.U=(P_1+R_1).U=R_1.U$, from which it follows (since R_1 is scattered) that $E.U$ is scattered. But since $p \in P$ and $p \in U$, the set $P.U$ is not null and it is also dense-in-itself (as the product of a set dense-in-itself and an open set); the set $E.U \supset P.U$ cannot, therefore, be scattered, and so we have arrived at a contradiction. We have, therefore, proved that the division $E=P+R$ is unique.

Similarly, it can be proved more generally that every set E can be represented as the sum of two sets, one of which is dense-in-itself and closed in E and the other scattered (where either of the two sets may be null).

23. Theorem 33. *Every set E contains a finite or a countable subset T, such that $E \subset \overline{T}$.*

Proof. Let E be the given set. Corresponding to every member W_n of the sequence (1) (§ 21) such that $W_n.E \neq 0$ select an element $p_n \in W_n.E$, and let T be the set of elements so obtained. Evidently, T is at most countable, and $T \subset E$. Suppose q to be an element of $E-T$, and let U be any open set containing q. By Corollary 1 (§ 21), there exists a member of the sequence (1), say W_n, such that $q \in W_n \subset U$. Since $p_n \in W_n.T$, therefore, $p_n \in U.T$, where $p_n \neq q$, since q is not an element of T. Hence, every open set containing q contains an element of T different from q, from which it follows that $q \in T'$. We have thus proved that $E-T \subset T'$ and so $E \subset T+T' = \overline{T}$.

If, in particular, the set E be closed, then, since $T \subset E$, we have $T' \subset E$ and, therefore, $\overline{T} \subset E$, which with $E \subset \overline{T}$ gives $E = \overline{T}$. Hence, *every closed set is the enclosure of a certain set countable at most.* Furthermore, it can be easily proved that if E be perfect, then T is dense-in-itself, and so $T \subset T'$, and $\overline{T} = T'$, which gives $E = T'$. Hence, *a perfect set is the derived set of a certain set countable at most.*

24. Theorem 34. *Every set of open non-abutting sets is at most countable.*

Proof. Let M be a given set of open non-abutting sets. If U be a set belonging to M and p an element of U, there exists, by Corollary 1 (§ 21), a set of the sequence (1) containing p and contained in U. Hence, for every set U of M there exists a set of the sequence (1) contained in U; let every set U of M be correlated with the smallest index n such that $W_n \subset U$. Since the sets belonging to M are non-abutting, it is obvious that different sets of M will be thus correlated with different indices. If we order the sets of M according to increasing indices, we shall obtain a sequence (finite or countable) consisting of all the sets of M. Theorem 34 is, therefore, proved.

25. Theorem 35 (Lindelöf). *If E be a given set and M a set of open sets such that every element of E belongs to at least one of the sets of M, then there exists a finite or countable sequence of sets of M, the sum of which contains E.*

Proof. Let every set W_n of the sequence (1) which is contained in at least one of the sets of M be correlated with one such set, U_n say. The sets U_n of M thus obtained will form a finite or countable sequence. It will be shown that this sequence satisfies the required condition.

For, let p denote any given element of E. It follows from the hypothesis of the theorem that there exists an open set U belonging to M and such that $p \epsilon U$. Hence, by Corollary 1 (§ 21), there exists a set W_m of the sequence (1), such that $p \epsilon W_m \subset U_m$. From the definition of the sets U_n it follows that U_m is a member of our sequence, and since $W_m \subset U_m$ we have $p \epsilon U_m$. Hence, Theorem 35 is proved.

By means of Theorem 35, we may generalize Theorem 28 (Borel) as follows:

Theorem 36 (Borel-Lebesgue). *If E be a closed and compact set, and M an aggregate of open sets such that every element of E belongs to at least one of the sets of M, then there exists a finite number of sets of M, the sum of which contains E.*

26. Theorem 37. *Every transfinite sequence of different decreasing sets*

$$E_0 \supset E_1 \supset E_2 \supset \ldots \supset E_\omega \supset E_{\omega+1} \supset \ldots \supset E_\xi \supset E_{\xi+1} \supset \ldots,$$

such that the set $E_{\xi+1}$ is closed in E_ξ, is countable.

Proof. Consider any member, E_ξ say, of our sequence, which has at least one successor. Since $E_\xi \neq E_{\xi+1}$ and $E_\xi \supset E_{\xi+1}$, there exists an element p of E_ξ which does not belong to $E_{\xi+1}$. Furthermore, since $E_{\xi+1}$ is closed in E_ξ and since $p \epsilon E_\xi$ and $p \bar{\epsilon} E_{\xi+1}$, there exists an open set U such that $p \in U$ and $U.E_{\xi+1}=0$. Hence, by Corollary 1, there exists a set W_n of the sequence (1), such that $p \in W_n \subset U$, and so $W_n.E_{\xi+1}=0$.

Hence, for every such set E_ξ of our transfinite sequence there exists a set W_n of the sequence (1) for which $W_n.E_\xi \neq 0$, and $W_n.E_{\xi+1}=0$. Let every such set E_ξ be correlated with the smallest index n, such that $W_n.E_\xi \neq 0$ but $W_n.E_{\xi+1}=0$.

It is easily seen that different sets of our transfinite sequence will be correlated with different indices n. In fact, if two sets E_ξ and E_η, where $\xi<\eta$, were to be correlated with the same index m, we should then have

$$W_m.E_\xi \neq 0,\ W_m.E_{\xi+1}=0,\ W_m.E_\eta \neq 0,\ W_m.E_{\eta+1}=0,$$

which is impossible, since from $\xi<\eta$ we have $\xi+1 \leq \eta$, and so, since the sequence is decreasing, $E_{\xi+1} \supset E_\eta$; consequently, $W_m.E_{\xi+1}=0$ gives $W_m.E_\eta=0$.

If now the terms of the transfinite sequence be ordered according to the indices correlated with them, we thereby obtain a countable sequence containing all the terms of the transfinite sequence except, perhaps, the last. Theorem 37 is, therefore, proved.

The analogous theorem about an increasing transfinite sequence of sets, each of which is closed in the succeeding one, can be proved in like manner.[1]

The following corollary may be deduced immediately from Theorem 37:

Given a transfinite sequence of ordinal type Ω of decreasing sets

(2) $$E_0 \supset E_1 \supset E_2 \supset \ldots \supset E_\omega \supset \ldots \supset E_\xi \supset E_{\xi+1} \supset \ldots,\ (\xi<\Omega)$$

such that each E_ξ is closed in E_η, for $\eta<\xi<\Omega$, then there exists an ordinal number $\alpha<\Omega$, such that $E_\xi=E_\alpha$, for $\alpha<\xi<\Omega$.

In fact, if in the given sequence we consider only the terms which are different from all the preceding ones, we have a sequence satisfying all the conditions of Theorem 37 and so, a countable sequence. We have, therefore, at most a countable set of different terms in the sequence (2); let these terms be

(3) $$E_{\xi_1},\ E_{\xi_2},\ E_{\xi_3},\ \ldots,$$

[1]For the generalization of these theorems see my note in *Biuletyn Polskiej Akad. Um.* (1921), pp. 62-65.

where $\xi_1, \xi_2, \xi_3, \ldots$ are ordinal numbers $<\Omega$, and so there exists an ordinal number $\alpha<\Omega$, such that $\alpha>\xi_n$, for $n=1, 2, 3, \ldots$ Let now ξ be an ordinal number such that $\alpha<\xi<\Omega$. The set E_ξ is one of the sets (3), say $E_\xi=E_{\xi_k}$. From $\xi_k<\alpha<\xi$, and since the sequence (2) is decreasing, we have $E_{\xi_k}\supset E_\alpha\supset E_\xi$ and, therefore, $E_\xi=E_\alpha$ since $E_\xi=E_{\xi_k}$. The corollary is, therefore, proved.

27. Let E be a given set. We shall define by transfinite induction sets $E_\xi (0\leq\xi<\Omega)$ in the following manner. Put $E_0=E$. Let now α be an ordinal number >0 and $<\Omega$ and suppose that all sets E_ξ, where $0\leq\xi<\alpha$, have been defined. If α be a number of the second kind, put $E_\alpha=\prod_{\xi<\alpha} E_\xi$. If $\alpha=\beta+1$, put $E_\alpha=E_\beta.E_\beta'$.

It is evident that the sets E_ξ thus defined satisfy the conditions of the corollary to Theorem 37 (that the transfinite sequence $E_\xi(\xi<\Omega)$ is decreasing follows from the definition of the sets, and from $E_{\xi+1}=E_\xi.E_\xi'$ it follows that $E_{\xi+1}$ is closed in E_ξ (§ 7), since E_ξ' as a derived set is closed (Theorem 25)). There exists, therefore, an ordinal number $\alpha<\Omega$, such that $E_\xi=E_\alpha$, for $\alpha<\xi<\Omega$. We may, of course, suppose that α is the smallest ordinal number with the above property.

From $E_{\alpha+1}=E_\alpha$, and $E_{\alpha+1}=E_\alpha.E_\alpha'$, we conclude that $E_\alpha\subset E_\alpha'$; E_α is, therefore, dense-in-itself (or null).

Furthermore, it is easily seen that for $\xi<\alpha$ we have always $E_\xi\neq E_{\xi+1}$. For if $E_\beta=E_{\beta+1}$, for some $\beta<\alpha$, then from the definition of the sets E_ξ it could be easily concluded that $E_\xi=E_\beta$, for $\xi>\beta$, which, since $\beta<\alpha$, is in contradiction with the definition of α. Hence $E_\xi\neq E_\xi.E_\xi'$, for $\xi<\alpha$, and so none of the sets E_ξ is dense-in-itself, for $\xi<\alpha$. Furthermore, we have

$$E=E_\alpha+\sum_{\xi<\alpha}(E_\xi-E_{\xi+1}). \tag{4}$$

For, it is evident from (2) that the set on the right of (4) is a subset of the set $E_0=E$. On the other hand, let p be a given element of E. If $p \bar{\epsilon} E_\alpha$, there exist indices $\xi\leq\alpha$ such that $p\bar{\epsilon} E_\xi$; let η be the smallest of them. It is easily seen that η cannot be a number of the second kind, for then we should have $E_\eta=\prod_{\xi<\eta} E_\xi$, and since (from the definition of η) p belongs to every E_ξ for $\xi<\eta$, p would belong to E_η, contrary to the definition of η. Hence

$\eta = \zeta + 1$, where (from the definition of η) $p \in E_\zeta$ but $p \bar{\in} E_{\zeta+1}$, and so $p \in (E_\zeta - E_{\zeta+1})$. Since $\zeta < \eta \leqq \alpha$, p belongs to a term of the sum on the right of (4).

It follows from (2) that the terms of the sum (4) are mutually exclusive sets.

It will be shown that the set E_α is the nucleus of the set E.

For, let P denote a subset of E dense-in-itself. We have, therefore, $P \subset E = E_0$. Suppose that for some $\eta < \Omega$, $P \subset E_\xi$, for $\xi < \eta$. If η is a number of the second kind, we evidently have $P \subset E_\eta$ following from the definition of E_η. If $\eta = \xi + 1$, then $E_{\xi+1} = E_\xi . E_\xi'$. But from $\xi < \eta$, we have, by hypothesis, $P \subset E_\xi$ and so $P' \subset E_\xi'$; since P is dense-in-itself, $P \subset P'$; hence, $P \subset E_\xi'$, and so $P \subset E_\xi . E_\xi' = E_{\xi+1}$. We have proved, therefore, by transfinite induction that $P \subset E_\eta$ for every ordinal number $\eta < \Omega$; hence, in particular, $P \subset E_\alpha$.

E_α contains, therefore, every subset of E dense-in-itself, and since, as we have seen, E_α is dense-in-itself, it is the nucleus of E.

It follows that the set

$$E - E_\alpha = \sum_{\xi < \alpha} (E_\xi - E_{\xi+1}) \tag{5}$$

is scattered. Relation (4) furnishes a division of the set E into its nucleus and a scattered subset.

With reference to the terms of the sum (5) we note that $E_\xi - E_{\xi+1} = E_\xi - E_\xi . E_\xi' = E_\xi - E_\xi'$ consists of all the isolated elements of E_ξ.

Hence, it follows from (4) that in order to obtain the nucleus of a set E it is necessary to remove from E its isolated elements, from the set E_1 thus obtained to remove the isolated elements of E_1 and to proceed similarly with the set E_2 thus obtained, repeating the process of removing isolated elements from the sets obtained at each stage transfinitely. After a countable number of such operations we shall arrive at a set dense-in-itself (which may be null). Thus the expression in (4) given by Cantor illustrates the structure of sets.

28. We shall define now by transfinite induction sets $P^{(\alpha)}$ for every set P and every ordinal number α, as follows. $P^{(1)}$ denotes the derived set of P. Let now α be an ordinal number > 1, and suppose that all sets $P^{(\xi)}$, where $\xi < \alpha$, have been already defined.

If α is a number of the second kind, we put $P^{(\alpha)} = \prod_{\xi<\alpha} P^{(\xi)}$; if $\alpha = \xi + 1$, we put $P^{(\alpha)} = (P^{(\xi)})'$.

The set $P^{(\alpha)}$, so defined, is called *the derived set of P of order α.* It follows from the definition of this derived set and from Theorems 3 and 25, that the sets $P^{(\alpha)}$ are always closed (for every ordinal number $\alpha > 0$). We have, therefore, $P^{(\alpha)} \supset (P^{(\alpha)})' = P^{(\alpha+1)}$ (for every ordinal number $\alpha > 0$), which proves (from the definition of $P^{(\alpha)}$ for α, a number of the second kind) that the transfinite sequence of successive derived sets

$$P' \supset P'' \supset P''' \supset \ldots \supset P^{(\omega)} \supset P^{(\omega+1)} \supset \ldots \supset P^{(\xi)} \supset \ldots$$

is a decreasing sequence of closed sets. The corollary to Theorem 37 (§ 26) may, therefore, be applied to that sequence. Hence, for every set P, there exists an ordinal number $\alpha < \Omega$, such that $P^{(\xi)} = P^{(\alpha)}$, for $\alpha < \xi < \Omega$, and so also $P^{(\Omega)} = P^{(\alpha)}$. Moreover, it may be obviously supposed that α is the smallest ordinal number with that property.

Hence *every set P possesses at most a countable number of different derived sets of transfinite order; for every set P there exists an ordinal number $\alpha < \Omega$, such that $P^{(\alpha)} = P^{(\Omega)}$.* Obviously, if α be the smallest ordinal number with the above property, then the derived sets of order $\leq \alpha$ are all different.

Let now E be a closed set and recall the definition of the sets E_α from § 27. Since E is closed, we have $E_1 = E.E' = E'$; hence, E_1 is closed and so also is $E_2 = E_1.E_1' = E_1' = (E')' = E''$; and, finally, by transfinite induction we have $E_\alpha = E^{(\alpha)}$, for every ordinal number $\alpha < \Omega$. Hence for a closed set E relation (4) takes the form

$$E = E^{(\alpha)} + \sum_{\xi<\alpha} (E^{(\xi)} - E^{(\xi+1)}),$$

where α denotes the smallest ordinal number (always $< \Omega$) such that $E^{(\alpha)} = E^{(\alpha+1)}$. Hence the nucleus of a closed set E is the set $E^{(\Omega)}$.

It follows at once from the above that *for a closed set E to be scattered, it is necessary and sufficient that $E^{(\Omega)} = 0$.* The set $E^{(\Omega)}$ is always perfect (or null).

29. The set E is said to be *bicompact* (Urysohn, Alexandroff), if for every infinite subset E_1 of E there exists an element a (belonging to E or not), such that every open set containing a, contains a subset of E_1 which has the same potency as E_1.

Theorem 38. *Every compact set is bicompact.*

It will obviously be sufficient to prove, that if E be compact and non-countable, there exists an element a (belonging to E or not), such that every open set containing a contains a subset of E of the same potency as E.

Let $\mathfrak{m}$ be the cardinal number of E. We shall consider two cases.

1. The cardinal number $\mathfrak{m}$ is not the sum of a countable series of cardinal numbers smaller than itself. In that case, it can be easily shown that the set E contains an element a such that every open set containing a contains a subset of E of cardinal $\mathfrak{m}$. For, if not, then for every element p of E there exists a rational set W containing p and such that the set $W.E$ has cardinal $< \mathfrak{m}$. Let $U_1, U_2, \ldots$ be the successive members of the sequence (1) (§ 21), such that $U_n.E$ has cardinal $\mathfrak{m}_n < \mathfrak{m}$. Evidently $E \subset U_1 + U_2 + \ldots$; hence, $E = U_1.E + U_2.E + \ldots$, and $\mathfrak{m} \leq \mathfrak{m}_1 + \mathfrak{m}_2 + \ldots$, which, along with $\mathfrak{m} \geq \mathfrak{m}_1 + \mathfrak{m}_2 + \ldots$ (since $\mathfrak{m} \geq \aleph_0$, and so $\mathfrak{m} = \mathfrak{m}\aleph_0$ and $\mathfrak{m}_n < \mathfrak{m}$, for $n = 1, 2, 3, \ldots$), gives $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \ldots$, contrary to the hypothesis about the number $\mathfrak{m}$.

2. $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \ldots$, where $\mathfrak{m}_n < \mathfrak{m}$, for $n = 1, 2, \ldots$. Put $\mathfrak{s}_n = \mathfrak{m}_1 + \mathfrak{m}_2 + \ldots + \mathfrak{m}_n$ (for $n = 1, 2, 3, \ldots$). Since $\mathfrak{m}_n < \mathfrak{m}$, we shall have $\mathfrak{s}_n < \mathfrak{m}$ (since, as is well known, a transfinite cardinal number is not the sum of a finite number of cardinal numbers smaller than itself).

Since $\mathfrak{s}_n < \mathfrak{m}$, there exists an element p_n of E such that for every open set U containing p_n, the set $U.E$ has a cardinal number $\geq \mathfrak{s}_n$. For, if not, we should have $E = U_1.E + U_2.E + \ldots$, where $U_k.E$ has cardinal $\leq \mathfrak{s}_n$, for $k = 1, 2, 3, \ldots$, from which it would follow that the cardinal of E is $\leq \mathfrak{s}_n.\aleph_0$ and, therefore, $< \mathfrak{m}$ (for, if $\mathfrak{s}_n < \aleph_0$, then $\mathfrak{s}_n.\aleph_0 \leq \aleph_0 < \mathfrak{m}$; and if $\mathfrak{s}_n \geq \aleph_0$, then $\mathfrak{s}_n.\aleph_0 = \mathfrak{s}_n < \mathfrak{m}$), which is impossible.

From the fact that E is compact we conclude further that there exists an element a of the class K such that every open set containing a contains an infinity of terms of the sequence $p_1, p_2,$

$p_3, \ldots$. This conclusion obviously applies also in the case when only a finite number of terms of the sequence are different.

Let U denote any open set containing a, n any given integer. It follows from the property of the element a that there exists an element p_k in U such that $k > n$, and so from the property of the element p_k, it follows that $U.E$ has potency $\geq \mathfrak{s}_k \geq \mathfrak{s}_n$. The set $U.E$ has, therefore, potency $\geq \mathfrak{s}_n$, for every integer n, from which, since $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \ldots$, and $\mathfrak{s}_n = \mathfrak{m}_1 + \ldots + \mathfrak{m}_n$ $(n = 1, 2, \ldots)$, it follows, as is known, that $U.E$ has potency $\geq \mathfrak{m}$.

Theorem 38 is, therefore, proved.

CHAPTER IV

TOPOLOGICAL SPACES WHICH SATISFY THE AXIOM OF COUNTABILITY

30. We shall now strengthen somewhat axiom (*iv*) (§ 15) in replacing it by the axiom:

$(iv)_a$. *If p and q be two different elements of the class K, there exist two open, mutually exclusive sets, one of which contains p and the other q.*

We shall deduce in this chapter conclusions from axioms (*i*), (*ii*), (*iii*), $(iv)_a$, (*v*), and (*vi*).

We have seen in § 19 that if by a neighbourhood of an element is meant an open set containing this element, then the investigation of classes satisfying axioms (*i*), (*ii*), (*iii*), (*iv*), and (*v*) is equivalent to the investigation of Fréchet's classes (H). We have given in the same article the definition of Hausdorff's topological spaces, which differ from Fréchet's classes (H) in that the property (γ) of Fréchet is replaced by the more stringent property (γ_1).

If, however, a neighbourhood of an element be defined to be an open set containing that element, then properties (γ) and (γ_1) become axioms (*iv*) and $(iv)_a$ respectively. It follows then at once that the investigation of classes K satisfying axioms (*i*), (*ii*), (*iii*), $(iv)_a$, and (*v*), is equivalent to the investigation of Hausdorff's topological spaces.

Hence the investigation of classes K, satisfying axioms (*i*), (*ii*), (*iii*), $(iv)_a$, (*v*), and (*vi*), which will be the subject of this chapter, is equivalent to the investigation of Hausdorff's topological spaces, satisfying the axiom of countability (*vi*).

31. Definition. An element a is said to be the *limit* of an infinite sequence of elements $p_1, p_2, p_3, \ldots$, written

$$\lim_{n\to\infty} p_n = a, \text{ or } p_n \to a, \text{ as } n \to \infty,$$

if for every open set U containing a, there exists an integer μ such that

$$p_n \in U, \text{ for } n > \mu.$$

The following properties of the limit of a sequence follow immediately from the above definition.

PROPERTY 1. *If* $p_n = p$, *for* $n = 1, 2, 3, \ldots$, *then* $\lim_{n \to \infty} p_n = p$.

In other words, an infinite sequence whose terms are all equal to the same element has that element for its limit. The proof follows directly from the definition of the limit of a sequence.

PROPERTY 2. *If* $\lim_{n \to \infty} p_n = a$, *and* $\lim_{n \to \infty} p_n = b$, *then* $a = b$.

In other words, an infinite sequence cannot have two different limits. In fact, if $\lim_{n \to \infty} p_n = a$, and $b \neq a$, then by $(iv)_a$ there exist open sets U and V such that $a \in U$, $b \in V$, and $U.V = 0$. Since $\lim_{n \to \infty} p_n = a$, $p_n \in U$ for $n > \mu$, and so (since $U.V = 0$), $p_n \bar{\in} V$, for $n > \mu$, and, therefore, we cannot have $\lim_{n \to \infty} p_n = b$.

PROPERTY 3. *If* $\lim_{n \to \infty} p_n = a$, *and if* $n_1, n_2, n_3, \ldots$ *is an infinite sequence of increasing indices, then* $\lim_{k \to \infty} p_{n_k} = a$.

In other words, if a is the limit of the sequence $p_n (n = 1, 2, \ldots)$, then a is also the limit of every sequence obtained from the given one. The proof follows at once from the definition of the limit of a sequence.

Fréchet considers in his thesis[1] a class (L) to be a set of any elements in which the limit is defined in such a manner that, for every infinite sequence $p_1, p_2, p_3, \ldots$ of the elements of the given set and every element a of that set, it is possible to state whether a is the limit of the sequence $p_n (n = 1, 2, 3, \ldots)$ or not. The convention which establishes this test is quite arbitrary except that properties 1, 2, and 3 are retained. In the paper referred

[1]*Rendiconti del Circolo Matematico di Palermo*, vol. XXII (1906); see also *Bull. des Sciences Math.*, vol. XLII (1918).

to above Fréchet investigates the conclusions following from these hypotheses and from the corresponding definitions.

It is obvious that every class K satisfying axioms (*i*), (*ii*), (*iii*), $(iv)_a$, and (*vi*) may be considered as a class (L) of Fréchet. (It could be shown, however, that the converse is not true.)

Some of the other properties of a limit (which follow from our hypotheses and definitions but do not hold necessarily in the classes (L) of Fréchet) are as follows:

The limit of a sequence (or its existence) does not depend on the order of the terms of the sequence. The proof follows immediately from the definition of the limit.

The limit (or its existence) is not affected if we remove, add, or change a finite number of terms of the sequence. The proof is immediate.

If $\lim_{n\to\infty} p_n = a$, *and* $\lim_{n\to\infty} q_n = a$, *then the sequence* $p_1, q_1, p_2, q_2, \ldots$ *also has the limit* a. The proof follows easily from the definition of a limit.

32. Theorem 39. *In order that an element a be a limit element of a set E, it is necessary and sufficient that there exist an infinite sequence $p_1, p_2, p_3, \ldots,$ such that*

$$\lim_{n\to\infty} p_n = a, \tag{1}$$

and

$$a \neq p_n \in E, \text{ for } n = 1, 2, 3, \ldots \tag{2}$$

Proof. That the condition is sufficient follows directly from the definition of a limit (§ 31). It remains, therefore, to prove the necessity of the condition.

Hence suppose that a is a limit element of E. Let $V_1, V_2, V_3, \ldots$ denote an infinite sequence of open sets satisfying the conditions of Corollary 2 of § 21 with reference to the element a. From $a \in V_n$ and the hypothesis that a is a limit element of E, it follows that there exists an element p_n of E different from a and contained in V_n. The infinite sequence $p_1, p_2, p_3, \ldots$ is the required sequence. In fact, condition (2) is evidently satisfied. Let now U denote any open set containing a. From the property of the sequence

$V_1, V_2, V_3, \ldots$ it follows that $V_n \subset U$ for $n > \mu$; hence, $p_n \in U$ for $n > \mu$, and so (1) follows. Theorem 39 is thus proved.

REMARK. We note that we may add in Theorem 39 the condition that the terms of the sequence $p_1, p_2, p_3, \ldots$ be all different.

In fact, the sequence $p_1, p_2, p_3, \ldots$ must contain an infinity of different terms, for, otherwise, one of the terms, p_s say, would be repeated an infinite number of times, and so from (1) and properties 3, 1, and 2 of a limit we should have $p_s = a$, contrary to (2). If $p_{n_1}, p_{n_2}, p_{n_3}, \ldots$ be the infinite sequence obtained from the sequence $p_1, p_2, \ldots$ on removing from it every term duplicating some preceding term, then from (1) and property 3 of limits we shall have $\lim_{k\to\infty} p_{n_k} = a$, which proves the truth of our remark.

33. **Theorem 40.** *In order that a function f be continuous in a set E at an element p_0 of that set, it is necessary and sufficient that for every infinite sequence $p_1, p_2, p_3, \ldots$ of elements of E, for which*

$$\lim_{n\to\infty} p_n = p_0, \tag{3}$$

we should have

$$\lim_{n\to\infty} f(p_n) = f(p_0). \tag{4}$$

Proof. Suppose that the function $f(p)$ defined in the set E is continuous at p_0, an element of E, and let p_n $(n=1, 2, 3, \ldots)$ be a sequence of elements of E for which (3) holds. Let further V denote any open set such that $f(p_0) \in V$. By the definition of the continuity of a function at a given element (§ 10) there exists an open set U, such that $p_0 \in U$, and the conditions $p \in U.E$ implies that $f(p) \in V$. But from (3) and the definition of a limit (§ 31), we have $p_n \in U$, for $n > \mu$; hence also $f(p_n) \in V$ for $n > \mu$, from which, since V is any open set and from the definition of a limit, (4) follows. The condition of our theorem is, therefore, necessary.

Suppose now that the function $f(p)$ is not continuous in E at an element p_0 of E. It follows then from the definition of continuity of a function, that there exists an open set V containing $f(p_0)$ such that in every open set U containing p_0 there is an element p of E for which $f(p) \bar{\in} V$.

By Corollary 2 of § 21 there exists an infinite sequence of open sets $U_1, U_2, U_3, \ldots$ containing p_0 and such that for every open set U containing p_0, we have $U_n \subset U$, for $n > \mu$ (where μ is an integer dependent on U). By the above, there exists an element p_n in U_n such that $f(p_n) \bar{\epsilon} V$.

Since $p_n \epsilon U_n$ and from the property of the sequence $U_1, U_2, \ldots$, it is clear that (3) holds. However, (4) is not true, since $f(p_n) \bar{\epsilon} V$, for $n=1, 2, 3, \ldots$ Hence the condition of the theorem is sufficient.

Theorem 40 is, therefore, proved completely.

We shall deduce now an important corollary from Theorems 40 and 33.

Let E be any given set. There exists by Theorem 33 a finite or countable subset P of E such that $E \subset \overline{P}$.

Furthermore, let $f(p)$ and $\phi(p)$ be two functions continuous in E and such that $f(p) = \phi(p)$, for $p \epsilon P$. *Then $f(p) = \phi(p)$ in the whole set E.*

Let p_0 denote an element of $E - P$; from $E \subset \overline{P} = P + P'$, it follows that $p_0 \epsilon P'$, and so by Theorem 39, there exists an infinite sequence $p_n (n = 1, 2, \ldots)$ of elements of P such that

$$\lim_{n \to \infty} p_n = p_0,$$

whence, since f and ϕ are continuous in E, by Theorem 40,

$$\lim_{n \to \infty} f(p_n) = f(p_0), \text{ and } \lim_{n \to \infty} \phi(p_n) = \phi(p_0).$$

Since $p_n \epsilon P$, for $n = 1, 2, \ldots, f(p_n) = \phi(p_n)$ and so we get at once

$$f(p_0) = \phi(p_0).$$

Hence $f(p) = \phi(p)$, for $p \epsilon E - P$, and since by hypothesis $f(p) = \phi(p)$ for $p \epsilon P$, therefore, $f(p) = \phi(p)$ in the whole set E. Hence, *where E is a given set and P a finite or countable subset such that $E \subset \overline{P}$, there is at most only one function continuous in E and having assigned values in P.*

Since in a finite or countable set there can be defined at most a continuum of different functions (the values of the functions being given by the elements of K, which by Corollary 3 of § 21 is of potency of the continuum at most) therefore:

There is at most a continuum of different continuous functions in every set E.

It follows from the above that *every set possesses at most a continuum of different continuous transforms*, and furthermore, that *there exists at most a continuum of different sets homeomorphic with a given set.*

Let now all sets contained in a given class K be divided into topological types, with two sets belonging to the same type if and only if they are homeomorphic. To each topological type will then belong a continuum of different sets at most. If the class K has the potency of the continuum, then there are $2^{\mathfrak{c}}$ different subsets of K, and since each topological type consists of at most $\mathfrak{c}$ different subsets of K, it may be concluded that K contains $2^{\mathfrak{c}}$ different topological types.[2]

34. Theorem 41. *A continuous transform of a closed and compact set is a closed and compact set.*

Proof. Let E_0 denote a given closed and compact set, $f(p)$ a function defined and continuous in E_0. Let T be any infinite subset of $T_0=f(E_0)$. There exists, therefore, an infinite sequence $q_n(n=1, 2, \ldots)$ consisting of different elements of T. Since

$$q_n \epsilon T \subset T_0=f(E_0),$$

there exists an element p_n of E_0 for every integer n, such that

$$q_n=f(p_n),$$

with the terms of the sequence $p_n(n=1, 2, \ldots)$ all different (since the terms of the sequence $q_n(n=1, 2, \ldots)$ are all different).

Denote by E_1 the set of all terms of the sequence $p_n(n=1, 2, \ldots)$; E_1 is, therefore, an infinite set contained in E_0 and so has a derived set which is not null, since E_0 is compact by hypothesis (§ 16). Let a be an element such that $a \epsilon E_1'$; since $E_1 \subset E_0$, and E_0 is closed, $a \epsilon E_0$. Since $a \epsilon E_1'$, there exists by Theorem 39 an infinite sequence of elements of E_1 different from a, with a as its limit; this sequence will differ in order only from a sequence obtained from $p_n(n=1, 2, \ldots)$, and, since (§ 31) the limit of a

[2]For, if $\mathfrak{m}$ be the potency of the set of all different topological types contained in K, and $\overline{\overline{K}}=\mathfrak{c}$, then $\mathfrak{m} \leq 2^{\mathfrak{c}} \leq \mathfrak{m}^{\mathfrak{c}}$, and so $\mathfrak{m}=2^{\mathfrak{c}}$ (Appendix, § 4).

sequence is independent of the order of the terms, we conclude that there exists an infinite sequence $p_{n_k}(k=1, 2, \ldots)$ obtained from the sequence $p_n(n=1, 2, \ldots)$, such that

$$\lim_{k\to\infty} p_{n_k}=a.$$

From the continuity of f in E_0 and from Theorem 39, it follows that

$$\lim_{k\to\infty} f(p_{n_k})=f(a),$$

i.e.

$$\lim_{k\to\infty} q_{n_k}=f(a),$$

and since the terms of the sequence q_n (and, therefore, those of the sequence q_{n_k}) are all different elements of T, it follows from the above that $f(a) \epsilon T'$, and, hence, $T'\neq 0$.

We have, therefore, proved that T_0 is a compact set.

Let now b denote any element such that $b \epsilon T_0'$.

By Theorem 39 (and Remark to Theorem 39) there exists an infinite sequence $q_n(n=1, 2, \ldots)$ of different elements of T_0, such that

(5) $$\lim_{n\to\infty} q_n=b.$$

Since

$$q_n \epsilon T_0=f(E_0),$$

there exists an element p_n for every integer n, such that

$$p_n \epsilon E_0, \text{ and } f(p_n)=q_n,$$

in which the terms of the sequence $p_n(n=1, 2, \ldots)$ are all different (since the terms of the sequence $q_n(n=1, 2, \ldots)$ are all different). Denote by E_1 the set of all terms of the sequence $p_n(n=1, 2, \ldots)$; E_1 will be an infinite set contained in the compact set E_0, and, hence, $E_1'\neq 0$. There exists, therefore, an element a, which is a limit element of E_1. We conclude, as before, that a is the limit of some sequence obtained from the sequence $p_n(n=1, 2, \ldots)$, and that $\lim_{k\to\infty} p_{n_k}=a$, where n_k $(k=1, 2, \ldots)$ is an infinite sequence of increasing integers.

Since E_0 is closed (and since $p_n \epsilon E_1\subset E_0$), we have $a \epsilon E_0$;

from the continuity of the function f in E_0, and from $\lim_{k\to\infty} p_{nk}=a$, it follows that

$$\lim_{k\to\infty} f(p_{nk})=f(a),$$

i.e.

$$\lim_{k\to\infty} q_{nk}=f(a),$$

which, from (5) (and from the properties 2 and 3 of limits) gives $b=f(a)$ and proves (since $a \epsilon E_0$) that $b \epsilon f(E_0)=T_0$.

We have thus proved that if $b \epsilon T_0'$, then $b \epsilon T_0$, which proves that T_0 is closed.

Theorem 41 is, therefore, proved.

A closed, compact, and connected set which contains more than one element is called a *continuum*. It follows at once from Theorems 41 and 21 that *a continuous transform of a continuum, if it contains more than one element is again a continuum*. It follows, in particular, that *the property of being a continuum is a topological property.*

35. Theorem 42. *If a function $f(p)$ defined at the elements of a closed and compact set E_0 is continuous and biuniform in that set, then the inverse function of $f(p)$ is continuous in the set $T_0=f(E_0)$.*

Proof. Let $f(p)$ be a continuous and biuniform function in a closed and compact set E_0 and let $\phi(q)$ be the inverse function of $f(p)$; $\phi(q)$ will, therefore, be defined for $q \epsilon T_0=f(E_0)$.

Suppose that $\phi(q)$ is not continuous at an element q_0 of T_0. Hence, by Theorem 40, there exists an infinite sequence $q_1, q_2, \ldots$ of elements of T_0 for which

$$\lim_{n\to\infty} q_n=q_0, \tag{6}$$

but not

$$\lim_{n\to\infty} \phi(q_n)=\phi(q_0).$$

Put

$$p_n=\phi(q_n), \text{ for } n=0, 1, 2, 3, \ldots; \tag{7}$$

the relation

$$\lim_{n\to\infty} p_n=p_0$$

will, therefore, not be true. It follows from the definition of the limit of a sequence (§ 31) that there exists an open set U containing p_0, such that

$$p_n \bar{\epsilon} U$$

is true for an infinite number of values of the index n. There exists, therefore, an infinite sequence of increasing indices $n_k (k=1, 2, \ldots)$ such that

(8) $$p_{n_k} \bar{\epsilon} U, \text{ for } k=1, 2, 3, \ldots$$

If among the terms of the sequence $p_{n_k}(k=1, 2, \ldots)$ a certain term, π say, were repeated an infinite number of times, then

$$f(p_{n_k}) = q_{n_k},\ f(p_0) = q_0$$

(which follows from (7) and the fact that ϕ is the inverse function of f) and from (6), we should have

$$f(\pi) = q_0 = f(p_0),$$

and so, since f is biuniform, $\pi = p_0 \epsilon U$, whereas for $\pi = p_{n_k}$ for some k, (8) gives $\pi \bar{\epsilon} U$. Hence, the set of all the terms of the sequence $p_{n_k}(k=1, 2, \ldots)$ is infinite and so as a subset of the compact set E_0 has a derived set which is not null. Hence, there exists an infinite sequence of increasing indices $k_r (r=1, 2, \ldots)$ such that

(9) $$\lim_{r \to \infty} p_{n_{k_r}} = p,$$

and, since E_0 is closed, we have $p \epsilon E_0$. The function f, being continuous in E_0, (9) gives

$$\lim_{r \to \infty} f(p_{n_{k_r}}) = f(p),$$

i.e., since $f(p_n) = q_n$,

$$\lim_{r \to \infty} q_{n_{k_r}} = f(p),$$

and so from (6) (and properties 2 and 3 of limits)

$$f(p) = q_0;$$

hence, from $f(p_0) = q_0$ and the fact that f is biuniform,

(10) $$p = p_0.$$

But from $p_0 \epsilon U$ and from (9) and (10), it follows that, for r sufficiently great,

$$p_{nk_r} \epsilon U,$$

contrary to (8).

We have thus proved that the hypothesis that $\phi(q)$ is not continuous in T_0 leads to a contradiction. Theorem 42 is, therefore, proved.

It follows from the above theorem that a continuous and biuniform transform of a closed and compact set is homeomorphic with that set.

36. A set E is said to *possess the Borel property* if, for every infinite sequence of open sets $Q_1, Q_2, Q_3, \ldots$ such that $E \subset Q_1+Q_2+\ldots$, there exists an integer n such that $E \subset Q_1+Q_2+\ldots+Q_n$.

By Theorem 28 every closed and compact set has the Borel property. We shall prove the converse, *i.e.* a set possessing the Borel property must be closed and compact.

To prove it, suppose that the given set E is not compact; hence it contains an infinite subset E_1 such that $E_1'=0$. Let $p_1, p_2, p_3, \ldots$ be an infinite sequence of different elements of E_1; denoting by P_n the set of all terms of the sequence $p_n, p_{n+1}, p_{n+2}, \ldots$ we shall have $P_n \subset E_1$, hence $P_n' \subset E_1'$, and so $P_n'=0$; the sets $P_n (n=1, 2, \ldots)$ are, therefore, closed, and so the sets $Q_n=K-P_n$ are open. Furthermore, it is clear that $E \subset Q_1+Q_2+Q_3+\ldots$ (For if $p \epsilon E-P_1$, then obviously $p \epsilon Q_1=K-P_1$, and if $p \epsilon P_1$, for instance $p=p_\kappa$, then $p \epsilon Q_{k+1}=K-P_{k+1}$, since P_{k+1} consists of the elements $p_{k+1}, p_{k+2}, \ldots$ which are different from p_k.) On the other hand, it is impossible to have $E \subset Q_1+Q_2+\ldots+Q_n$ for any n, for $p_n \epsilon E$ but $p_n \bar{\epsilon} Q_k$, for $k \leq n$ (since $Q_k=K-P_k$, and $p_n \epsilon P_k$ for $k \leq n$).

Hence a set which is not compact does not possess the Borel property. Suppose now that the set E is not closed. There exists, therefore, a limit element a of E which does not belong to E. There exists, by Theorem 39, an infinite sequence $p_1, p_2, p_3, \ldots$ of elements of E different from a, such that $\lim_{n\to\infty} p_n=a$. Denote by P_n the set of all different terms of the sequence $p_n, p_{n+1}, p_{n+2}, \ldots$. It is easily seen that the set P_n' consists of one element a only.

(That a is the limit element of P_n follows from the property of the sequence $p_1, p_2, \ldots$ and from property 3 of a limit. If there were a $b \in P_n'$ then, by Theorem 39, a certain sequence obtained from the sequence $p_1, p_2, \ldots$ would have the limit b and so, by properties 2 and 3 of limits, we should have $b=a$.) Put $Q_n = K - \overline{P}_n$, where $\overline{P}_n = P_n + P_n'$; these will be open sets, for $n=1, 2, \ldots$. It is easily seen that $E \subset Q_1 + Q_2 + Q_3 + \ldots$. For if $p \in E - P_1$, then from $a \bar{\epsilon} E$ and $P_1' = (a)$, we have $p \in E - \overline{P}_1$ and certainly $p \in K - \overline{P}_1 = Q_1$; and if $p \in P_1$, for instance $p = p_k$, then $p \in Q_{k+1}$. On the other hand, $E \subset Q_1 + Q_2 + \ldots + Q_n$ is not true for any n, since $p_n \in E$, but $p_n \bar{\epsilon} Q_k$, for $k \leq n$.

Hence, a set which is not closed does not possess the Borel property.

By virtue of Theorem 28, it can now be stated that

In order that a set possess the Borel property it is necessary and sufficient that it be closed and compact.

37. **Theorem 43.** *The set of all open (closed) sets has the potency of the continuum* (for classes K containing an infinite number of elements).

Proof. Taking into consideration Corollary 4 from axiom (*vi*) (§ 21) it will be sufficient to prove that the set of all open sets of a class K has the potency of the continuum at least. It will be sufficient for that purpose to show that *there exists in K an infinite sequence of open, mutually exclusive, non-null sets.* Different subsets of such a sequence (and there is, as is well known, a continuum of such) will determine different sums, which will be open sets.

We shall distinguish two cases for purposes of proof.

1. The class K does not contain elements which are limit elements of K. Hence, for every element $p \in K$ there exists an open set $U(p)$ which contains only the element p, *i.e.* $U(p) = (p)$. It follows, therefore, that sets consisting of the single elements of K (one by one) form an infinite set of open, mutually exclusive, non-null sets.

2. An element a of K is a limit element of K. Let p_1 be an element of K different from a. (Such an element exists since we suppose that the class K is infinite.)

There exist by axiom $(iv)_a$ two open sets U_1 and V_1 such that $p_1 \epsilon U_1$, $a \epsilon V_1$, and $U_1.V_1=0$. Since $a \epsilon V_1$ and $a \epsilon K'$, there exists in V_1 an element of K different from a, say p_2.

By axiom $(iv)_a$ there exist open sets U_2 and V_2 such that $p_2 \epsilon U_2$, $a \epsilon V_2$, and $U_2.V_2=0$, and it may be supposed that $U_2 \subset V_1$ and $V_2 \subset V_1$, for otherwise we could take the products of the sets U_2 and V_2 respectively by V_1, which are open sets by axiom (v).

Suppose, in general, that we have determined a sequence of elements $p_1, p_2, \ldots, p_n$, and the sequences of open sets $U_1, U_2, \ldots, U_n$, and $V_1, V_2, \ldots, V_n$, where $p_n \epsilon U_n \subset V_{n-1}$, $a \epsilon V_n \subset V_{n-1}$, and $U_n.V_n=0$. Since $a \epsilon V_n$ and $a \epsilon K'$, there exists an element p_{n+1} of K different from a in V_n, and by axiom $(iv)_a$ there exist open sets U_{n+1} and V_{n+1} such that $p_{n+1} \epsilon U_{n+1}$, $a \epsilon V_{n+1}$, and $U_{n+1}.V_{n+1}=0$; as before, it may be supposed that $U_{n+1} \subset V_n$ and $V_{n+1} \subset V_n$.

The infinite sequence of open sets $U_1, U_2, U_3, \ldots$ is thus defined by induction; we have also $V_1 \supset V_2 \supset V_3, \ldots, U_{n+1} \subset V_n$ but $U_n.V_n=0$, for $n=1, 2, 3, \ldots$, from which it is readily seen that the sets $U_1, U_2, U_3, \ldots$ are mutually exclusive.

Theorem 43 may, therefore, be considered as proved completely.

CHAPTER V

NORMAL CLASSES

38. To the axioms introduced so far in the investigation of classes K we shall add the so-called axiom of *regularity* (Tychonoff):

(*vii*) *If an element p belongs to an open set U, there exists an open set V which contains p and whose enclosure is contained in U (i.e. $\overline{V} \subset U$).*

It is easily seen that *every compact class satisfies the axiom of regularity.*[1] In fact, let K denote a compact class, p a given element of K which belongs to an open set U. The set $F=K-U$ is, therefore, closed and compact (since it is a subset of the compact class K) and $p \bar{\epsilon} F$. For every $q \epsilon F$ (and so $q \neq p$) there exist by $(iv)_a$ open sets $V(q)$, and $W(q)$ such that $p \epsilon V(q)$, $q \epsilon W(q)$, and $V(q).W(q)=0$. Since F is closed and compact there exists, by Theorem 36, a finite number of elements $q_1, q_2, \ldots, q_n$ of F such that $F \subset W = W(q_1)+W(q_2)+\ldots+W(q_n)$. If we let $V=V(q_1).V(q_2)\ldots.V(q_n)$, we shall have, as is easily seen, two open sets W and V such that $F \subset W$, $p \epsilon V$, and $W.V=0$ (since $W(q_i).V(q_i)=0$ by the definition of the sets $V(q)$ and $W(q)$, for $i=1, 2, \ldots, n$). From $W.V=0$ we get $V \subset K-W$, and since $K-W$ is closed we have $\overline{V} \subset K-W$ and so $\overline{V} \subset K-F=U$, since $F \subset W$. Hence $p \epsilon V$ and $\overline{V} \subset U$, and so, V being any open set, the class K satisfies the axiom of regularity.

We note that axioms $(iv)_a$ and (vii) can be expressed as a single axiom in the following form:

If p and q be two different elements of K, there exist two open sets U and V such that $p \epsilon U$, $q \epsilon V$, and $\overline{U}.V=0$. The proof that this axiom is equivalent to axioms $(iv)_a$ and (vii) combined, does not offer any difficulty.

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[1]See E. W. Chittenden, *Bull. of the Amer. Math. Soc.*, vol. XXXIII (1927), p. 23.

A class K is said to be *normal* (Urysohn) if it satisfies the following

CONDITION OF NORMALITY: If *P and Q be two closed, mutually exclusive sets, there exist two mutually exclusive, open sets, one of which contains P and the other Q.*

Axioms $(iv)_a$ and (vii) follow, as is easily seen, from the condition of normality (and hypotheses (I) and (II)). In fact, to obtain axiom $(iv)_a$ it is sufficient to apply the condition of normality (for two different elements p and q of the class considered) by letting $P=(p)$ and $Q=(q)$. To deduce axiom (vii) from the condition of normality it will be sufficient to put $P=(p)$ for $p \epsilon U$, where U is an open set, and $Q=K-U$; since P and Q are closed and mutually exclusive sets there exist by the condition of normality open sets V and W such that $P \subset V$, $Q \subset W$, and $V.W=0$, and so, since W is open, $\overline{V}.W=0$; hence, $\overline{V} \subset K-W \subset K-Q=U$.[2]

39. Theorem 44. *A class K which satisfies axioms (i)-(vii) is normal.*[3]

We shall even prove a stronger property than that of normality of a class K which satisfies axioms (i)-(vii), namely:

If P and Q be two mutually exclusive sets neither of which contains limit elements of the other, then there exist open sets U and V such that $P \subset U$, $Q \subset V$, *and* $U.V=0$.

Suppose, therefore, that the class K satisfies axioms (i)-(vii), and let P and Q be two sets contained in K and such that $P.\overline{Q}=\overline{P}.Q=0$.

Thus if $p \epsilon P$, then $p \epsilon K-\overline{Q}$, and since $K-\overline{Q}$ is open ($\overline{Q}$ being closed) there exists by (vii) an open set V such that $p \epsilon V$ and $\overline{V} \subset K-\overline{Q}$, and so $\overline{V}.\overline{Q}=0$. By Corollary 1 of § 21, there exists a rational set W such that $p \epsilon W \subset V$; hence, $\overline{W} \subset \overline{V}$; and $\overline{W}.\overline{Q}=0$.

Hence for every element p of P there exists a rational set W, such that $p \epsilon W$, and $\overline{W}.\overline{Q}=0$.

Let

$$U_1,\ U_2,\ U_3,\ \ldots \tag{2}$$

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[2]Hypotheses (I) and (II) are required here to be able to conclude that a set consisting of one element only is closed, and the complement of an open set is closed (Theorems 24 and 2).

[3]A. Tychonoff, *Math. Annalen*, vol. XCV (1925), pp. 139 *et seq.*

denote a sequence of successive terms W_n of the sequence (1) of § 21, such that $\overline{W}_n.\overline{Q} = 0$. We shall have then

(3) $$P \subset U_1 + U_2 + U_3 + \ldots\,; \quad \overline{U}_n.\overline{Q} = 0, \text{ for } n = 1, 2, \ldots$$

Similarly, let

(4) $$V_1, V_2, V_3, \ldots.$$

denote a sequence of successive terms W_n of the sequence (1) of § 21, such that $\overline{W}_n.\overline{P} = 0$. Then

(5) $$Q \subset V_1 + V_2 + V_3 + \ldots; \quad \overline{V}_n.\overline{P} = 0, \text{ for } n = 1, 2, \ldots$$

We may suppose that the sequences (2) and (4) are infinite, repeating one of the terms an infinite number of times if necessary.

Let now

(6) $$G_1 = U_1,\ H_1 = V_1 - \overline{G}_1,$$

and for $n > 1$,

(7) $$G_n = U_n - (\overline{H}_1 + \overline{H}_2 + \ldots + \overline{H}_{n-1}),\ H_n = V_n - (\overline{G}_1 + \overline{G}_2 + \ldots + \overline{G}_n);$$

finally,

(8) $$G = G_1 + G_2 + G_3 + \ldots,\ H = H_1 + H_2 + H_3 + \ldots.$$

It follows at once from (6), (7), and (8) that the sets G and H are open (since the sets (2) and (4) are open and because of Theorem 26). It will be shown that

(9) $$P \subset G,\ Q \subset H, \text{ and } G.H = 0.$$

Suppose $p \in P$; there exists an index m by (3) such that $p \in U_m$. But by (6) and (7) $H_n \subset V_n$, and so $\overline{H}_n \subset \overline{V}_n$, and by (5), $\overline{H}_n.P \subset \overline{V}_n.\overline{P}$; hence, $\overline{H}_n.P = 0$, for $n = 1, 2, 3, \ldots$; since $p \in P$ and $p \in U_m$, it follows from (6) and (7) that $p \in G_m$ and so $p \in G$ by (8). We have thus proved that $P \subset G$. Similarly, it may be proved that $Q \subset H$.

Evidently, from (8), in order to prove that $G.H = 0$, it will be sufficient to show that $G_k.H_l = 0$ for every pair of integers, k and l. If $k \leq l$, then by (7) $H_l = V_l - (\overline{G}_1 + \overline{G}_2 + \ldots + \overline{G}_l) \subset V_l - \overline{G}_k \subset V_l - G_k$, and so $G_k.H_l = 0$. If $k > l$, then by (7) $G_k = U_k - (\overline{H}_1 + \overline{H}_2 + \ldots + \overline{H}_{k-1}) \subset U_k - \overline{H}_l \subset U_k - H_l$, and so $G_k.H_l = 0$.

Hence (9) is proved. The class K is, therefore, normal.

40. Theorem 45. *Every perfect, compact, non-null set has the potency of the continuum.*

To prove the above we shall first establish the following

LEMMA. *For every perfect, non-null set P there exist perfect sets P_0 and P_1 such that*

$$0 \neq P_0 \subset P,\ 0 \neq P_1 \subset P \text{ and } P_0.P_1 = 0.$$

In fact, let P be a given perfect set, p_0 and p_1 two different elements of P.[4] By axiom $(iv)_a$ there exist open sets U_0 and U_1 such that $p_0 \epsilon U_0$, $p_1 \epsilon U_1$ and $U_0.U_1 = 0$. In virtue of (vii) we conclude that there exist open sets V_0 and V_1 such that $p_0 \epsilon V_0$, $p_1 \epsilon V_1$, $\overline{V}_0 \subset U_0$, $\overline{V}_1 \subset U_1$, and so $\overline{V}_0.\overline{V}_1 = 0$ since $U_0.U_1 = 0$.

Since P is perfect the set $V_0.P$ is dense-in-itself (§ 22) and not null (since $p_0 \epsilon P$ and $p_0 \epsilon V_0$). But, by the corollary to Theorem 6 (§ 6), the enclosure of a set dense-in-itself is dense-in-itself and, therefore, perfect; the set $P_0 = \overline{(V_0.P)}$ is, therefore, perfect, not null (since $p_0 \epsilon V_0.P \subset P_0$), and $P_0 \subset \overline{P} = P$. Moreover, $P_0 \subset \overline{V}_0$ since $V_0.P \subset V_0$.

Similarly, it may be shown that there exists a perfect set P_1 not null, such that $P_1 \subset P$ and $P_1 \subset \overline{V}_1$. Since $\overline{V}_0.\overline{V}_1 = 0$, the conditions of the lemma are obviously satisfied by the sets P_0 and P_1; the lemma is, therefore, proved.

We shall apply this lemma to prove Theorem 45. Let P denote a given perfect, compact set. There exist by the above lemma two perfect sets P_0 and P_1 such that

$$0 \neq P_0 \subset P,\ 0 \neq P_1 \subset P, \text{ and } P_0.P_1 = 0.$$

Applying the lemma to the perfect set P_0 we find that there exist two perfect sets P_{00} and P_{01} such that

$$0 \neq P_{00} \subset P_0,\ 0 \neq P_{01} \subset P_0, \text{ and } P_{00}.P_{01} = 0.$$

Suppose now that all the perfect, non-null sets $P_{\alpha_1 \alpha_2 \ldots \alpha_k}$ have been defined, where $\alpha_1 \alpha_2 \ldots \alpha_k$ denote any combination of k numbers each of which is equal to zero or one. By $P_{\alpha_1 \alpha_2 \ldots \alpha_k 0}$ and $P_{\alpha_1 \alpha_2 \ldots \alpha_k 1}$ we shall mean perfect sets such that

[4] Such elements exist since a non-null, perfect set cannot consist of one element only.

(10) $0 \neq P_{\alpha_1 \alpha_2 \dots \alpha_k 0} \subset P_{\alpha_1 \alpha_2 \dots \alpha_k}$, $0 \neq P_{\alpha_1 \alpha_2 \dots \alpha_k 1} \subset P_{\alpha_1 \alpha_2 \dots \alpha_k}$,

and

(11) $$P_{\alpha_1 \alpha_2 \dots \alpha_k 0} \cdot P_{\alpha_1 \alpha_2 \dots \alpha_k 1} = 0;$$

these sets will certainly exist by our lemma.

The sets $P_{\alpha_1 \alpha_2 \dots \alpha_k}$, where $\alpha_1 \alpha_2 \dots \alpha_k$ denote any finite combination of the numbers 0 and 1, are thus defined by induction. Let now

(12) $$\alpha_1, \alpha_2, \alpha_3, \dots$$

denote any infinite sequence obtained from the numbers 0 and 1. Consider the infinite product of the sets

(13) $$P_{\alpha_1} \cdot P_{\alpha_1 \alpha_2} \cdot P_{\alpha_1 \alpha_2 \alpha_3} \dots$$

It follows from (10) that the factors of the above product form a descending sequence of non-null, compact sets, since they are subsets of a compact set; they are also closed (since perfect). The conditions of Theorem 27 (Cantor) are therefore satisfied, and so (13) is not a null set. Let $p(\alpha_1, \alpha_2, \alpha_3, \dots)$ denote any element of this product.

To every infinite sequence (12) which consists of the two numbers 0 and 1 (and there is, as is well known, a continuum of such sequences) there corresponds a certain element $p(\alpha_1, \alpha_2, \dots)$ of P. It is easily seen that to different sequences (12) there will correspond different elements of P. In fact, let

$$\beta_1, \beta_2, \beta_3, \dots$$

denote an infinite sequence obtained from the numbers 0 and 1 and different from the sequence (12); hence, there exist indices n for which $\alpha_n \neq \beta_n$. Let m be the smallest of these indices; we shall have then

(14) $$\alpha_i = \beta_i, \text{ for } i = 1, 2, \dots, m-1, \text{ and } \alpha_m \neq \beta_m,$$

and so if *e.g.* $\alpha_m < \beta_m$, then

(15) $$\alpha_m = 0, \ \beta_m = 1.$$

From the property of the elements $p(\alpha_1, \alpha_2, \dots)$ and $p(\beta_1, \beta_2, \dots)$ it follows that

(16) $$p(\alpha_1, \alpha_2, \ldots) \epsilon P_{\alpha_1 \alpha_2 \ldots \alpha_m}$$
and
(17) $$p(\beta_1, \beta_2, \ldots) \epsilon P_{\beta_1\beta_2\ldots\beta_m}.$$

But from (14) and (15) we have $P_{\beta_1\beta_2\ldots\beta_m} = P_{\alpha_1\alpha_2\ldots\alpha_{m-1}1}$, and $P_{\alpha_1\alpha_2\ldots\alpha_m} = P_{\alpha_1\ldots\alpha_{m-1}0}$; hence, from (11)

$$P_{\alpha_1\alpha_2\ldots\alpha_m} \cdot P_{\beta_1\beta_2\ldots\beta_m} = 0,$$

and so from (16) and (17)

$$p(\alpha_1, \alpha_2, \ldots) \neq p(\beta_1, \beta_2, \ldots).$$

The set of elements $p(\alpha_1, \alpha_2, \ldots)$ which are correlated with the infinite sequences $\alpha_1, \alpha_2, \alpha_3, \ldots$ consisting of the two numbers 0 and 1, is, therefore, a subset of P of the potency of the continuum. Hence the set P has the potency of the continuum at least. On the other hand, the set P formed of elements of the class K has, by Corollary 3 to axiom (*vi*) (§ 21), the potency of the continuum at most. The set P has, therefore, the potency of the continuum; this proves Theorem 45.

Let now p denote any element of a perfect and compact set P, and U any open set containing p.

It follows from the lemma that there exists in U a perfect non-null subset of P and so, by Theorem 45, a subset of the potency of the continuum (since a subset of the compact set P is compact).

If, therefore, p be an element of a perfect and compact set P, then every open set containing p contains a continuum of elements of P.

It follows immediately from the above that *every element of a perfect and compact set is an element of condensation of that set.*

From Theorems 32 and 45 we get immediately

Theorem 46. *Every non-countable, closed, and compact set has the potency of the continuum.*

In fact, it follows from Theorem 32 that a closed, non-countable set is the sum of a non-countable perfect set and a set countable at most; Theorem 45 will apply to the above perfect subset, which is compact, since it is a subset of a compact set.

41. We shall now prove an auxiliary theorem which will be made use of in the present as well as in the following chapter.

URYSOHN'S LEMMA. *If P and Q be two mutually exclusive, closed sets, there exists a real function $f(p)$ defined and continuous in the set K, and such that $0 \leq f(p) \leq 1$ throughout, with $f(p)=0$ for $p \in P$, and $f(p)=1$ for $p \in Q$.*

Proof. Let P and Q be two mutually exclusive, closed sets. Put $G(1)=K-Q$; hence, $G(1)$ will be an open set and $P \subset G(1)$ (since $P.Q=0$). By the condition of normality (§ 38; Theorem 44, § 39) there exist open sets U and V such that $P \subset U$, $Q \subset V$, and $U.V=0$; put $G(\frac{1}{2})=U$; hence,

$$P \subset G(\tfrac{1}{2}) \text{ and } \overline{G(\tfrac{1}{2})} \subset G(1).$$

(Since $U.V=0$ and V is open, we have $\overline{U}.V=0$, and so certainly $\overline{U}.Q=0$; therefore, $\overline{U} \subset K-Q=G\ (1)$.)

Let now m denote a given integer and suppose that all sets $G\left(\frac{k}{2^n}\right)$ have been defined, where n is an integer $\leq m$, and $k=1$, 2, 3, . . . , 2^n-1, and where

$$P \subset G\left(\frac{k}{2^n}\right) \overline{\subset G\left(\frac{k}{2^n}\right)} \subset G\left(\frac{k+1}{2^n}\right), \tag{18}$$

for $n \leq m$, $k=1, 2, \ldots, 2^n-1$.

We have then from (18) $P \subset G\left(\frac{1}{2^m}\right)$, and so, as formerly in the case when $P \subset G\ (1)$, there exists an open set which will be denoted by $G\left(\frac{1}{2^{m+1}}\right)$ and such that

$$P \subset G\left(\frac{1}{2^{m+1}}\right), \text{ and } \overline{G\left(\frac{1}{2^{m+1}}\right)} \subset G\left(\frac{1}{2^m}\right).$$

If, however, $1 \leq k \leq 2^{m-1}$, then by (18) $\overline{G\left(\frac{k}{2^m}\right)} \subset G\left(\frac{k+1}{2^m}\right)$, and so there exists an open set, which we shall denote by $G\left(\frac{2k+1}{2^{m+1}}\right)$, such that

$$\overline{G\left(\frac{k}{2^m}\right)} \subset G\left(\frac{2k+1}{2^{m+1}}\right) \subset \overline{G\left(\frac{2k+1}{2^{m+1}}\right)} \subset G\left(\frac{k+1}{2^m}\right).$$

The open sets $G\left(\frac{k}{2^n}\right)$ are thus defined by induction for every integer n and for $k \leq 2^n$, and they all satisfy relation (18).

We shall define now a real function f of the elements p of K as follows.

Let p denote a given element of K. If $p \in Q$, then put $f(p)=1$. If $p \bar{\in} Q$, *i.e* $p \in K-Q=G\,(1)$, then there are numbers $\frac{k}{2^n}$ such that $p \in G\left(\frac{k}{2^n}\right)$; if t be the lower bound of such numbers $\frac{k}{2^n}$, put $f(p)=t$.

Obviously $f(p)=0$ for $p \in P$, since by (18) $P \subset G\left(\frac{1}{2^n}\right)$, for $n=1, 2, \ldots$; the lower bound of the numbers $\frac{k}{2^n}$ such that $p \in G\left(\frac{k}{2^n}\right)$ is in this case equal to zero. It is also clear that we have $0 \leq f(p) \leq 1$ throughout (since the numbers $\frac{k}{2^n}$ are all >0 and ≤ 1).

Hence, it only remains to prove that the function $f(p)$ is continuous in the set K.

Let p_0 denote a given element of K, ϵ any positive number. We shall consider three cases:

1. $f(p_0)=1$. Let m be an integer such that $\frac{1}{2^m}<\epsilon$, and put $U=K-\overline{G\left(\frac{2^m-1}{2^m}\right)}$; U will be an open set containing p_0 $\Big($for if $p_0 \in \overline{G\left(\frac{2^m-1}{2^m}\right)} \subset G\left(\frac{2^{m+1}-1}{2^{m+1}}\right)$ we should have from the definition of the function f, $f(p_0) \leq \frac{2^{m+1}-1}{2^{m+1}} <1$, contrary to hypothesis$\Big)$. If $p \in U$ then $p \bar{\in} \overline{G\left(\frac{2^m-1}{2^m}\right)}$ and so certainly $p \bar{\in} G\left(\frac{2^m-1}{2^m}\right)$; by (18) the relation $p \bar{\in} G\left(\frac{k}{2^n}\right)$ implies that $p \bar{\in} G\left(\frac{r}{2^s}\right)$ for $\frac{r}{2^s} < \frac{k}{2^n}$ and, therefore, from the definition of the function f it will follow that $f(p) \geq \frac{2^m-1}{2^m}=1-\frac{1}{2^m}>1-\epsilon$, and since, on the other hand, we have

$f(p) \leq 1$ throughout, hence $1-\epsilon < f(p) \leq 1$ for $p \epsilon U$. Since ϵ (which determines the open set U containing p_0) is an arbitrary number, the function f is continuous at p_0.

2. $f(p_0)=0$. Again let m be an integer such that $\frac{1}{2^m} < \epsilon$, and put $U=G\left(\frac{1}{2^m}\right)$; U will be an open set containing p_0. For it follows from the definition of the function that if $f(p_0)=0$, there exists a number of the type $\frac{k}{2^n} < \frac{1}{2^m}$ and such that $p_0 \epsilon G\left(\frac{k}{2^n}\right)$ and so by (18), since $\frac{k}{2^n} < \frac{1}{2^m}$, we have $G\left(\frac{k}{2^n}\right) \subset G\left(\frac{1}{2^m}\right)$. Hence for $p \epsilon U$ we shall have, from the definition of f, $f(p) \leq \frac{1}{2^m} < \epsilon$, and since $f(p) \geq 0$, we get $0 \leq f(p) < \epsilon$ whenever $p \epsilon U$; this proves the continuity of the function f in the set K at the element p_0.

3. $0 < f(p) < 1$. In this case there exist, as is easily seen, integers n and k both >1 and such that $\frac{1}{2^{n-1}} < \epsilon$ and $\frac{k}{2^n} < f(p_0) < \frac{k+1}{2^n} \leq 1$. Hence, from the definition of f and from (18), $p_0 \bar{\epsilon}\, G\left(\frac{k}{2^n}\right)$ and certainly $p_0 \bar{\epsilon}\, \overline{G\left(\frac{k-1}{2^n}\right)}$, but $p_0 \epsilon G\left(\frac{k+1}{2^n}\right)$. Put $U=G\left(\frac{k+1}{2^n}\right) - \overline{G\left(\frac{k-1}{2^n}\right)}$; then $p_0 \epsilon U$, and U is obviously an open set. If $p \epsilon U$, then $p \epsilon G\left(\frac{k+1}{2^n}\right)$ and $p \bar{\epsilon}\, \overline{G\left(\frac{k-1}{2^n}\right)}$, and so certainly $p \bar{\epsilon}\, G\left(\frac{k-1}{2^n}\right)$; from the definition of the function it will then follow that $\frac{k-1}{2^n} \leq f(p) \leq \frac{k+1}{2^n}$. Since $p_0 \epsilon U$ and $\frac{1}{2^{n-1}} < \epsilon$, we have $|f(p)-f(p_0)| \leq \epsilon$ whenever $p \epsilon U$; the function f is, therefore, continuous in K at the element p_0.

Urysohn's lemma is, therefore, proved.

42. Let S be a connected set containing more than one element, and let p_0 be a given element of S and U an open set containing

p_0. If there were no other elements of S in U than p_0, the sets (p_0) and $S-(p_0)$ would be, as is easily seen, separated, contrary to the supposition that S is connected. There exists in U, therefore, an element p_1 of S different from p_0. Put $P=(p_0)$, $Q=(p_1)+(K-U)$; P and Q are clearly two closed, mutually exclusive sets. Hence, by Urysohn's lemma, there exists a real function $f(p)$ defined and continuous in K and such that $0 \leq f(p) \leq 1$, where $f(p)=0$ for $p \epsilon P$ (*i.e.* for $p=p_0$), and $f(p)=1$ for $p \epsilon Q$. $f(p)$ being continuous in K will certainly be continuous in $S \subset K$, where $f(p_0)=0$, and $f(p_1)=1$ (since $p_1 \epsilon Q$).

By the Corollary to Theorem 21 (§ 11) the function f will take in the connected set S every value between 0 and 1. But $K-U \subset Q$, and $f(p)=1$ for $p \epsilon Q$; hence f can take values $\neq 1$ in U only. Hence f must take every value between 0 and 1 in the set $S.U$. The set $S.U$ has, therefore, the potency of the continuum. Hence

Theorem 47. *If S be a connected set containing more than one element, and if U be an open set such that $S.U \neq 0$, then $S.U$ has the potency of the continuum.*

We may remark (with the hypothesis of Theorem 47) that the set $S.U$ need not be connected and that it may not even contain any connected subsets consisting of more than one element.[5]

As an immediate result from Theorem 47 it follows that

Every connected set containing more than one element has the potency of the continuum.

From the fact that a continuum is a closed, connected, and compact set containing more than one element (§ 34), we may deduce by means of Theorem 47 the following

COROLLARY. *If a set C be a continuum, then every open set containing an element of C contains a continuum of elements of C.*

This property justifies the name of a continuum.

We shall note here that the property of a continuum deduced above follows also immediately from an analogous property of perfect sets (deduced in § 40), namely from the fact that every continuum is a perfect and compact set (since it is compact and closed, also connected and containing more than one element, hence not containing isolated points).

[5]See *Fund. Math.*, vol. II, p. 244.

CHAPTER VI

METRIC SPACES

43. An aggregate M is called a *metric space* (Hausdorff) if to every two elements a and b of M there is assigned a certain real non-negative number $\rho(a, b)$, called the distance of the elements a and b, in a perfectly arbitrary manner apart from requiring it to satisfy the following three conditions (so-called distance axioms):

1) $\rho(b, a) = \rho(a, b)$ (law of symmetry);
2) $\rho(a, b) = 0$, when and only when $a = b$;
3) $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$ for every three elements a, b, c of M (the triangle law).[1]

A subset of a metric space is evidently a metric space.

Let E denote a set contained in M. An element a of M (belonging to E or not) is said to be a *limit element* of E, if corresponding to every positive number ϵ there exists at least one element p of E such that

(1) $$0 < \rho(a, p) < \epsilon.$$

The set of all limit elements of E is called the derived set of E and is denoted by E'. A set E, such that $E \supset E'$, is said to be closed. The complement of a closed set (with respect to M) is called an open set.

44. Denote by $K(p, r)$ the set of all elements q of the aggregate M which satisfy the condition

$$\rho(p, q) < r,$$

for a given element p and a given positive number r.

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[1] A. Lindenbaum remarked that conditions 1) and 3) can be replaced (retaining, of course, condition 2)), by one condition:

$$\rho(a, c) \leq \rho(a, b) + \rho(c, b),$$

for every three elements a, b, c, of M (*Fund. Math.*, vol. VIII, p. 211).

Then *for a set $E \subset M$ to be open, it is necessary and sufficient that for every element $p \in E$ there should exist a positive number r such that $K(p, r) \subset E$.*

In fact, if the existence of a number r for a certain element p of E is denied, it follows that the set

$$K\left(p, \frac{1}{n}\right) . (M-E) \neq 0, \text{ for } n=1, 2, 3, \ldots,$$

and for every integer n there exists an element q_n such that

$$(2) \qquad q_n \in K\left(p, \frac{1}{n}\right) . (M-E), \text{ for } n=1, 2, 3, \ldots;$$

hence, from the definition of $K(p, r)$

$$\rho(p, q_n) < \frac{1}{n}, \text{ for } n=1, 2, 3, \ldots,$$

and, therefore, since by (2) $q_n \in M-E$, for $n=1, 2, 3, \ldots$, from the definition of a limit element and a derived set, we have

$$p \in (M-E)'.$$

But $p \in E$; hence, $M-E$ is not a closed set, and so E is not an open set. The condition of the theorem is, therefore, necessary.

Suppose, on the other hand, that the condition is satisfied, and suppose that E is not open. Then $M-E$ is not closed, and so there exists a limit element p of $M-E$, which does not belong to $M-E$. Thus $p \in (M-E)'$, and $p \in E$. Since $p \in E$ and from the fact that the condition is satisfied for E, there exists a positive number r such that $K(p, r) \subset E$. But, since $p \in (M-E)'$, it follows from the definitions of a derived set and a limit element that there exists an element $q \in M-E$ such that $\rho(p, q) < r$, and so $q \in K(p, r)$; from this it follows that $K(p, r) . (M-E) \neq 0$, contrary to the result $K(p, r) \subset E$ deduced above. The condition is, therefore, sufficient.

We shall prove next that *every set $K(p, r)$* (where $p \in M$, and $r > 0$) *is open.*

Let p denote a given element of the aggregate M, r a given positive number, and suppose that $q \in K(p, r)$; we have then from the definition of the set $K(p, r)$

$$\rho(p, q) < r.$$

The number $r_1 = r - \rho(p, q)$ is, therefore, positive. It will be shown that $K(q, r_1) \subset K(p, r)$. In fact, if $q_1 \in K(q, r_1)$ we have

$$\rho(q, q_1) < r_1,$$

and so, by the triangle law (condition 3)),

$$\rho(p, q_1) \leq \rho(p, q) + \rho(q, q_1) < \rho(p, q) + r_1 = r;$$

hence $\rho(p, q_1) < r$, *i.e* $q_1 \in K(p, r)$.

Hence, from the condition for a set to be open deduced above, it follows at once that the set $K(p, r)$ is open.

The set $K(p, r)$ (where $p \in M$, and $r > 0$) is called an *open sphere* of centre p and radius r.

We shall next show that *for an element $p \in M$ to be a limit element of a set $E \subset M$, it is necessary and sufficient that every open set containing p contain an element of E different from p.* The sufficiency of the condition follows at once from the definitions of a limit element and the set $K(p, r)$, and from the fact that the latter is an open set (for $p \in M$, and $r > 0$).

Suppose now that $p \in E'$, and let U be an open set such that $p \in U$. By the property of open sets deduced above, there exists a positive number r such that $K(p, r) \subset U$. But, since $p \in E'$, there exists an element $q \in E$ such that $0 < \rho(p, q) < r$; hence $q \neq p$, and $q \in K(p, r)$, and so certainly $q \in U$. The condition is, therefore, necessary.

45. It will now be shown that *the distance $\rho(p, q)$ is a continuous function of the two variables p and q* (in the whole metric space M). In other words, it will be shown that for every two elements p_0 and q_0 (different or not) of the metric space considered, and for every positive number ϵ there exist open sets U and V such that $p_0 \in U$, $q_0 \in V$, and for which the relations

$$p \in U \text{ and } q \in V$$

imply the inequality

$$|\rho(p, q) - \rho(p_0, q_0)| < \epsilon.$$

In fact, put $U = K\left(p_0, \frac{\epsilon}{2}\right)$, $V = K\left(q_0, \frac{\epsilon}{2}\right)$ for the given elements p_0 and q_0 and a given positive number ϵ. If $p \in U$ and $q \in V$, then $\rho(p_0, p) < \frac{\epsilon}{2}$, and $\rho(q_0, q) < \frac{\epsilon}{2}$, and so, by the law of symmetry and the triangle law,

$$\rho(p, q) \leq \rho(p, p_0) + \rho(p_0, q_0) + \rho(q_0, q) < \rho(p_0, q_0) + \epsilon,$$

and

$$\rho(p_0, q_0) \leq \rho(p_0, p) + \rho(p, q) + \rho(q, q_0) < \rho(p, q) + \epsilon,$$

whence

$$-\epsilon < \rho(p, q) - \rho(p_0, q_0) < \epsilon.$$

46. We shall next show that open sets of a metric space satisfy hypotheses (I) (§ 1) and (II) (§ 15), and the condition of normality.

Axioms (*i*) and (*ii*) follow immediately from the definitions of an open, closed, and derived set respectively and from that of a limit element. Axiom (*iii*) follows from the necessary and sufficient condition for a set to be open, deduced in § 44.

To prove axiom (*iv*), it will be sufficient to remark that, if p and q are two different elements, then $\rho(p, q) = r > 0$; the set $K(p, r)$ is, therefore, an open set which contains p but does not contain q.

To prove axiom (*v*), suppose that $T = U.V$, where U and V are two open sets. If $p \epsilon U.V$, then, since $p \epsilon U$ and from the condition for an open set (§ 44), it follows that there exists a number $r_1 > 0$, such that $K(p, r_1) \subset U$. Similarly, since $p \epsilon V$ (and V is open), there exists a number $r_2 > 0$, such that $K(p, r_2) \subset V$. Denote by r a positive number $\leq r_1$ and $\leq r_2$; then, obviously, $K(p, r) \subset K(p, r_1)$, and $K(p, r) \subset K(p, r_2)$, and so $K(p, r) \subset U.V = T$. Hence, for every element $p \epsilon T$ there exists a number $r > 0$, such that $K(p, r) \subset T$, and this, as we know, proves that T is an open set.

We shall prove now that metric spaces satisfy the condition of normality (§ 38).

Hence let P and Q be two mutually exclusive, closed sets (contained in a given metric space M).

If p is an element of the set P, then, since $P.Q = 0$ and Q is closed, we have $P.\bar{Q} = 0$; there exists, therefore, by the definition of a limit element, a positive number $r = \phi(p)$, such that $K(p, r) . Q = 0$. Similarly, there exists for every element $q \epsilon Q$ a positive number

$r=\phi(q)$, such that $K(q, r) \cdot P=0$.[2] Denote by U the sum of all sets $K(p, \frac{1}{2}\phi(p))$, where $p \in P$, and by V the sum of all sets $K(q, \frac{1}{2}\phi(q))$, where $q \in Q$. The sets U and V are evidently open (since they are sums of open sets); also $P \subset U$, and $Q \subset V$ (since $p \in K(p,r)$, for every $r>0$). It remains to be proved that $U.V=0$.

Suppose, on the contrary, that $U.V \neq 0$; there exists an element a, such that $a \in U$, and $a \in V$. Since $a \in U$ and from the definition of the set U, there exists an element $p \in P$, such that $a \in K(p, \frac{1}{2}\phi(p))$. Similarly, since $a \in V$ and from the definition of Q, we conclude the existence of an element $q \in Q$, such that $a \in K(q, \frac{1}{2}\phi(q))$. We have then simultaneously

$$\rho(p, a) < \tfrac{1}{2}\phi(p), \text{ and } \rho(q, a) < \tfrac{1}{2}\phi(q),$$

and so (by the law of symmetry and the triangle law)

$$\rho(p, q) < \tfrac{1}{2}\phi(p) + \tfrac{1}{2}\phi(q).$$

If $\phi(p) \geq \phi(q)$, we have then

$$\rho(p, q) < \phi(p);$$

hence, $q \in K(p, \phi(p))$, contrary to the definition of the number $\phi(p)$ (since $q \in Q$, and $K(p, \phi(p)) \cdot Q=0$). Similarly, we arrive at a contradiction if we assume that $\phi(q) \geq \phi(p)$, which gives $\rho(p, q) < \phi(q)$, and $p \in K(q, \phi(q))$. We must, therefore, have $U.V=0$.

We have thus proved that metric spaces satisfy the condition of normality. From this and hypotheses (I) and (II) we conclude (§ 38) that axioms $(iv)_a$ and (vii) are satisfied.

Looking over the proof of normality of metric spaces, we realize that we have proved a property of them stronger even than that of normality, namely the property expressed in § 39.

It follows at once from the properties of open sets proved in this article and from the necessary and sufficient condition for a given element to be a limit element, deduced towards the close of § 44, that every metric space is a class K which satisfies

[2]To avoid the axiom of selection it could be here assumed that $r=\phi(p)$ is the first term of a given infinite sequence of rational numbers satisfying the condition $K(p, r) \cdot Q=0$ with a similar convention for $\phi(q)$.

axioms (*i*), (*ii*), (*iii*), $(iv)_a$, (*v*), and (*vii*). It can also be stated that, if open sets be taken to mean neighbourhoods of elements a (in a metric space), then a metric space will be a normal (§§ 19, 38) topological space (Hausdorff); it will also be a normal class (H) of Fréchet.

Axiom (*vi*) may not, however, be satisfied in a metric space. This follows already from the fact that the set of elements of a metric space may have any potency. In fact, every aggregate may become a metric space if the distance between every two elements of the aggregate be appropriately defined. It would be sufficient to put $\rho(p, q)=1$ for every two different elements of the given aggregate. (Conditions 1), 2), and 3) of the distance function (§ 43) would here be satisfied, as could be easily shown.) Moreover, we shall show in the next article that axiom (*vi*) is satisfied in every compact, metric space.

47. Let M be a given metric space, E a given compact set contained in M (*i.e.* such that every infinite subset of E has a non-null derived set), and let ϵ be a given positive number. Take any element p_1 of E. If there are elements p in E such that $\rho(p_1, p)\geq\epsilon$, then let p_2 denote one of them. If, further, there are elements p of E such that

$$\rho(p_1, p)\geq\epsilon, \text{ and } \rho(p_2, p)\geq\epsilon,$$

let p_3 be one of them.

Generally, suppose that we have obtained the elements p_1, $p_2, \ldots, p_n$ of E; if there are elements p of E such that

$$\rho(p_k, p)\geq\epsilon, \text{ for } k=1, 2, \ldots, n,$$

let p_{n+1} be one of them.

It will be shown that the sequence of elements $p_1, p_2, p_3, \ldots$ thus obtained, cannot be infinite. In fact, if it were infinite, then (from the fact that the terms of the sequence are all different, since every element is at a distance ϵ at least from the preceding one) the set E_1 of the terms, as an infinite subset of the compact set E, would have a non-null derived set, and so there would exist a limit element a of the set E_1. The sphere $K(a, \epsilon/2)$ would have to contain an infinite number of elements of E_1; in any case, there would exist indices k and $l > k$, such that

$$\rho(a, p_k) < \epsilon/2 \text{ and } \rho(a, p_l) < \epsilon/2,$$

and so, by the law of symmetry and the triangle law,

$$\rho(p_k, p_l) \leq \rho(p_k, a) + \rho(a, p_l) < \epsilon,$$

whereas from the definition of the sequence $p_n (n = 1, 2, \ldots)$ it follows that (for $l > k$) we must have

$$\rho(p_k, p_l) \geq \epsilon.$$

We have thus proved

Theorem 48. *If E be a compact set, then for every positive number ϵ there exists a finite sequence $p_1, p_2, \ldots, p_n$ of elements of E, such that every element p of E is at a distance less than ϵ from at least one of the elements of that sequence.*

Let now in Theorem 48 $\epsilon = \dfrac{1}{m}$, where m is a positive integer, and denote the corresponding sequence by $p_1^m, p_2^m, \ldots, p_{n_m}^m$. Denote by P the set of all different terms of the infinite sequence

$$p_1^1, p_2^1, \ldots, p_{n_1}^1, p_1^2, p_2^2, \ldots, p_{n_2}^2, p_1^3, \ldots, p_{n_3}^3, p_1^4, \ldots$$

For every element p of E and every positive integer m we shall have for some index $k \leq n_m$ (dependent on p and m)

$$\rho(p, p_k^m) < \frac{1}{m},$$

and so, from the definition of a limit element (§ 43) and the above, we conclude that p is either an element or a limit element of P. In any case, we have $E \subset P + P' = \overline{P}$. Hence, we obtain

Theorem 49. *Every compact set* (in a metric space) *contains a finite or a countable subset P such that*

$$E \subset \overline{P}.$$

REMARK. Theorem 49 is a particular case of Theorem 33 of chapter III; the theorems of chapter III, however, deduced as they are from axiom (*vi*), which axiom may not apply in a metric space, may not hold in a metric space, and in any case, any attempt at their proof in a metric space should avoid the use of axiom (*vi*).

Theorem 33 itself is not true in some metric spaces (*e.g.* in spaces of potency greater than that of the continuum).

As an immediate corollary to Theorem 49 we get:

A closed and compact set is the enclosure of a certain set countable at most.

We shall note also the following immediate corollary to Theorem 48:

The set of all distances between pairs of elements of a compact set E is always bounded.

If E be a given set, then the upper bound of the set of all distances between pairs of elements of E (*i.e.* the upper bound of the set of all numbers $\rho(p, q)$, where $p \epsilon E$ and $q \epsilon E$) is called the *diameter* of the set E, and is denoted by $\delta(E)$. Hence the diameter of a set is a non-negative, real number, finite or infinite, and completely defined for every given set E (contained in the metric space considered). It is easily seen that for a set to be *bounded* (*i.e.* for a set of all the distances between pairs of elements to be bounded), it is necessary and sufficient that it should have a finite diameter.

Diameters of sets E possess the following properties:

$\delta(E)=0$ *when and only when* E *is a null set or contains one element only.*

If $E_1 \subset E$, *then* $\delta(E_1) \leq \delta(E)$.

(This follows at once from the fact that the upper bound of a subset of a given set of real numbers cannot be greater than the upper bound of the whole set.)

If $E_1.E_2 \neq 0$, *then* $\delta(E_1+E_2) \leq \delta(E_1)+\delta(E_2)$;

(the proof is left to the reader).

For every set E

$$\delta(E)=\delta(\bar{E}).$$

(This follows from the continuity of the function ρ and the definition of a diameter; the complete proof is left to the reader.)

Theorem 48, as is easily seen, can be also expressed as follows:

Every compact set can be divided into a finite number of sets of arbitrarily small diameters.

The diameter of the sphere $K(p, r)$ is certainly $\leq 2r$, but may even be $< 2r$ (*e.g.* if in the metric space considered there is no element at a distance less than r from p, then the diameter of $K(p, r)$ is 0). We note also that the diameter of a set E is not in general the lower bound of the diameters of the spheres which contain E.

Theorem 49 is true also for semi-compact sets in metric spaces. (To prove this, it will be sufficient to refer to the definition of semi-compact sets, § 16.)

Let now M denote a compact, metric space. By Theorem 49, there exists a set $P \subset M$ countable at most, such that

$$M = \overline{P}.$$

Let $p_1, p_2, p_3, \ldots$ be a sequence (finite or countable) consisting of all the elements of P. Consider the spheres $K\left(p_k, \frac{1}{n}\right)$, k and n positive integers, to be rational sets.

It will be shown that every open set of the space M is the sum of a certain aggregate of rational sets, namely those which it contains. It will obviously be sufficient to show that, if p is an element of the open set U, there exists a rational set which contains p and is contained in U.

Hence, suppose that U is an open set, and $p \in U$. There exists, therefore (§ 44), a number $r > 0$, such that $K(p, r) \subset U$.

Let n be an integer greater than $\frac{2}{r}$. Since $M = \overline{P}$, there exists an element p_k of P, such that $\rho(p, p_k) < \frac{1}{n}$. If now q be an element of the aggregate M, such that $\rho(p_k, q) < \frac{1}{n}$, then

$$\rho(p, q) \leq \rho(p, p_k) + \rho(p_k, q) < \frac{1}{n} + \frac{1}{n} < r;$$

this proves that $q \in K(p, r) \subset U$, *i.e.* $q \in U$. We have, therefore, $K\left(p_k, \frac{1}{n}\right) \subset U$.

On the other hand, $p \in K\left(p_k, \frac{1}{n}\right)$, since $\rho(p, p_k) < \frac{1}{n}$.

The sphere $K\left(p_k, \frac{1}{n}\right)$ is, therefore, a rational set, which contains p and is contained in U.

We have thus proved that *a compact, metric space satisfies axiom* (*vi*). Hence, all theorems proved in chapters I-V are true for compact, metric spaces.

Since Theorem 49, as remarked above, is true also for semi-compact metric spaces, it follows that these too satisfy axiom (*vi*) and all theorems of the preceding chapters.

It should be noted that we have also proved that, if a metric space M contains a subset P countable at most, such that $M = \overline{P}$ (*i.e.* a subset *everywhere dense* and countable at most), then M satisfies axiom (*vi*). On the other hand, every metric space satisfies axioms (*i*)-(*v*), and so, if it also satisfies axiom (*vi*), Theorem 33 of chapter III will be true in that space, which implies that the latter will contain a subset everywhere dense and countable at most. Hence:

For a metric space axiom (*vi*) *is equivalent to the condition of separability, i.e. that the space considered contains a subset countable at most and everywhere dense* (Fréchet, séparabilité).

It follows readily from the above (by Theorem 33) that, if a metric space contains a subset countable at most and everywhere dense, then every set contained in that space contains a subset countable at most and *dense on this set* (*i.e.* such that the set is contained in the enclosure of the subset).

Furthermore, we shall prove that *in a metric space the following three properties of a set E are equivalent:*

(*A*) *The set E possesses a countable subset dense on E.*

(*B*) *The set E possesses the Lindelöf property* (*i.e.* Theorem 35 is satisfied in E).

(*C*) *Every non-countable subset of E has at least one element of condensation which belongs to E.*

It will be sufficient to show that the property (*A*) implies (*B*), (*B*) implies (*C*), and that (*A*) follows from (*C*).

Suppose then, that the set E (contained in a metric space) has the property (*A*), and let P $(p_1, p_2, p_3, \ldots)$ denote a countable subset of E dense on E, *i.e.*, such that $E \subset \overline{P}$. Let M be an aggregate of open sets, the sum of which contains E. To each pair of positive integers (k, n), such that $K\left(p_k, \frac{1}{n}\right)$ is contained in at least one of the sets of M, let there be assigned one of those sets, $U_{k,n}$ say. Let p denote a given element of E. It follows from the property of the aggregate M that there exists an open set U which belongs to M and contains the element p.

Since $p \epsilon U$, there exists a number $r>0$, such that $K(p, r) \subset U$. Choose an integer $n>\frac{2}{r}$; it follows from the property of the set P that there exists an index k such that $\rho(p, p_k)<\frac{1}{n}$, and so, since $n>\frac{2}{r}$, we conclude readily that $p \epsilon K\left(p_k, \frac{1}{n}\right) \subset K(p, r) \subset U$, which proves that $p \epsilon U_{k, n}$.

Assume now that the set E has the property (B).

If E does not possess the property (C) there exists a non-countable subset N of E such that no element of E is an element of condensation of N. Hence, for every element p of E there exists a sphere $K(p, r)$, $r>0$, which contains a subset of N countable at most. By the property (B) the set E will be contained in the sum S of at most a countable number of such spheres. We should then have $N \subset E \subset S$, which is impossible, since the sum S contains a countable set at most of elements of N. We have thus proved that the property (B) implies the property (C).

Suppose now that the set E has the property (C).

Let n be a given positive integer. To establish the property (A) it will evidently be sufficient to show that there exists a subset P of E countable at most, and such that every element of the set E is at a distance less than $\frac{1}{n}$ from some element of P. Suppose, therefore, that such a subset P does not exist.

Let q_1 be an element of E. Let now α be an ordinal number such that $1<\alpha<\Omega$, and suppose that all q_ξ have been defined for $\xi<\alpha$. The set of all elements q_ξ, $\xi<\alpha$, is at most countable (since $\alpha<\Omega$). It follows, therefore, from our assumption that there exists an element q_α of E, such that $\rho(q_\alpha, q_\xi) \geq \frac{1}{n}$ for $\xi<\alpha$.

Denote by N the set of all elements q_ξ, where $\xi<\Omega$; the set N will obviously be a non-countable subset of E. Clearly, no element of E is a limit element of N, for if p were such an element, there would exist an element q_α of N, such that $0<\rho(p, q_\alpha)<\frac{1}{2n}$, and an element q_β of N, such that $\rho(p, q_\alpha)>\rho(p, q_\beta)<\frac{1}{2n}$, and so $\alpha \neq \beta$, and $\rho(p_\alpha, q_\beta)<\frac{1}{n}$, contrary to the property of the elements of N.

We have thus proved that the property (C) implies the property (A). We note that we have proved rather more, namely, that the property

(C') *Every non-countable subset of E contains at least one limit element belonging to E* implies the property A.[3]

With regard to any metric space we can merely state that it satisfies a condition not quite as strong as axiom (*vi*), namely

[3]Proved by W. Gross in 1914 (*Fund. Math.*, vol. VIII, p. 234).

Corollary 2 to axiom (*vi*) (§ 21), when open sets be substituted for rational sets in that corollary. In fact, it will be sufficient to put $V_n = K\left(a, \frac{1}{n}\right)$ for every element a of a metric space to obtain an infinite sequence of open sets which satisfy the conditions of Corollary 2.

We note, that Hausdorff treats this property as a distinct condition which he refers to as *the first axiom of countability*, in distinction to (the more stringent) axiom (*vi*) which he calls *the second axiom of countability*.

Thus all the theorems of chapters III, IV, and V, the proofs of which are based on axioms (*i*), (*ii*), (*iii*), $(iv)_a$, (*v*), and (*vii*) and on the property just proved (and which do not require axiom (*vi*) in all its implications) are true for every metric space. Such are all the theorems of chapter IV except Theorem 43, and all the theorems of chapter V except Theorem 47; a modification of this last theorem will be true for every metric space whereby the expression "potency of the continuum" is replaced by the expression "potency not less than that of the continuum". But none of the theorems of chapter III are true for every metric space, except Theorem 38, which follows, as is easily seen, from Theorem 49 after a slight modification in the proof of Theorem 38.

47a. Two elements p and q of a given set E are said to be connected by an ϵ-chain in E if there exists a finite sequence p_0, p_1, $p_2, \ldots, p_n$ of elements of E such that

$$p_0 = p,\ p_n = q, \text{ and } \rho(p_{k-1}, p_k) < \epsilon, \text{ for } k = 1, 2, \ldots, n.$$

It will be shown that *if E be a connected set, then for each* $\epsilon > 0$, *every two elements of E can be connected by an ϵ-chain in E.*

To prove the above, suppose that the elements a and b of E cannot be connected by an η-chain in E for some $\eta > 0$. Denote by A the set of all those elements of E (not excluding a itself) which can be connected with a by an η-chain in E; let $B = E - A$. We then have $a \in A$, $b \in B$; hence A and B are not null sets.

It follows that $A.B' = 0$. For, if there should exist an element $p \in A.B'$, then, since $p \in B'$, there would exist an element $q \in B$ such that $\rho(p, q) < \eta$. Since $p \in A$, then by the definition of A,

p can be connected with a by an η-chain in E; and since $\rho(p, q) < \eta$, it follows at once that q can be connected with a by an η-chain in E, contrary to the fact that $q \in B = E - A$. Similarly, should there exist an element $p \in A'.B$, then, since $p \in A'$, there would exist an element $q \in A$, such that $\rho(p, q) < \eta$, and so p could be connected with a by an η-chain in E, contrary to the fact that $p \in B$. We have, therefore, $A \neq 0$, $B \neq 0$, $A.B = A.B' = A'.B = 0$, *i.e.* E is a sum of two separated sets, and hence is not a connected set. Our theorem is, therefore, proved.

The converse is, as is easily seen, not true; *e.g.* the set of all rational numbers is not connected, although every two elements of the set can be connected by an ϵ-chain for every $\epsilon > 0$. We shall prove, however, that

If a set E is closed and compact, and if every pair of its elements can be connected by an ϵ-chain in E, for every $\epsilon > 0$, then E is connected.

To prove it, suppose that E is not connected. The set E being closed and not connected is by Theorem 13 the sum of two mutually exclusive, closed, non-null sets A and B. Since $A \neq 0$, and $B \neq 0$, there exist elements a and b such that $a \in A$ and $b \in B$. Suppose that a and b can be connected by an ϵ-chain in E for every ϵ. In such a chain, in which the first term is a (hence an element of A) and the last term is b (and so an element of B), there exist two neighbouring terms, one of which belongs to A, the other to B. Hence, for every positive integer n there exist elements p_n and q_n such that $p_n \in A$, $q_n \in B$, and $\rho(p_n, q_n) < \frac{1}{n}$. If only a finite number of the terms of the sequence $p_n (n = 1, 2, \ldots)$ were different, then one of them, p say, would be repeated an infinite number of times, and we should have $\rho(p, q_n) < \frac{1}{n}$ for an infinity of n's, from which it would follow (since $q_n \in B$, for $n = 1, 2, \ldots$) that $p \in \bar{B}$; this is impossible, since p, being one of the terms of the sequence p_n $(n = 1, 2, \ldots)$, belongs to A, and $A.B = A.B' = 0$. The set of all different terms of the sequence $p_n (n = 1, 2, \ldots)$ is, therefore, infinite and so, as a subset of the compact set E, has a limit element p. Since A is closed, and $p_n \in A$ for $n = 1, 2, \ldots$, we have $p \in A$. It follows from the definition of p that for every

$\sigma > 0$ there exists an index $n > \frac{1}{\sigma}$, such that $\rho(p, p_n) < \sigma$, and since at the same time $\rho(p_n, q_n) < \frac{1}{n} < \sigma$, we, therefore, have $\rho(p, q_n) < 2\sigma$. From $q_n \in B$, for $n = 1, 2, \ldots$, we conclude that for every $\sigma > 0$ there exists an element q of B such that $\rho(p, q) < 2\sigma$. Since B is closed, this leads to the result that $p \in B$, which is impossible, for, as shown above, $p \in A$ and $A.B = 0$.

We have thus proved that the elements a and b cannot be connected by an ϵ-chain in E for every $\epsilon > 0$. Hence the condition of our theorem would not be satisfied. The theorem may, therefore, be considered as proved.[4]

In view of the theorems proved above we are now in a position to state that *in order that a closed and compact set be connected it is necessary and sufficient that every pair of elements of the set can be connected by an ϵ-chain in it for every $\epsilon > 0$.*

We shall also prove the following property of metric spaces.

The derived set of a compact set (contained in a metric space) *is compact.*

Proof. Let E be a compact set (which consists of elements of a given metric space), and let T be an infinite subset of E', the derived set of E. There exists, therefore, an infinite sequence $q_1, q_2, q_3, \ldots$ of different elements of E'. Since $q_1 \in E'$, it follows, from the definitions of a derived set and a limit element, that there exists an element p_1 of E, such that $\rho(q_1, p_1) < 1$. Let n denote an integer > 1, and suppose that all elements $p_1, p_2, \ldots, p_{n-1}$ of E have been defined. Since $q_n \in E'$, there exists, as is easily seen, an element p_n of E different from $p_1, p_2, \ldots, p_{n-1}$ and such that $\rho(q_n, p_n) < \frac{1}{n}$. The infinite sequence $p_1, p_2, p_3, \ldots$ is thus defined by induction, and the terms being all different, this sequence is an infinite subset E_1 of E. But E is compact; there exists, therefore, a limit element a of E_1. Let η denote an arbitrary positive number. Since $a \in E_1'$ there exists, as is easily seen, an index n such that $\frac{1}{n} < \frac{\eta}{2}$ and $\rho(a, p_n) < \frac{\eta}{2}$, and so from $\rho(q_n, p_n) < \frac{1}{n} < \frac{\eta}{2}$ and the triangle law, we find that $\rho(a, q_n) < \eta$; it may obviously be supposed that $q_n \neq a$, since the terms of the sequence $q_1, q_2, \ldots$ are all different, and so the inequality $q_n \neq a$ will certainly be true for sufficiently large n. The element a is, therefore, a limit element of T, and so $T' \neq 0$.

[4]This theorem is not true for closed sets which are not compact. *E.g.* the set consisting of all points of a hyperbola and its asymptotes is not connected although it is closed and every pair of its elements can be connected by an ϵ-chain in it for every $\epsilon > 0$.

Since the sum of two compact sets is a compact set (§ 16), it follows at once from the above that

The enclosure of a compact set is a compact set.

48. In § 31 we have given the definition of the limit of an infinite sequence of elements of a topological space. This definition applies also to sequences of elements of a metric space, where properties 1, 2, and 3 of § 31, and Theorems 39, 40, 41, and 42 (see last paragraph of § 46) still hold good. We shall next prove

Theorem 50. *The relation,* $\lim_{n\to\infty} p_n = p$ *is equivalent to the relation* $\lim_{n\to\infty} \rho(p, p_n) = 0$.

Proof. Suppose that $\lim_{n\to\infty} p_n = p$, and let ϵ be an arbitrary positive number. From the definition of the limit of an infinite sequence of elements (§ 31) and the fact that $K(p, \epsilon)$ is an open set, we conclude that there exists a positive integer μ such that

(3) $$p_n \in K(p, \epsilon) \text{ for } n > \mu,$$

i.e.

(4) $$\rho(p, p_n) < \epsilon, \text{ for } n > \mu,$$

and so $\lim_{n\to\infty} \rho(p, p_n) = 0$.

Again, suppose that $\lim_{n\to\infty} \rho(p, p_n) = 0$, and let U be any open set such that $p \in U$. There exists, by the property established in § 44, a positive number ϵ such that $K(p, \epsilon) \subset U$. But, since $\lim_{n\to\infty} \rho(p, p_n) = 0$, there exists an index μ (determined by ϵ) such that (4) and therefore also (3) hold, which, since $K(p, \epsilon) \subset U$, gives $p_n \in U$, for $n > \mu$; this proves that $\lim_{n\to\infty} p_n = p$.

Theorem 50 is thus proved.

Suppose now that we have two metric spaces M and M_1 consisting of the same elements, but such that, if $\rho(p, q)$ be the distance of two elements p and q in M, and $\rho_1(p, q)$ their distance in M_1, then we do not necessarily always have $\rho(p, q) = \rho_1(p, q)$. We then say that there are two metrics, which may be different, established in the space M. If, however, the relation $\lim_{n\to\infty} \rho(p, p_n)$

$=0$ implies the relation $\lim_{n\to\infty} \rho_1(p, p_n)=0$ (for all elements of the aggregate M) and conversely, then we say that the two metrics are equivalent.

EXAMPLES.

1. In every metric space M a metric, which is equivalent to the given one, can be established, and such that the new distances between the elements of M are all ≤ 1. It will be sufficient to put $\rho_1(p, q)=\rho(p, q)$ ($\rho(p, q)$ the former distance) whenever $\rho(p, q)\leq 1$ and $\rho_1(p, q)=1$, if $\rho(p, q)>1$. The proof, that the new metric is equivalent to the former one, does not offer any difficulties.

2. The aggregate of all sets of real numbers $x_1, x_2, \ldots, x_m$ will be a metric space if by the distance $\rho(p, q)$ of two sets $p(x_1, x_2, \ldots, x_m)$ and $q(y_1, y_2, \ldots, y_m)$ we shall mean the number

$$\rho(p, q)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_m-y_m)^2}$$

(since, then, the distance properties 1, 2, and 3 of § 43 will be satisfied). The above will be the so-called Euclidian m-dimensional space.

An equivalent metric will be obtained by putting *e.g.*

$$\rho_1(p, q)=|x_1-y_1|+|x_2-y_2|+\ldots+|x_m-y_m|;$$

the proof does not offer any difficulties. The reader can find easily the geometrical meaning of the new distances and the new "spheres" (for $m=2$, and $m=3$).

Let E and E_1 be two given sets, the first contained in the metric space M, in which the distance is denoted by ρ, and the second in the metric space M_1, in which the distance is ρ_1. (In particular, we may have $M=M_1$ and $\rho=\rho_1$.) If it is possible to establish a (1, 1) correspondence f between the elements of E and E_1, such that for every pair p and q of elements of E we have

$$\rho_1(f(p), f(q))=\rho(p, q), \tag{5}$$

we say, then, that the set E is *congruent* to the set E_1. Obviously, the set E_1 is then also congruent to the set E. The sets E and E_1 are then said to be congruent; in symbols $E\cong E_1$.

The function f which satisfies condition (5) (for every pair p and q of E) is said to establish an *isometric* mapping of the set E on the set $E_1 = f(E)$. It follows easily from (5) that the function f is biuniform and continuous in the set E.

If each of two given sets A and B can be expressed as a sum ($A = A_1 + A_2 + \dots + A_n$, $B = B_1 + B_2 + \dots + B_n$) of the same finite number of mutually exclusive, congruent subsets ($A_1 \cong B_1$, $A_2 \cong B_2, \dots, A_n \cong B_n$), then A and B are said to be equivalent by division (into a finite number of subsets).

If a set P is congruent to a subset of Q, and Q is congruent to a subset of P, then the sets P and Q are not necessarily congruent to each other; it can, however, be shown that P and Q are then equivalent by division.[5] Two sets, which are equivalent by division to a third, are equivalent by division to each other. (The proof follows readily.) If $A_1 \subset A$, $B_1 \subset B$, $A \cong B$, $A_1 \cong B_1$, then the sets $A - A_1$ and $B - B_1$ are not necessarily congruent, and may not be even equivalent by division.

49. The aggregate of all infinite sequences of real numbers $x_1, x_2, x_3, \dots$, such that the series

$$x_1^2 + x_2^2 + x_3^2 + \dots$$

is convergent and where the distance between two elements $p(x_1, x_2, x_3, \dots)$ and $q(y_1, y_2, y_3, \dots)$ is given by the number

$$\rho(p, q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + \dots}$$

(which is always finite, since, owing to the convergence of the series of squares of the coordinates, the series under the radical sign is always convergent) is called a *Hilbert space*.

An m-dimensional Euclidian space is evidently congruent to a certain subset of the Hilbert space, namely to that one which consists of all infinite sequences $x_1, x_2, x_3, \dots$ such that $x_k = 0$ for $k > m$.

Let P and Q be two metric spaces in which the distances are denoted by ρ_1 and ρ_2 respectively. Fréchet denotes by $[[P, Q]]$ a space which consists of all the pairs (p, q), $p \in P$ and $q \in Q$ and where the distance ρ between two pairs (p_1, q_1) and (p_2, q_2) is defined by the relation

[5] See S. Banach and A. Tarski, *Fund. Math.*, vol. VI, p. 251.

$$\rho[(p_1, q_1), (p_2, q_2)] = \sqrt{(\rho_1(p_1, p_2))^2 + (\rho_2(q_1, q_2))^2}$$ [6]

Fréchet calls the sum of the dimensions dP and dQ the dimension of the space $[[P, Q]]$.

If H be a Hilbert space, then, as is easily seen, the space $[[H, H]]$ is isometric with the space H. In order to obtain an isometric mapping it will be sufficient to correlate with each sequence $(x_1, x_2, x_3, \ldots)$, which is an element of a Hilbert space, the pair of sequences

$$(x_1, x_3, x_5, \ldots) \text{ and } (x_2, x_4, x_6, \ldots).$$

Theorem 51 (Urysohn). *A topological space which satisfies the axiom of countability and the condition of normality is homeomorphic with a certain subset of a Hilbert space.*

Proof. Let T denote a given topological space, which satisfies axioms (*vi*) and (*vii*). Consider all the pairs W_i, W_j of rational sets (*i.e.* terms of the sequence (1), § 21) such that $\overline{W_i} \subset W_j$. Let

$$(6) \qquad (W_{k_1}, W_{l_1}), (W_{k_2}, W_{l_2}), \ldots, (W_{k_n}, W_{l_n}), \ldots$$

be an infinite sequence consisting of such pairs.[7]

Let n be a given index. The sets $\overline{W_{k_n}}$ and $T - W_{l_n}$ are closed and mutually exclusive (since $\overline{W_{k_n}} \subset W_{l_n}$ and W_{l_n} is open). There exists, therefore, by the lemma of Urysohn (§ 41) a real function $f_n(p)$ defined and continuous in the whole set T, and such that $0 \leq f_n(p) \leq 1$ throughout, $f_n(p) = 0$ for $p \in \overline{W_{k_n}}$, and $f_n(p) = 1$ for $p \in T - W_{l_n}$.

For every element p of T denote by $\phi(p)$ the infinite sequence

$$(7) \qquad f_1(p), \ \frac{1}{2} f_2(p), \ \frac{1}{2^2} f_3(p), \ \ldots, \ \frac{1}{2^{n-1}} f_n(p), \ \ldots$$

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[6]It is left to the reader to prove that the distance ρ so defined satisfies the three required conditions (provided these conditions are satisfied by the distances ρ_1 and ρ_2).

[7]Such pairs do exist. For let p denote an element which belongs to an open set U. By axiom (*vi*) there exists a rational set W such that $p \in W \subset U$; by axiom (*vii*) there exists an open set V such that $p \in V$ and $\overline{V} \subset W$; hence, by (*vi*) there exists a rational set W^* such that $p \in W^* \subset V$, and so $\overline{W^*} \subset \overline{V} \subset W$. We can always suppose that the sequence (6) is infinite, repeating if necessary one of the terms an infinite number of times.

The sequence (7) is an element of a Hilbert space, since, on account of $0 \leq f_n(p) \leq 1$, the sum of the squares of the terms of (7) is convergent.

It will be shown that T is homeomorphic with the set $\phi(T)$.

In order to prove it we shall show first that ϕ establishes a (1, 1) correspondence between the elements of the sets T and $\phi(T)$. It will be sufficient for that purpose to show that, if p and q be two different elements of the set T, then $\phi(p) \neq \phi(q)$. Hence, suppose that $p \in T$, $q \in T$, and $p \neq q$.

By axiom (*iv*) and Corollary 1, § 21, there exists a rational set U, such that $p \in U$ and $q \bar{\in} U$; by axiom (*vii*) (and Corollary 1, § 21) there exists a rational set V such that $p \in V$, and $\overline{V} \subset U$. Since U and V are terms of sequence (1) of § 21, and $\overline{V} \subset U$, and, from the definition of the sequence (6), the pair of sets U, V is a term of the sequence (6); we have, therefore, for some n, $U = W_{l_n}$ and $V = W_{k_n}$; hence $p \in W_{k_n}$, and $q \bar{\in} W_{l_n}$ (since $q \bar{\in} U$, and $\overline{V} \subset U$), *i.e.* $q \in T - W_{l_n}$. It follows, therefore, from the definition of the function f_n that $f_n(p) = 0$ and $f_n(q) = 1$. Hence the sequences $\phi(p)$ and $\phi(q)$ differ in their n^{th} terms and thus represent different elements in the Hilbert space. We have proved, therefore, that from $p \neq q$ follows $\phi(p) \neq \phi(q)$.

In virtue of Corollary 4 to Theorem 22, it will be sufficient for the proof of the relation $T\, h_\phi \phi(T)$ to show that, if E be any subset of T, then

$$\phi(T.E') = \phi(T) \,.\, [\phi(E)]'. \tag{8}$$

Let, therefore, E denote any subset of T. Let, further, b denote an element of the Hilbert space such that $b \in \phi(T.E')$; there exists, therefore, an element $a \in T.E'$ such that $b = \phi(a)$. Let η be an arbitrary positive number. Take a positive integer m such that

$$\frac{1}{2^{2m-2}} \leq \eta^2. \tag{9}$$

Since the real functions $f_n(p)$ are continuous in T at a, there exists for every positive integer n an open set $U_n \subset T$ such that $p \in U_n$, and

$$|f_n(a) - f_n(p)| < \frac{\eta}{2m}, \text{ for } p \in U_n. \tag{10}$$

Put $U = U_1.U_2 \ldots U_m$; this will be an open set containing a, and from (10)

$$(11) \qquad |f_n(a) - f_n(p)| < \frac{\eta}{2m}, \text{ for } p \in U, \ n = 1, 2, \ldots, m.$$

Since $a \in T.E' \subset E'$, and $a \in U$, there exists an element $p \in U.E$, different from a; (11) will, therefore, hold for such an element p. From the definition of distance in a Hilbert space we have

$$(12) \qquad \rho(\phi(a), \phi(p)) = \sqrt{\sum_{n=1}^{\infty} \frac{1}{2^{2n-2}} [f_n(a) - f_n(p)]^2}.$$

But $|f_n(a) - f_n(p)| \leq 1$, for $n = 1, 2, 3, \ldots$ (since f_n is contained in the interval (0, 1)); we have, therefore, from (11) and (9)

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n-2}} [f_n(a) - f_n(p)]^2 \leq \sum_{n=1}^{m} \frac{1}{2^{2n-2}} [f_n(a) - f_n(p)]^2 + \sum_{n=m+1}^{\infty} \frac{1}{2^{2n-2}}$$

$$< m.\frac{\eta^2}{4m^2} + \frac{1}{2^{2m-1}} \leq \frac{\eta^2}{4} + \frac{\eta^2}{2} < \eta^2,$$

and so from (12)

$$(13) \qquad \rho(\phi(a), \phi(p)) < \eta;$$

since $p \neq a$, and the function ϕ is biuniform, we have $\phi(p) \neq \phi(a)$, and since $p \in U.E \subset E$, we have $\phi(p) \in \phi(E)$. We have thus proved that for every positive number η there exists an element $\phi(p)$ of $\phi(E)$ different from $\phi(a)$, and for which (13) holds. It follows from the definition of a limit element and a derived set of a metric space (a Hilbert space is such) that $b = \phi(a) \in [\phi(E)]'$, and so $b \in \phi(T).[\phi(E)]'$ (since $a \in T$, and $b = \phi(a) \in \phi(T)$).

Since b is any element of the Hilbert space belonging to the set $\phi(T.E')$, we have proved that

$$(14) \qquad \phi(T.E') \subset \phi(T).[\phi(E)]'.$$

Let now b denote an element of the Hilbert space such that $b \in \phi(T).[\phi(E)]'$. We have then $b \in \phi(T)$, and so there exists an element $a \in T$ such that $b = \phi(a)$. Let U be any open set contained in T such that $a \in U$. Since $a \in U$, by axiom (*vii*) and Corollary 1, § 21, there exist two rational sets V_1 and V_2 such that $a \in V_1$, and $\overline{V_1} \subset V_2 \subset U$. Thus the pair (V_1, V_2) is a term of the sequence (6), and so we have for some positive integer m, $V_1 = W_{k_m}$ and $V_2 = W_{l_m}$. Put $\eta = \frac{1}{2^{m-1}}$; since $\phi(a) = b \in [\phi(E)]'$, there exists an element e of the Hilbert space different from b and belonging to $\phi(E)$, and such that

$$(15) \qquad \rho(b, e) < \eta.$$

Since $e \in \phi(E)$, there exists an element $p \in E$ such that $e = \phi(p)$.

It follows that $p \in U$. In fact, if $p \bar{\in} U$, then certainly $p \bar{\in} W_{l_m}$, since $W_{l_m} = V_2 \subset U$, and so $p \in T - W_{l_m}$; from the definition of f_m it would follow that $f_m(p) = 1$, while $f_m(a) = 0$, since $a \in V_1 \subset \overline{V}_1 = \overline{W}_{k_m}$.

Hence, from (12)

$$\rho(b, e) = \rho(\phi(a), \phi(p)) \geq \frac{1}{2^{m-1}} = \eta,$$

contrary to (15). We must, therefore, have $p \in U$. From $\phi(p) = e \neq b = \phi(a)$ we get $p \neq a$. We have thus proved that in every open set $U \subset T$, and containing a, there exists an element p of E different from a. This proves that $a \in E'$, and so also $a \in T.E'$ (since $a \in T$); hence, $b = \phi(a) \in \phi(T.E')$. We have thus proved that

$$\phi(T).[\phi(E)]' \subset \phi(T.E'),$$

which, on account of (14), gives (8).

The relation $T\, h_\phi\, \phi(T)$ is, therefore, proved and with it also Theorem 51.

It follows from Theorem 51 that *in a topological space satisfying the axioms of countability and normality a metric can be established, i.e.*, for every pair of elements of such a space a function $\rho(a, b)$ can be defined such that the conditions 1), 2), 3) of § 43 are satisfied and such that the definition of a limit element given in § 43 is in accordance with that given in a topological space.

Since, as was proved in § 46, every metric space (with a proper definition of neighbourhoods) is a normal topological space, Theorem 51 leads at once to the following

COROLLARY. *Every metric space which satisfies the axiom of countability, is homeomorphic with a certain subset of a Hilbert space.*

It may be easily concluded from the above that *of all metric spaces which satisfy the axiom of countability, a Hilbert space has the greatest dimensional type.* To prove the above it will be sufficient to show that a Hilbert space satisfies the axiom of countability.

Hence, let every element $(x_1, x_2, x_3, \dots)$, for which the numbers $x_i (i=1, 2, 3, \dots)$ are all rational with only a finite number of them different from zero, be a rational element of the Hilbert space. The set of all rational elements is, as is easily seen, countable. Furthermore, let a rational sphere be a sphere $K(p, r)$, in which p is a rational element of the Hilbert space and r a rational number. The aggregate of all rational spheres is evidently countable. Hence, it will be sufficient to show that every open set contained in the Hilbert space is a sum of a certain number of rational spheres, *i.e.* that if q is an element of the open set U, there exists a rational sphere containing q and contained in U.

Hence, let q be an element of the open set U. By the condition for an open set, deduced in § 44, there exists a positive number ϵ such that $K(q, \epsilon) \subset U$. The element q, being an element of the Hilbert space, is an infinite sequence of real numbers

$$x_1, x_2, x_3, \dots$$

such that the series $x_1^2+x_2^2+x_3^2+ \dots$ is convergent. There exists, therefore, an integer m such that

$$(16) \qquad x_{m+1}^2 + x_{m+2}^2 + \dots < \frac{\epsilon^2}{12}.$$

For every real number x_i there exists a rational number w_i such that

$$(17) \qquad |x_i - w_i| \leq \frac{\epsilon}{6m}.$$

Let r be a rational number such that

$$(18) \qquad \epsilon/_3 < r < \epsilon/_2 .$$

Denote by p the infinite sequence

$$w_1, w_2, \dots, w_m, 0, 0, 0, \dots;$$

p will be a rational element of the Hilbert space, and from (17), (16), and the definition of distance in a Hilbert space,

$$(19) \qquad \rho(p, q) < \epsilon/_3 .$$

It follows from (19) and (18) that $q \epsilon K(p, r)$, where the sphere $K(p, r)$ is obviously rational. But from (18) and (19) it follows readily that $K(p, r) \subset K(q, \epsilon)$ and so $\subset U$. We have proved, therefore, that a Hilbert space satisfies the condition of countability.

50. A set which is the sum of a countable aggregate of closed sets is called (Hausdorff) an F_σ and its complement, *i.e.* the product of a countable aggregate of open sets, a G_δ. Obviously, the sum of a countable aggregate of sets F_σ is an F_σ, and the product of a countable aggregate of sets G_δ is a G_δ.

Theorem 52. *Every closed set* (in a metric space) *is a* G_δ.

Proof. Let F be a given closed set. Denote by Γ_n the sum $\Sigma K\left(p, \frac{1}{n}\right)$, where the summation extends to all elements p of F. The set Γ_n is evidently open (as a sum of open sets).

It will be shown that

(16) $$F = \Gamma_1.\Gamma_2.\Gamma_3 \ldots$$

Since obviously $F \subset \Gamma_n$, for $n = 1, 2, \ldots$, it will be sufficient for the proof of (16), to show that if $q \epsilon \Gamma_n$, for $n = 1, 2, 3, \ldots$, then $q \epsilon F$. Hence, let q be an element such that $q \epsilon \Gamma_n$, for $n = 1, 2, 3, \ldots$ It follows from the definition of Γ_n that, if $q \epsilon \Gamma_n$, there exists an element p_n of F, such that $q \epsilon K\left(p_n, \frac{1}{n}\right)$, *i.e.* $\rho(p_n, q) < \frac{1}{n}$.

Since this inequality holds for $n = 1, 2, \ldots$, q is either an element or a limit element of F and so is an element of F in any case, F being closed. (16) is, therefore, proved; and this establishes the truth of Theorem 52.

Passing to complements, we obtain at once from Theorem 52

Theorem 52a. *Every open set* (in a metric space) *is an* F_σ.

51. Suppose a function $f(p)$ to be defined at the elements of a set E contained in a metric space so that the values of f are elements in the same or another metric space.

Let p_0 be an element of the set $\bar{E} = E + E'$, *i.e.* of the enclosure of E. (Hence, if $p_0 \bar{\epsilon} E$, then $f(p_0)$ may not be defined.)

Denote by $\omega(p_0, \epsilon)$, for every positive ϵ, the upper bound of all the numbers

$$\rho(f(p), f(q)),$$

where p and q are any two elements of the set $E.K\ (p_0, \epsilon)$. Clearly

$$\omega(p_0, \epsilon') \leq \omega(p_0, \epsilon), \text{ for } \epsilon' < \epsilon;$$

hence, the limit

(17) $$\omega(p_0) = \lim_{\epsilon \to 0} \omega(p_0, \epsilon)$$

exists and is a non-negative number, finite or infinite.

This limit is called *the oscillation of the function f in the set E at the element p_0*.

Theorem 53. *If $f(p)$ be a function defined in a set E, then the set of all those elements of $\overline{E}$, at which the oscillation of f in E is $\geq \alpha$, is closed.*

Proof. Let $f(p)$ be a function defined in E, α a given real number ≥ 0. Denote by P the set of those elements p of E, at which $\omega(p) \geq \alpha$, where $\omega(p)$ is the oscillation of f in E at the element p. Let p_0 be a limit element of P and ϵ an arbitrary positive number. Since $p_0 \epsilon P'$, there exists an element p of P such that $p \epsilon K(p_0, \epsilon)$, and, since $K(p_0, \epsilon)$ is open, there exists a number $\eta > 0$, such that $K(p, \eta) \subset K(p_0, \epsilon)$. From $p \epsilon P$ and the definition of P, we have $\omega(p) \geq \alpha$, and so certainly $\omega(p, \eta) \geq \alpha$. It follows from the definition of the number $\omega(p, \eta)$ that there exist two elements p_1 and p_2 of the set $E.K(p, \eta)$ such that

$$\rho(f(p_1), f(p_2)) > \alpha - \epsilon; \tag{18}$$

and, since $K(p, \eta) \subset K(p_0, \epsilon)$, p_1 and p_2 belong to the set $E.K(p_0, \epsilon)$, and so from (18)

$$\omega(p_0, \epsilon) > \alpha - \epsilon;$$

since ϵ is an arbitrary number, we conclude from (17) that $\omega(p_0) \geq \alpha$, *i.e.* that $p_0 \epsilon P$. We have thus proved that P is closed. Theorem 53 is, therefore, established.

COROLLARY. *If $f(p)$ be a function defined in a set E, then the set of all those elements of $\overline{E}$, at which the oscillation of f in E is zero, is a G_δ.*

In fact, denote by P the set of all elements p of $\overline{E}$, at which $\omega(p) = 0$, and by P_n the set of all elements p of $\overline{E}$, at which $\omega(p) \geq \frac{1}{n}$. The sets $P_n (n = 1, 2, 3, \ldots)$ are all closed by Theorem 53; hence, the set $S = P_1 + P_2 + \ldots$ is an F_σ, and so CS is a G_δ. But, obviously, $P = \overline{E} - S = \overline{E}.CS$. Since $\overline{E}$ is closed, it is a G_δ by Theorem 52. The set P is, therefore, a product of two G_δ's and so itself a G_δ.

Theorem 54. *In order that a function $f(p)$ defined in a set E be continuous in E at an element p_0, it is necessary and sufficient that the oscillation of f in E at the element p_0 be equal to zero.*

Proof. Suppose that the function f is continuous in E at an element p_0 of that set. Let ϵ be an arbitrary positive number. Since f is continuous in E at the element p_0 and the set $K(f(p_0), \epsilon)$ is open and contains $f(p_0)$, there exists an open set U containing p_0 and such that the relation

$$p \in U.E$$

implies the relation $f(p) \in K(f(p_0), \epsilon)$.

Since $p_0 \in U$, and U is open, there exists a number $\eta > 0$, such that $K(p_0, \eta) \subset U$. For $p \in E.K(p_0, \eta)$ we shall have $p \in U.E$, and so $f(p) \in K(f(p_0), \epsilon)$, *i.e.* $\rho(f(p_0), f(p)) < \epsilon$.

If then $p_1 \in E.K(p_0, \eta)$ and $p_2 \in E.K(p_0, \eta)$, we have

$$\rho(f(p_1), f(p_2)) \leq \rho(f(p_0), f(p_1)) + \rho(f(p_0), f(p_2)) < 2\epsilon.$$

It follows from the above and from the definition of the number $\omega(p_0, \eta)$ that $\omega(p_0, \eta) < 2\epsilon$, and so certainly $\omega(p_0) < 2\epsilon$; hence, since ϵ is arbitrary, $\omega(p_0) = 0$. The condition of our theorem is, therefore, necessary.

Suppose now that $\omega(p_0) = 0$ at a given element p_0 of the set E. Let V be any open set containing $f(p_0)$. There exists, therefore, a number $\epsilon > 0$ such that $K(f(p_0), \epsilon) \subset V$. Since $\omega(p_0) = 0$, there exists by (17) a number $\eta > 0$, such that $\omega(p_0, \eta) < \epsilon$. From the definition of the number $\omega(p_0, \eta)$ it follows that, if $p \in U.E$, where $U = K(p_0, \eta)$, then $\rho(f(p_0), f(p)) \leq \omega(p_0, \eta) < \epsilon$, and so $f(p) \in K(f(p_0), \epsilon) \subset V$. We have thus proved that, if $\omega(p_0) = 0$, then for every open set V containing $f(p_0)$, there exists an open set U containing p_0 and such that the relation $p \in U.E$ implies the relation $f(p) \subset V$. This proves that the function f is continuous in E at p_0. The condition of the theorem is, therefore, sufficient.

52. **Theorem 55.** *If $f(p)$ be a function defined at the elements p of a closed and compact set E, and continuous in that set, then for every positive ϵ there exists a positive number η such that the conditions*

(1) $$p \in E,\ q \in E,\ \rho(p, q) < \eta$$

imply the inequality

(2) $$\rho(f(p), f(q)) < \epsilon.$$

Proof. Let ϵ be a given positive number and a any given element of the set E. Since f is continuous in E at a and since

$f(a) \epsilon K\left(f(a), \frac{\epsilon}{2}\right)$, there exists an open set U containing a, such that the condition $b \epsilon U.E$ implies the relation $f(b) \epsilon K\left(f(a), \frac{\epsilon}{2}\right)$. From $a \epsilon U$ and the condition for an open set as in § 44, it follows that there exists a positive number $r = r(a)$ such that $K(a, r(a)) \subset U$.

Hence, for every element a of E there exists a sphere $K(a, r(a))$ such that

(3) $$\rho(f(a), f(b)) < \frac{\epsilon}{2}, \text{ for } b \epsilon E.K(a, r(a)).$$

For every element a of E denote by $Q(a)$ the set

(4) $$Q(a) = K(a, \tfrac{1}{2} r(a)).$$

Let M be the aggregate of all spheres $Q(a)$ corresponding to the elements a of E. Hence, every element of E belongs to at least one of the open sets of the aggregate M (since we have from (4) $a \epsilon Q(a)$ for $a \epsilon E$). By the Borel-Lebesgue theorem (Theorem 36, § 25) there exists a finite number of sets $Q(a_1), Q(a_2), \ldots, Q(a_n)$ of the aggregate M such that

(5) $$E \subset Q(a_1) + Q(a_2) + \ldots + Q(a_n).$$

Let η be a positive number satisfying the inequalities

(6) $$\eta \leq \tfrac{1}{2} r(a_i), \text{ for } i = 1, 2, \ldots, n.$$

Let now p and q be any two elements of E, which satisfy conditions (1). By (5), there exists an index $k \leq n$, such that $p \epsilon Q(a_k)$, and so from (4), $p \epsilon K(a_k, \frac{1}{2} r(a_k))$, *i.e.*

(7) $$\rho(a_k, p) < \tfrac{1}{2} r(a_k),$$

and since from (1) and (6), $\rho(p, q) < \eta < \frac{1}{2} r(a_k)$, therefore,

$$\rho(a_k, q) \leq \rho(a_k, p) + \rho(p, q) < r(a_k);$$

i.e.

(8) $$q \epsilon K(a_k, r(a_k)),$$

and so from (7) certainly,

(9) $$p \epsilon K(a_k, r(a_k)).$$

It follows from (8) and (3) that

(10) $$\rho(f(a_k), f(q)) < \frac{\epsilon}{2}$$

and from (9) and (3), that

(11) $$\rho(f(a_k), f(p)) < \frac{\epsilon}{2};$$

relations (10) and (11) give at once (2).

Theorem 55 is, therefore, proved. We express it also by stating that *a function continuous in a closed, compact set is uniformly continuous in that set.*

53. We shall consider now, in particular, countable metric spaces. These are of importance owing to the fact that many metric spaces have countable subsets everywhere dense, which are themselves countable metric spaces. We proceed to prove

Theorem 56. *Every countable metric space is homeomorphic with a certain set of real numbers.*[8]

Proof. Let P be a given countable, metric space, and let $\rho(p, q)$ denote the distance of two elements p and q of the space P. For every element p of P and for every positive real number r, denote by $S(p, r)$ the set of all elements q of P such that $\rho(p, q) = r$ (in particular, the set $S(p, r)$ may be a null set).

Let p be a given element of P, and ϵ a given positive number. The sets $S(p, r)$, where $0 < r < \epsilon/2$, constitute a non-countable aggregate of sets (since r can take a continuum of different values) and, as is easily seen, $S(p, r) . S(p, r') = 0$, for $r \neq r'$. Since P is countable there exists (for every element $p \in P$ and for every positive number ϵ) a real number $r = \phi(p, \epsilon)$, such that $0 < r < \epsilon/2$, and $S(p, r) = 0$.

The sets $K(p, \phi(p, \epsilon))$ are, as is easily seen, both open and closed (since the sets $K(p, r)$ are open, and the sets $K(p, r) + S(p, r)$ are closed),[9] and for $r = \phi(p, \epsilon)$ we have $S(p, r) = 0$. This is true also

[8]Every set of real numbers constitutes a metric space, if by the distance of two numbers x and y belonging to the set in question, we mean the number $|x - y|$.

[9]The set $K(p, r) + S(p, r)$ is evidently the set of all elements q of P, which satisfy the condition $\rho(p, q) \leq r$; it is, therefore, a closed set (since the distance $\rho(p, q)$ is a continuous function of the variables p and q, § 45).

in case $K(p, \phi(p, \epsilon))$ is a null set (since a null set is both open and closed).

LEMMA. *If Q be a subset of P, which is both open and closed, and ϵ be a given positive number, there exists a division $Q = Q_1 + Q_2 + Q_3 + \ldots$ of Q into the sum of mutually exclusive sets $Q_n (n = 1, 2, \ldots)$, which are both open and closed and such that $\delta(Q_n) < \epsilon$, for $n = 1, 2, \ldots$.* (where $\delta(Q_n)$ is the diameter of the set Q_n, § 47).

Proof. If Q is a null set it is sufficient to put $Q_n = 0$, for $n = 1, 2, \ldots$. If Q is a finite set, *e.g.* a set consisting of the elements $q_1, q_2, \ldots, q_m$, it is sufficient to put $Q_n = (q_n)$, for $n = 1, 2, \ldots, m$ and $Q_n = 0$, for $n > m$. It will be sufficient, therefore, to consider the case when Q is countable. Hence let

(1) $$q_1, q_2, q_3, \ldots$$

denote an infinite sequence, consisting of all the (different) elements of Q.

Put $Q_1 = Q.K(q_1, \phi(q_1, \epsilon))$; Q_1 will be a set both open and closed (as a product of two such sets) where obviously $q_1 \epsilon Q_1$.

Let now n be a given integer > 1, and suppose that we have defined already the sets $Q_1, Q_2, \ldots, Q_{n-1}$, which are both open and closed, and whose sum contains the elements $q_1, q_2, \ldots, q_{n-1}$. If there exist terms of the sequence (1) which do not belong to the set $Q_1 + Q_2 + \ldots + Q_{n-1}$, then let q_s denote the first of them, and put

(2) $$Q_n = Q.K(q_s, \phi(q_s, \epsilon)) - (Q_1 + Q_2 + \ldots + Q_{n-1});$$

otherwise put $Q_n = 0$.

The set $Q_1 + Q_2 + \ldots + Q_n$ will, in any case, contain the elements $q_1, q_2, \ldots, q_n$, since the set $Q_1 + Q_2 + \ldots + Q_{n-1}$ contains the elements $q_1, q_2, \ldots, q_{n-1}$, and, if it does not contain the element q_n, then from the definition of the element q_s, we must have $q_s = q_n$, where q_s is an element of the set (2).

Since the sets $Q_1, Q_2, \ldots, Q_{n-1}$ are both open and closed, the same property is possessed also by their sum $Q_1 + Q_2 + \ldots + Q_{n-1}$; furthermore, since the sets Q and $K(q_s, \phi(q_s, \epsilon))$, and, therefore, also their product, are both open and closed, it follows from (2) that Q_n is both open and closed.

Finally, (if Q_n is not null) we have from (2) and the definition of the numbers $\phi(p, \epsilon)$

$$Q_n \subset K(q_s, \phi(q_s, \epsilon)) \subset K\left(q_s, \frac{\epsilon}{2}\right),$$

and so (from the definition of the sets $K(p, r)$)

$$\delta(Q_n) < \epsilon.$$

The sets $Q_n (n=1, 2, \ldots)$ are thus defined by induction, and, as is easily seen, they satisfy all the conditions of our lemma. The lemma is, therefore, proved.

The set P itself is obviously both open and closed (in the set P); we can apply, therefore, the above lemma, putting $Q=P$ and $\epsilon=1$. We thus obtain a division

$$P=P_1+P_2+P_3+\ldots, \tag{3}$$

where $P_n (n=1, 2, \ldots)$ are mutually exclusive sets, both open and closed, and for which $\delta(P_n)<1$, for $n=1, 2, \ldots$

Since each of the sets $P_n (n=1, 2, \ldots)$ is both open and closed, we can apply to it the above lemma, putting $\epsilon=\frac{1}{2}$, which gives (for every integer n) a division

$$P_n=P_{n,1}+P_{n,2}+P_{n,3}+\ldots, \tag{4}$$

where $P_{n,k} (k=1, 2, \ldots)$ are mutually exclusive sets, both open and closed, and where $\delta(P_{n,k})<\frac{1}{2}$, for $k=1, 2, \ldots.$

Let now k be an integer >1, and suppose that we have defined already all sets $P_{n_1, n_2, \ldots, n_{k-1}}$ both open and closed, where $n_1, n_2, \ldots, n_{k-1}$ is any combination of $k-1$ integers. Let $n_1, n_2, \ldots, n_{k-1}$ be any combination of $k-1$ integers. Since the set $P_{n_1, n_2, \ldots, n_{k-1}}$ is both open and closed, we may apply to it our lemma, putting $\epsilon=\frac{1}{k}$, which gives the division

$$P_{n_1, n_2, \ldots, n_{k-1}}=P_{n_1, n_2, \ldots, n_{k-1}, 1}+P_{n_1, n_2, \ldots, n_{k-1}, 2}+P_{n_1, n_2, \ldots, n_{k-1}, 3}+\ldots,$$

where $P_{n_1, n_2, \ldots, n_{k-1}, n_k} (n_k=1, 2, 3, \ldots)$ are mutually exclusive sets, both open and closed, and where

$$\delta(P_{n_1, n_2, \ldots, n_{k-1}, n_k})<\frac{1}{k}, \tag{5}$$

for $n_k=1, 2, 3, \ldots$

The sets $P_{n_1, n_2, \ldots, n_k}$, where $n_1, n_2, \ldots, n_k$ denote any combination of k integers, are thus defined by induction.

Let now p be any element of the set P. Since the terms of the sum (3) are mutually exclusive sets and $p \in P$, there exists a definite index n_1 such that $p \in P_{n_1}$. Similarly, we conclude from (4) the existence of a definite index n_2 such that $p \in P_{n_1, n_2}$. Repeating this argument indefinitely, we arrive at an infinite sequence of indices

(6) $$n_1, n_2, n_3, \ldots$$

completely defined (by the element p), and such that

(7) $$p \in P_{n_1, n_2, \ldots, n_k}, \text{ for } k=1, 2, 3, \ldots$$

Put

(8) $$f(p) = \frac{1}{n_1+} \frac{1}{n_2+} \frac{1}{n_3+} \cdots;$$

this will be a certain irrational number defined by the element p.

It is easily seen that not only does (7) imply (8), but conversely (8) implies (7) (since for every element p of P there exists a unique infinite sequence (6) of indices, for which (7) holds).

It is also easily seen that, if p and p' be two different elements of P, then $f(p) \neq f(p')$. In fact, if $f(p) = f(p')$, the numbers $f(p)$ and $f(p')$ would have the identical developments as continued fractions, and so from (8) and (7)

$$p \in P_{n_1, n_2, \ldots, n_k} \text{ and } p' \in P_{n_1, n_2, \ldots, n_k}, \text{ for } k=1, 2, \ldots;$$

hence, from (5)

$$\rho(p, p') < \frac{1}{k}, \text{ for } k=1, 2, \ldots,$$

which is impossible, since $p \neq p'$.

The function $f(p)$ establishes, therefore, a (1, 1) correspondence between the elements of the sets P and $f(P)$.

It will be shown that $P\, h_f\, f(P)$.

Let p_0 be a given element of P, and

$$n_1^0, n_2^0, n_3^0, \ldots$$

a sequence of indices corresponding to that element, *i.e.* such that

(9) $$p_0 \in P_{n_1^0, n_2^0, \ldots, n_k^0}, \text{ for } k=1, 2, 3, \ldots$$

Hence,

$$f(p_0) = \frac{1}{n_1^0+} \; \frac{1}{n_2^0+} \; \frac{1}{n_3^0+} \cdots \tag{10}$$

Let ϵ be an arbitrary positive number. It follows from the well-known properties of continued fractions that there exists a positive integer k such that for every infinite sequence (6) of integers, for which

$$n_i = n_i^0, \; i = 1, 2, \ldots, k,$$

the inequality

$$\left| \left(\frac{1}{n_1+} \; \frac{1}{n_2+} \; \frac{1}{n_3+} \cdots \right) - \left(\frac{1}{n_1^0+} \; \frac{1}{n_2^0+} \; \frac{1}{n_3^0+} \cdots \right) \right| < \epsilon \tag{11}$$

is satisfied.

Since the set $P_{n_1^0, n_2^0, \ldots, n_k^0}$ is open, it follows from (9) that there exists a positive number σ such that $K(p_0, \sigma) \subset P_{n_1^0, n_2^0, \ldots, n_k^0}$. Hence, if p be an element of P such that

$$\rho(p_0, p) < \sigma, \tag{12}$$

then

$$p \in P_{n_1^0, n_2^0, \ldots, n_k^0},$$

and so, certainly,

$$p \in P_{n_1^0, n_2^0, \ldots, n_i^0}, \text{ for } i = 1, 2, \ldots, k.$$

The first k terms of the sequence (6) associated with the element p, are, therefore, $n_1^0, n_2^0, \ldots, n_k^0$, which fact, as has been proved, implies the inequality (11), and so from (8) and (10), the inequality

$$|f(p) - f(p_0)| < \epsilon. \tag{13}$$

We have thus proved that, for every element p_0 of P, and for every positive number ϵ, there exists a positive number σ such that the inequality (12) implies the inequality (13) for the elements of P. The function $f(p)$ is, therefore, continuous in P.

As previously shown, $f(p) \neq f(p')$, for $p \in P$, $p' \in P$, and $p \neq p'$. The function $f(p)$ is, therefore, biuniform in P; let $g(x)$ denote the inverse function of f. Hence, g will be a function defined in the set $f(P)$ of real numbers. To prove that $P \; h_f \; f(P)$ it will be sufficient to show that the function $g(x)$ is continuous in the set $f(P)$.

Hence, let x_0 denote a given element of $f(P)$ and let

$$(14) \qquad x_0 = \frac{1}{n_1^0 +} \; \frac{1}{n_2^0 +} \; \frac{1}{n_3^0 +} \; \cdots$$

be a development of x_0 as a continued fraction. Put $p_0 = g(x_0)$; from (14) we get (10), which, as we know, implies the relations (9).

Let ϵ be an arbitrary positive number. Choose a positive integer k such that $\frac{1}{k} < \epsilon$. There exists, by the well-known property of continued fractions, a positive number σ such that every irrational number x which satisfies the inequality

$$(15) \qquad |x - x_0| < \sigma,$$

may be written as a continued fraction in the form

$$(16) \qquad x = \frac{1}{n_1 +} \; \frac{1}{n_2 +} \; \frac{1}{n_3 +} \; \cdots,$$

where

$$(17) \qquad n_i = n_i^0, \text{ for } i = 1, 2, \ldots, k.$$

Hence, if x is a number of the set $f(P)$ satisfying the inequality (15), then, putting $p = g(x)$, we shall have $f(p) = x$ and, therefore, from (16) and (17) (and the fact that relation (8) implies (7))

$$(18) \qquad p \in P_{n_1^0, n_2^0, \ldots, n_k^0}.$$

From (18), (9), and (5), and since $\frac{1}{k} < \epsilon$, we find that $\rho(p, p_0) < \epsilon$, *i.e.*

$$(19) \qquad \rho(g(x), g(x_0)) < \epsilon.$$

We have thus proved that for every number x_0 of the set $f(P)$, and for every positive number ϵ, there exists a positive number σ such that the inequality (15) results in the inequality (19) for the numbers x of the set $f(P)$. This proves that the function $g(x)$ is continuous in the set $f(P)$.

We have thus proved that the function f establishes a homeomorphic mapping of the set P on the set of real numbers $f(P)$. Theorem 56 is, therefore, proved.

Every countable set of real numbers is, as is well known, homeomorphic with a certain set of rational numbers;[10] hence Theorem 56 leads to the following

COROLLARY. *Every countable metric space is homeomorphic with a certain set of rational numbers.*

Every countable set of real numbers which is dense-in-itself is, as we know, homeomorphic with the set of all rational numbers; it follows, therefore, from Theorem 56, that *every countable metric space which is dense-in-itself is homeomorphic with the set of all rational numbers.*

Thus *e.g.* the set of all rational points of the Hilbert space (§ 49) (which is everywhere dense in that space) is homeomorphic with the set of all rational numbers.

Hence, the set of all rational numbers has the greatest dimensional type (§ 12) of all metric spaces.

If a countable metric space P contains a subset dense-in-itself (not null), then this subset is homeomorphic with the set W of all rational numbers, which results in the fact (by the corollary deduced above) that P and W have the same dimensional type.

It follows from the above that if a countable metric space P has a dimensional type different from that of W (hence a smaller one), then it does not contain a non-null subset dense-in-itself, and so it must be scattered.

54. The Hilbert space is a natural generalization of the m-dimensional space for $m=\aleph_0$ (owing to the definition of distance), but it possesses a somewhat artificial limitation on the coordinates, namely the condition of convergency for the sum of the squares (a condition which is necessary to assure that the distance between two elements is always finite). Fréchet raised the following question: Let E be a set whose elements are the infinite sequences of real numbers

$$x_1, x_2, x_3, \ldots$$

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[10] In fact, let X denote a countable set of real numbers, and W the set of all rational numbers. The set $X+W$ is, therefore, countable, and so (see appendix, § 9) similar to the set W. But, as is easily seen, the similar mapping of the set $X+W$ on the set W is at the same time homeomorphic.

Is it possible to establish a metric in E so that, in the metric space thus obtained, the necessary and sufficient condition for the element

$$(1) \qquad p(x_1, x_2, x_3, \ldots)$$

to be the limit of the infinite sequence of elements $p_n (n=1, 2, \ldots)$, where p_n is an infinite sequence of real numbers,

$$(2) \qquad p_n(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots),$$

is that the relations

$$(3) \qquad \lim_{n \to \infty} x_i^{(n)} = x_i$$

hold for

$$i=1, 2, 3, \ldots ?$$

Fréchet has proved that the answer is in the affirmative, although the definition of distance which he adopted for the set E is somewhat artificial. He chose, namely, the distance between two elements

$$p(x_1, x_2, x_3, \ldots) \text{ and } q(y_1, y_2, y_3, \ldots)$$

to be the number

$$(4) \qquad \rho(p, q) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{n!\,(1+|x_n - y_n|)} .$$

It will be necessary to show first of all, that the function (4) possesses the three required properties of distance. It is evident that it possesses the first two; it will be sufficient, therefore, to prove the triangle property.

Let a and b be any two real numbers. We have, as is well known,

$$|a+b| \leq |a| + |b|,$$

and so

$$\frac{|a+b|}{1+|a+b|} = 1 - \frac{1}{1+|a+b|} \leq 1 - \frac{1}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|}$$

$$+ \frac{|b|}{1+|a|+|b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|};$$

hence, for $a = x_n - y_n$, $b = y_n - z_n$,

(5) $$\frac{|x_n - z_n|}{1+|x_n - z_n|} \leq \frac{|x_n - y_n|}{1+|x_n - y_n|} + \frac{|y_n - z_n|}{1+|y_n - z_n|}.$$

Hence, if r be the sequence $(z_1, z_2, z_3, \ldots)$, then (4) gives at once (on account of (5)) the inequality

$$\rho(p, r) \leq \rho(p, q) + \rho(q, r).$$

The set of all infinite sequences of real numbers, in which the distance function is given by (4), becomes a metric space which is denoted according to Fréchet by E_ω.

Let now $p_n (n=1, 2, \ldots)$ be an infinite sequence of elements of the space E_ω such that $\lim_{n\to\infty} p_n = p$, where p_n and p are the sequences (2) and (1) respectively. Since $\lim_{n\to\infty} p_n = p$ we have

(6) $$\lim_{n\to\infty} \rho(p, p_n) = 0.$$

But evidently from (4)

$$\frac{1}{i!} \frac{|x_i - x_i^{(n)}|}{1+|x_i - x_i^{(n)}|} \leq \sum_{i=1}^{\infty} \frac{1}{i!} \frac{|x_i - x_i^{(n)}|}{1+|x_i - x_i^{(n)}|} = \rho(p, p_n),$$

for $i=1, 2, \ldots$, and so from (6)

$$\lim_{n\to\infty} \frac{|x_i - x_i^{(n)}|}{1+|x_i - x_i^{(n)}|} = 0, \text{ for } i=1, 2, \ldots;$$

i.e.

$$\lim_{n\to\infty} \left(1 - \frac{1}{1+|x_i - x_i^{(n)}|}\right) = 0;$$

hence,

$$\lim_{n\to\infty} \frac{1}{1+|x_i - x_i^{(n)}|} = 1,$$

and so

$$\lim_{n\to\infty} (1+|x_i - x_i^{(n)}|) = 1,$$

and, finally,

$$\lim_{n\to\infty} |x_i - x_i^{(n)}| = 0, \text{ for } i=1, 2, \ldots;$$

this gives (3).

On the other hand, let p_n $(n=1, 2, \ldots)$ be a sequence of elements of E_ω, for which (3) holds. Let ϵ be any positive number. Choose a number m sufficiently large so that

$$\sum_{i=m+1}^{\infty} \frac{1}{i!} < \frac{\epsilon}{2} \tag{7}$$

(which is possible, since the series $e-1=\sum_{1}^{\infty} \frac{1}{i!}$ is convergent). Since $\frac{|a|}{1+|a|} < 1$ for every real a, we have certainly

$$\sum_{i=m+1}^{\infty} \frac{1}{i!} \frac{|x_i - x_i^{(n)}|}{1+|x_i - x_i^{(n)}|} < \frac{\epsilon}{2}, \tag{8}$$

as a result of (7).

In virtue of (3), there exists an index μ such that

$$|x_i - x_i^{(n)}| < \frac{\epsilon}{2m}, \text{ for } n > \mu,\ i=1, 2, \ldots, m,$$

and so, since

$$\frac{|a|}{i!\,(1+|a|)} \leq |a|, \text{ for } i=1, 2, \ldots$$

we have

$$\sum_{i=1}^{m} \frac{|x_i - x_i^{(n)}|}{i!\,(1+|x_i - x_i^{(n)}|)} < m \cdot \frac{\epsilon}{2m} = \frac{\epsilon}{2}, \text{ for } n > \mu;$$

hence, from (8) and the definition of distance, we see that

$$\rho(p, p_n) < \epsilon, \text{ for } n > \mu,$$

from which it follows that $\lim_{n \to \infty} p_n = p$.

Hence, in order that an element p be the limit of an infinite sequence of elements of the space E_ω, it is necessary and sufficient that the i^{th} coordinates of the terms of the sequence approach the i^{th} coordinate of the element p, for every index i.

Let the infinite sequences of rational numbers, with all but a finite number of terms in each sequence equal to zero, be called the *rational elements* of E_ω. It follows from the property deduced above that the set of all rational elements of the space E_ω is everywhere dense in that space. In fact, let $(x_1, x_2, \ldots)$ denote any infinite sequence of real numbers. There exist, as is well known, infinite sequences of rational numbers $x_i^{(n)}$ $(n=1, 2, \ldots)$ for which (3) holds. Denote by q_n the sequence $q_n(\xi_1^{(n)}, \xi_2^{(n)}, \ldots)$, where $\xi_i^{(n)}=x_i^{(n)}$ for $i \leq n$, and $\xi_i^{(n)}=0$ for $i>n$. As is easily seen, we shall have from (3)

$$\lim_{n\to\infty} \xi_i^{(n)}=x_i, \text{ for } i=1, 2, \ldots$$

and so $\lim\limits_{n\to\infty} q_n=p$, where $q_n(n=1, 2, \ldots)$ are rational elements of E_ω.

E_ω has, therefore, a countable subset, everywhere dense.

Let now S denote any metric space which possesses a countable subset everywhere dense. We shall prove that S is homeomorphic with a certain subset of E_ω.

Let

$$p_1, p_2, p_3, \ldots \tag{9}$$

be an infinite sequence which consists of all the different elements of the countable, everywhere dense subset P of S. We shall denote by $\rho_1(p, q)$ the distance between any two elements in S. Let every element p of S be correlated with an element $q=\phi(p)$ of E_ω, in such a manner that the coordinates of q are the numbers

$$\rho_1(p_1, p), \rho_1(p_2, p), \rho_1(p_3, p), \ldots$$

(*i.e* $\phi(p)$ is an infinite sequence of real numbers $x_1, x_2, x_3, \ldots$, where $x_i=\rho_1(p_i, p)$, for $i=1, 2, 3, \ldots$).

It will be shown that $S\, h_\phi\, \phi(S)$. We shall prove first that the function ϕ establishes a (1, 1) correspondence between the elements of the sets S and $\phi(S)$. To do so, it will obviously be sufficient to show that the function ϕ is biuniform in S, *i.e* that if $p \epsilon S$, $p' \epsilon S$, and $p \neq p'$, then $\phi(p) \neq \phi(p')$.

Hence, suppose that p and p' are two different elements of S; we have, therefore, $\rho_1(p, p')>0$, and so from the property of the sequence (9), there exists a term p_k of that sequence such that

$$\rho_1(p, p_k) < \tfrac{1}{2}\rho_1(p, p'),$$

from which we have

$$2\rho_1(p, p_k)<\rho_1(p, p') \leq \rho_1(p, p_k)+\rho_1(p_k, p'),$$

and so

$$\rho_1(p, p_k) < \rho_1(p', p_k);$$

this proves that the elements $\phi(p)$ and $\phi(p')$ differ in their k^{th} coordinates; hence, $\phi(p) \neq \phi(p')$.

Assume now that we have $\lim_{n\to\infty} p^{(n)} = p$ in space S. Let i be a given index. Since the distance $\rho_1(p, q)$ is a continuous function in S, the relation $\lim_{n\to\infty} p^{(n)} = p$ implies the relation $\lim_{n\to\infty} \rho_1(p_i, p^{(n)}) = \rho_1(p_i, p)$. Hence, the i^{th} coordinate of the element $\phi(p^{(n)})$ approaches the i^{th} coordinate of the element $\phi(p)$, for $i = 1, 2, \ldots$, and so (from the property of the space E_ω) $\lim_{n\to\infty} \phi(p^{(n)}) = \phi(p)$. The function $\phi(p)$ is, therefore, continuous in S.

Suppose now that for a certain sequence $p_n (n=1, 2, \ldots)$ of elements of a set S, and a certain element p of that set, we have $\lim_{n\to\infty} \phi(p^{(n)}) = \phi(p)$. From the property of the space E_ω and the definition of the function ϕ, we have, therefore,

$$\lim_{n\to\infty} \rho_1(p_i, p^{(n)}) = \rho_1(p_i, p), \text{ for } i=1, 2, \ldots \tag{10}$$

But

$$\rho_1(p, p^{(n)}) \leqq \rho_1(p, p_i) + \rho_1(p_i, p^{(n)});$$

hence, from (10) we have

$$\lim_{n\to\infty} \rho_1(p, p^{(n)}) \leq 2\rho_1(p_i, p), \text{ for } i=1, 2, \ldots \tag{11}$$

In virtue of the property of the sequence (9), the number $2\rho_1(p_i, p)$ can be made arbitrarily small for a suitable i; the inequality (11) gives, therefore,

$$\lim_{n\to\infty} \rho_1(p, p^{(n)}) = 0, \text{ } i.e. \lim_{n\to\infty} p^{(n)} = p.$$

We have thus proved that the inverse of the function ϕ is continuous in the set $\phi(S)$. The relation $S\ h_\phi\ \phi(S)$ is, therefore, proved. This gives

Theorem 57. *Every metric space which possesses a countable subset, everywhere dense, is homeomorphic with a certain subset of* E_ω.

As previously shown (§ 47) the axiom of countability (*vi*) for a metric space is equivalent to the existence of a countable subset, everywhere dense. Hence, connecting Theorem 57 with the corollary to Theorem 51 (§ 49) and considering that the Hilbert space and the space E_ω both possess countable subsets everywhere dense, we conclude that the Hilbert space is homeomorphic with a certain subset of E_ω, and conversely. From the definition of dimensional types (§ 12) and the above, we obtain, therefore, the following

COROLLARY. *The Hilbert space and the space E_ω of Fréchet have the same dimensional type.*

However, the question whether the Hilbert space is homeomorphic with Fréchet's space E_ω is not as yet settled.

Denote by E the set of all elements of E_ω, whose coordinates are all numerically ≤ 1.

It follows that E is a closed, compact set. In fact, let $p_n(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots)$ be a given infinite sequence of different elements of E, where $n=1, 2, \ldots$ The sequence $x_1^{(n)}$ $(n=1,2,\ldots)$ is a bounded sequence of real numbers (since $|x_1^{(n)}| \leq 1$, for $n=1, 2, \ldots$); there exists, therefore, an increasing sequence $n_k > 1$ $(k=1, 2, \ldots)$ of indices such that the sequence $x_1^{(n_k)} (k=1, 2, \ldots)$ is convergent, with limit x_1 say (where, obviously, $|x_1| < 1$). Similarly, since the sequence $x_2^{(n_k)} (k=1, 2, \ldots)$ is bounded, there exists an infinite increasing sequence of indices $k_l > 1$ $(l=1, 2, \ldots)$ such that the sequence $x_2^{n_{k_l}}$ $(l=1, 2, \ldots)$ is convergent, with limit x_2 say. Similarly, we conclude in general the existence of an infinite, increasing sequence of indices $l_m > 1$ $(m=1, 2, \ldots)$ such that the sequence $x_3^{n_{k_{l_m}}}$ $(m=1, 2, \ldots)$ is convergent, with limit x_3 say, and so on. It is easily seen that the infinite sequence $p_1, p_{n_1}, p_{n_{k_1}}, p_{n_{k_{l_1}}}, \ldots$ contains only different terms from the sequence p_n $(n=1, 2, \ldots)$, and that it tends as limit to $p(x_1, x_2, x_3, \ldots)$, which is an element of E. Hence, E is closed and compact.

Furthermore, it will be shown that E_ω is homeomorphic with a certain subset of the set E, namely with the set E_1 consisting of all the elements of E_ω, whose coordinates are numerically <1.

To prove the above we note first of all that the set of all real numbers is homeomorphic with the set of all numbers which are in the interval $(-1, 1)$; such a homeomorphic mapping may be established by the function $\phi(x) = \frac{x}{1+|x|}$ say $\left(\text{here the inverse function of } y=\phi(x) \text{ is } \psi(y) = \frac{y}{1-|y|}\right)$. If now every element $p(x_1, x_2, \ldots)$ of E_ω be correlated with the element $f(p) = q(\phi(x_1), \phi(x_2), \ldots)$ of E_1, then, as is easily seen, $E_\omega \, h_f \, E_1$.

Since E is compact, the set $E_1 \subset E$ is compact. Hence:

The space E_ω is homeomorphic with a certain compact subset of itself. Hence, it follows also from Theorem 58 that every metric space, which posesses a countable subset everywhere dense, is homeomorphic with a certain compact subset of E_ω.

Denote now (Fréchet) by H_ω the set of all elements of E_ω whose coordinates are irrational. We shall prove that H_ω is homeomorphic with the set H_1 of all irrational numbers.

The set H_1 is, as we know, homeomorphic with the set T_1 of all irrational numbers in the interval $(0, 1)$. It follows readily from this (in the same way as the relation $E_\omega \, h_f \, E_1$ previously obtained) that the set H_ω is homeomorphic with the set T of all elements of E_ω with coordinates irrational numbers in the interval $(0, 1)$. It will be sufficient, therefore, to show that $T \, h \, T_1$.

Let $p(x_1, x_2, \ldots)$ be a given element of the set T. The numbers $x_1, x_2, \ldots$ are, therefore, irrational and in the interval $(0, 1)$; let

$$x_i = \frac{1}{n_1^{(i)}+} \; \frac{1}{n_2^{(i)}+} \; \frac{1}{n_3^{(i)}+} \cdots$$

be a development of the number x_i as a continued fraction.

Employ the diagonal method to rearrange the double sequence

$$\begin{array}{l} n_1', \; n_2', \; n_3', \ldots \\ n_1'', \; n_2'', \; n_3'', \ldots \\ n_1''', \; n_2''', \; n_3''', \ldots \\ \ldots\ldots\ldots\ldots, \ldots \end{array}$$

into the single sequence

$$n_1', n_1'', n_2', n_1''', n_2'', n_3', n_1^{(4)}, \ldots$$

Put

$$f(p)=\frac{1}{n_1'+}\ \frac{1}{n_1''+}\ \frac{1}{n_2'+}\ \frac{1}{n_1'''+}\cdots$$

It is easily seen that $T\,h_f T_1$. It is sufficient here to base the proof on the property of sequences convergent in E_ω and on the following two properties of continued fractions: 1. For every irrational number x_0 and every positive integer k there exists a positive number ϵ such that every irrational number x, which satisfies the inequality $|x-x_0|<\epsilon$, possesses a development as a continued fraction which is identical in the first k convergents with that of the number x_0 itself. 2. For every irrational number x_0 and every positive number ϵ, there exists a positive integer k such that every irrational number x, whose development as a continued fraction has the first k convergents identical with the corresponding convergents in the development of x_0, satisfies the inequality $|x-x_0|<\epsilon$.

The relation $H_\omega\,h\,H_1$ may, therefore, be considered as proved.

TRANSLATIONS OF THE SPACE E_ω. Let

$$a_1,\ a_2,\ a_3,\ \ldots$$

denote a given infinite sequence of real numbers. Correlate with each element $p(x_1, x_2, \ldots)$ of E_ω the element

$$\phi(p)=q(x_1+a_1,\ x_2+a_2,\ x_3+a_3,\ \ldots). \tag{12}$$

It is easily seen that $E_\omega h_\phi \phi(E_\omega)$. The transformation ϕ is called a *translation* of the space E_ω (by analogy with the m-dimensional space). As is easily seen (from (4)), a translation of E_ω is an isometric transformation of E_ω into itself. Since by a suitable translation any element of E_ω can be transformed into any other, we may say that E_ω is not only topologically but also metrically homogenous.

Let now N be any set of elements of E_ω with potency less than that of the continuum.

It will be shown that there exists a translation of E_ω which transforms the set N into a certain subset of the set H_ω. In order to prove this, we shall first establish the following

LEMMA. *If Q be a set of real numbers of potency less than that of the continuum, there exists a real number a such that for every number x of Q the number $x+a$ is irrational.*

Proof. Let $\mathfrak{m}$ denote the potency of Q; hence $\mathfrak{m}<\mathfrak{c}$. Denote by S the set of all real numbers of the form $r-x$, where r is rational and $x \in Q$. Hence $\overline{\overline{S}} \leq \aleph_0 \mathfrak{m}$, and so $\overline{\overline{S}}<\mathfrak{c}$ (since $\mathfrak{m}<\mathfrak{c}$). There exists, therefore, a real number a, which does not belong to S. Obviously, for $x \in Q$, the number $x+a$ will be irrational. (For in case $x+a=r$, where r is rational, we would have $a=r-x$, and so $a \in S$, contrary to the definition of a.) The lemma is, therefore, proved.

If now $N \subset E_\omega$, and $\overline{\overline{N}}<\mathfrak{c}$, then the set N_i of all i^{th} coordinates of the elements of N has potency $<\mathfrak{c}$, and, therefore, by the above lemma, there exists a real number a_i such that for every $x_i \in N_i$, the number x_i+a_i is irrational. Furthermore, it is obvious that the translation (12) transforms the set N into a certain subset of H_ω.

From Theorem 57 and the relation $H_\omega \, h \, H_1$, the following theorem results:

Every metric space of potency less than that of the continuum, which possesses a countable subset, everywhere dense, is homeomorphic with a certain set of real numbers.

In case $\mathfrak{c}=\aleph_1$, the above theorem is obviously identical with Theorem 56.

54a. Baire calls the set of all infinite sequences of integers, where the sequence $C(a_1, a_2, a_3, \ldots)$ is considered to be the limit of the sequence of sequences $C^{(n)}(a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots)$ $(n=1, 2, \ldots)$, a 0-*dimensional space*, when and only when there exists for every integer k a number μ such that

$$a_i^{(n)}=a_i, \text{ for } i=1, 2, \ldots, k, \; n>\mu.$$

Baire's space may be slightly generalized. Let P denote any countable set consisting of (different elements) $p_1, p_2, p_3, \ldots$. Instead of sequences of integers, consider sequences of terms which are elements of the set P, and define the limit as Baire does. We shall show that every space Π thus obtained is homeomorphic with the set H_1 of all irrational numbers.

The set H_1 is, as we know, homeomorphic with the set T_1 of all irrational numbers in the interval (0, 1); hence, it will be sufficient to show that $T_1 \, h \, \Pi$. For this purpose, correlate the irrational number

$$x = \frac{1}{n_1+} \, \frac{1}{n_2+} \, \frac{1}{n_3+} \cdots$$

with the sequence

$$f(x) = (p_{n_1}, p_{n_2}, p_{n_3}, \ldots).$$

To prove that $T_1 \, h_f \, \Pi$, it will be sufficient to base the argument on a property of continued fractions, the definition of limit in the space Π, and the fact that the equality $p_m = p_n$ is equivalent to the equality $m = n$ (since the elements of the set P are all different).

Hence, in particular,

The 0-dimensional space of Baire is homeomorphic with the set of all irrational numbers.

It follows from the above that a metric can be established in the Baire space. Fréchet establishes it directly, considering the number $\frac{1}{r}$ as the distance between two different sequences $(a_1, a_2, a_3, \ldots)$ and $(b_1, b_2, b_3, \ldots)$, where r is the smallest index k such that $a_k \neq b_k$. (The proof, that the distance so defined satisfies the required properties of distance, and that it leads to a definition of limit equivalent to that adopted by Baire, is left to the reader.)

We remark, finally, that if P consists of two (or a finite number of) different elements, then the space Π thus obtained is homeomorphic with a perfect point-set obtained from real numbers, as can be easily seen. Hence, in particular, a subset of the Baire space, consisting of all the infinite sequences formed with 0 and 1, has the same dimensional type as the whole Baire space.

CHAPTER VII

METRIC SPACES IN WHICH BOUNDED SETS ARE COMPACT

55. We shall consider in this chapter metric spaces which satisfy the following condition

(W): *Every bounded set is compact.*

The converse, as is easily seen, is true for all metric spaces, *i.e. every compact set is bounded.* In fact, if a compact set E is supposed to be unbounded, and if p_0 be a given element of E, then for every positive integer n there exists an element p_n of E such that $\rho(p_0, p_n) > n$. It is easily seen that E_1, a subset of E, consisting of the different terms of the sequence $p_1, p_2, p_3, \ldots$, is an infinite set without a limit element, from which it follows that E is not compact. In metric spaces satisfying (W) the conditions of being compact and bounded are equivalent.

A metric space M which satisfies condition (W) is obviously semi-compact (but the converse is not necessarily true). In fact, let p_0 be a given element of the space M; we shall then have

$$M = K(p_0, 1) + K(p_0, 2) + K(p_0, 3) + \ldots;$$

M is, therefore, the sum of a countable aggregate of bounded sets, which are also compact by condition (W); hence, M is semi-compact. As already mentioned in § 47, every semi-compact metric space satisfies axiom (*vi*), and all the theorems of chapters I-V are valid. Moreover, it contains a countable subset everywhere dense (*i.e.* it is the enclosure of some countable set); it is therefore *separable.*

From condition (W) and the definition of a compact set (§ 16), we obtain at once

Theorem 58 (Bolzano-Weierstrass). *Every infinite bounded set has at least one limit element.*

On the other hand, a metric space for which Theorem 58 is true evidently satisfies condition (W). Hence, condition (W) and the Bolzano-Weierstrass theorem are equivalent.

Theorem 59 (Cauchy). *In order that an infinite sequence of elements $p_1, p_2, p_3, \ldots$ has a limit, it is necessary and sufficient that to every positive number ϵ there exists an index μ such that*

(1) $$\rho(p_{n+k}, p_n) < \epsilon, \text{ for } n > \mu,\ k = 1, 2, \ldots$$

Proof. Assume that the infinite sequence $p_1, p_2, p_3, \ldots$ of elements of the metric space considered has a limit a (§ 31), and let ϵ denote a given positive number. For the open set $K\left(a, \frac{\epsilon}{2}\right)$ (which contains a) there exists an index μ such that for $n > \mu$ we have $p_n \epsilon K\left(a, \frac{\epsilon}{2}\right)$ and so also $p_{n+k} \epsilon K\left(a, \frac{\epsilon}{2}\right)$; hence, $\rho(a, p_n) < \frac{\epsilon}{2}$, and $\rho(a, p_{n+k}) < \frac{\epsilon}{2}$, from which (by the triangle law) inequality (1) follows at once. The condition of the theorem is, therefore, necessary.

Suppose now that a given infinite sequence $p_1, p_2, p_3, \ldots$ satisfies the condition of Theorem 59. We shall consider next two cases.

1. Only a finite number of terms of the sequence p_n ($n = 1, 2, \ldots$) are different. In such case a certain term of the sequence, a say, will occur an infinite number of times. Let ϵ denote a given positive number. Since the condition of Theorem 59 is assumed to be satisfied, there exists an index μ for which (1) is true. Let n be any index $> \mu$. Since a occurs an infinite number of times in the sequence, there exists an integer k (corresponding to the index n) such that $p_{n+k} = a$, and so from (1)

(2) $$\rho(a, p_n) < \epsilon.$$

Thus for every number $\epsilon > 0$ there exists an index μ such that the inequality $n > \mu$ implies the inequality (2). This proves that

$$\lim_{n \to \infty} \rho(a, p_n) = 0,$$

and so, by Theorem 50, we get

$$\lim_{n \to \infty} p_n = a.$$

2. There is an infinite number of different terms among the terms of the sequence $p_n (n=1, 2, \ldots)$. The set E consisting of all the different terms of the given sequence is, therefore, infinite and since, by condition (1), it is bounded, it is compact by condition (W) and so has at least one limit element, a say. Let E_n denote the set of all the different terms of the sequence p_{n+1}, $p_{n+2}, \ldots$; then $E_n' = E'$ (§ 15), and so a is a limit element of E_n. Let ϵ denote a positive number; by the condition of Theorem 59, there exists an index μ for which (1) is true. Let n be an index $> \mu$. Since $a \epsilon E_n'$, there exists at least one element of the set E_n in the open set $K(a, \epsilon)$; hence, there exists by the definition of the set E_n an integer k such that $p_{n+k} \epsilon K(a, \epsilon)$, whence $\rho(a, p_{n+k}) < \epsilon$, and from (1) (and the triangle law) we have

$$\rho(a, p_n) < 2\epsilon. \tag{3}$$

We have thus proved that to every positive number ϵ there exists an index μ such that the inequality (3) holds for every $n > \mu$. This proves that $\lim_{n \to \infty} \rho(a, p_n) = 0$ and, hence, by Theorem 50, that $\lim_{n \to \infty} p_n = a$. The condition of Theorem 59 is, therefore, sufficient. Hence, Theorem 59 is proved.

Fréchet calls a space *complete* when a metric can be established for it such that Cauchy's Theorem is true. It therefore follows from Theorem 59 that a metric space which satisfies condition (W) is complete. But the converse is not necessarily true; for if a set N be infinite and the distance between any two different elements of N be always taken to be the number 1, then the metric space thus obtained will obviously be complete, but will not satisfy condition (W).

56. The following corollary follows immediately from condition (W), Theorem 41, and the fact that every compact set is bounded (§ 55):

A continuous transform of a closed and bounded set is closed and bounded.

Furthermore, we obtain easily the following:

Theorem 60. *A continuous transform of a set F_σ is a set F_σ.*[1]

Proof. Let E denote the given set F_σ. From the definition of sets F_σ (§ 50), we have $E=E_1+E_2+E_3+\dots$, where the sets $E_n (n=1, 2, \dots)$ are closed. Let p_0 denote a given element, k a given positive integer, and P_k the set of elements p of the space considered which satisfy the inequality $\rho(p_0, p)\leq k$; the set P_k is, as is easily seen, closed (from Theorem 39 and the continuity of the function ρ) and bounded, and so the sets $E_n.P_k$ are closed and bounded (for all integers n and k). But obviously

$$E=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} E_n.P_k;$$

thus every set F_σ is the sum of a countable aggregate of closed and bounded sets. From this and the fact that a continuous transform of a closed and bounded set is closed, Theorem 60 follows at once.

57. Let $f(p)$ be a function defined in a given set E and continuous in that set. Denote by T the set of all elements of $\overline{E}$ for which the oscillation of f in E is zero. By the corollary to Theorem 53, the set T will then be a set G_δ, and $E\subset T$ by Theorem 54. Let p_0 be an element of the set $T-E$; since $T\subset\overline{E}$, p_0 is a limit element of E, and so, by Theorem 39, there exists an infinite sequence $p_n (n=1, 2, \dots)$ of elements of E such that $\lim_{n\to\infty} p_n=p_0$. It follows that the sequence $f(p_n)$ $(n=1, 2, \dots)$ has a limit. For, let ϵ be a given positive number; since $\omega(p_0)=0$, $(p_0 \epsilon T)$ and from the definition of oscillation (§ 51), there exists a number $r>0$ such that for every two elements p and q of the set $E.K(p_0, r)$ the inequality

$$\rho(f(p), f(q))<\epsilon$$

is satisfied.

But, since $\lim_{n\to\infty} p_n=p_0$, and from the definition of a limit (§ 31), there exists an index μ such that $p_n \epsilon K(p_0, r)$, for $n>\mu$. We have, therefore,

$$\rho(f(p_{n+k}), f(p_n))<\epsilon, \text{ for } n>\mu,\ k=1, 2, \dots$$

[1]Analogous theorems for closed or open sets cannot be obtained without additional hypotheses on the space considered.

Hence, by Theorem 59, the sequence $f(p_n)$ $(n=1, 2, \ldots)$ has a limit; denote it by b.

The element b depends solely on the element p_0 and not on the choice of the sequence p_n of elements of E for which $\lim_{n\to\infty} p_n = p_0$. In fact, let q_n denote any sequence of elements of E such that $\lim_{n\to\infty} q_n = p_0$. The sequence $f(q_n)$ $(n=1, 2, \ldots)$ will have a limit, as proved above; let it be denoted by c. But, since $\lim_{n\to\infty} p_n = p_0$, and $\lim_{n\to\infty} q_n = p_0$, the sequence $p_1, q_1, p_2, q_2, \ldots$ will also have for its limit p_0 (§ 31), and so the sequence $f(p_1), f(q_1), f(p_2), f(q_2), \ldots$ will have a limit, which we may denote by g. But, from property 3 of a limit (§ 31), it follows that the sequences $f(p_1), f(p_2), \ldots$ and $f(q_1), f(q_2), \ldots$ being subsequences of a sequence whose limit is g, must also have the limit g. We have, therefore, $b=g$, $c=g$, and so $b=c$.

We have thus proved that to every element p_0 of the set $T-E$ corresponds a completely defined element $b=\phi(p_0)$ such that, if $p_n \epsilon E$, for $n=1, 2, \ldots$ and $\lim_{n\to\infty} p_n = p_0$, then $\lim_{n\to\infty} f(p_n) = \phi(p_0)$. The function $\phi(p)$ is, therefore, defined in the set $T-E$. Next put $\phi(p)=f(p)$, for $p \epsilon E$; the function $\phi(p)$ is then defined in the whole set T. It will be shown next that the function $\phi(p)$ is continuous in the set T.

For, let p_0 denote a given element of the set T and ϵ an arbitrary given positive number. Denote by $\omega(p_0, \epsilon)$ the upper bound of the set G of all the numbers

$$\rho(f(p), f(q)),$$

where p and q are any two elements of the set $E.K(p_0, \epsilon)$, and by $\omega_1(p_0, \epsilon)$ the upper bound of the set H of all the numbers

$$\rho(\phi(p), \phi(q)),$$

where p and q are any two elements of the set $T.K(p_0, \epsilon)$.

Let h be a number of the set H; there exist two elements p and q of the set $T.K(p_0, \epsilon)$ such that

$$h = \rho(\phi(p), \phi(q)). \tag{4}$$

Since $p \epsilon T \subset \overline{E}$, there exists an infinite sequence p_n $(n=1, 2, \ldots)$

of elements of E such that $\lim_{n\to\infty} p_n = p$, and since $p \epsilon K(p_0, \epsilon)$, then also $p_n \epsilon K(p_0, \epsilon)$, for $n > \mu$. Similarly, since $q \epsilon T.K(p_0, \epsilon)$, there exists an infinite sequence q_n $(n=1, 2, \ldots)$ of elements of E such that $\lim_{n\to\infty} q_n = q$, and $q_n \epsilon K(p_0, \epsilon)$, for $n > \nu$. Hence, for $n > \mu + \nu$, the numbers

$$\rho(f(p_n), f(q_n))$$

belong to the set G, and so, from the definition of $\omega(p_0, \epsilon)$, we shall have

$$(5) \qquad \rho(f(p_n), f(q_n)) \leq \omega(p_0, \epsilon), \text{ for } n > \mu + \nu.$$

But, since $p_n \epsilon E$, $q_n \epsilon E$, and $\lim_{n\to\infty} p_n = p$, $\lim_{n\to\infty} q_n = q$, we have

$$(6) \qquad \lim_{n\to\infty} f(p_n) = \phi(p), \text{ and } \lim_{n\to\infty} f(q_n) = \phi(q).$$

(In case either p or $q \epsilon T - E$, the above follows from the definition of the function ϕ in the set $T - E$, and if either p or $q \epsilon E$, the result follows from the continuity of the function f in the set E (Theorem 40) and the fact that $\phi(p) = f(p)$, for $p \epsilon E$.) From (5), (6), and the continuity of the function ρ (§ 45), we find

$$\rho(\phi(p), \phi(q)) \leq \omega(p_0, \epsilon),$$

i.e. from (4), $h \leq \omega(p_0, \epsilon)$. Since h is any number of the set H, it follows that the upper bound of that set, given by the number $\omega_1(p_0, \epsilon)$, does not exceed $\omega(p_0, \epsilon)$. Hence

$$(7) \qquad \omega_1(p_0) = \lim_{\epsilon\to 0} \omega_1(p_0, \epsilon) \leq \lim_{\epsilon\to 0} \omega(p_0, \epsilon) = \omega(p_0),$$

where (§ 51, (17)) $\omega(p_0)$ denotes the oscillation of the function f in the set E at the element p_0, and $\omega_1(p_0)$ (from the definition of the numbers $\omega_1(p_0, \epsilon)$) the oscillation of the function ϕ in the set T at the element p_0. But from the definition of the set T, and since $p_0 \epsilon T$, we have $\omega(p_0) = 0$; hence, from (7), $\omega_1(p_0) = 0$, and so, by Theorem 54, the function ϕ is continuous in the set T at the point p_0.

Since p_0 is any element of T, we have thus proved that ϕ is continuous in the whole set T.

We have, therefore, proved the following:

Theorem 61. *If $f(p)$ be a function defined in a set E and continuous in that set, there exists a function $\phi(p)$ defined in a certain set T, which is a G_δ, containing E and contained in $\overline{E}$, and this function is continuous in the set T and equals $f(p)$ for each element p of E.*

This theorem is expressed by stating that a function continuous in a given set E may be extended, the continuity being preserved, over a certain set G_δ containing E. Such a set could be chosen to be the set T of all the elements of $\overline{E}$ for which the oscillation of the given function in E is zero. It can be easily shown that, if a function $f(p)$ defined and continuous in a set E can be extended with the retention of continuity over a certain set S such that $E \subset S \subset \overline{E}$, then $S \subset T$. It is also easily seen that a given function (continuous in E) can be extended, continuity being preserved, over every set $S \subset T$ in one and only one way.

58. Let now E and F be two homeomorphic sets. Then there exists a function f defined and continuous in E such that $E\, h_f\, F$, and a function g, the inverse of f, continuous in F such that $F\, h_g\, E$.

By Theorem 61, there exists a function $\phi(p)$ defined and continuous in a certain set G_δ, T say, such that $E \subset T \subset \overline{E}$, and $\phi(p) = f(p)$, for $p \in E$. Similarly, by the same theorem, there exists a function $\psi(q)$ defined and continuous in a certain set G_δ, H say, such that $F \subset H \subset \overline{F}$, and $\psi(q) = g(q)$, for $q \in F$.

Let M denote the set of all elements p of T for which $\phi(p) \in H$, and denote by N the set of all elements q of H for which $\psi(q) \in T$. Then $M\, h_\phi\, N$. For, since $M \subset T$, and the function ϕ is continuous in T, it is certainly continuous in M. Similarly, the function ψ is continuous in the set $N \subset H$. To prove the relation $M\, h_\phi\, N$, it will be sufficient to show that the function ψ (considered in the set N) is inverse to the function ϕ (in the set M), or, in other words, that the relation

$$(8) \qquad p \in M,\ \phi(p) = q$$

is equivalent to the relation

$$(9) \qquad q \in N,\ \psi(q) = p;$$

on account of the symmetry of these relations it will obviously be sufficient to show that (8) implies (9).

Hence, suppose that (for a given p) relations (8) are satisfied. Since $p \epsilon M$, we have $\phi(p) \epsilon H$ by the definition of M, and so $q \epsilon H$ by (8).

Since $p \epsilon M \subset T \subset \overline{E}$, there exists an infinite sequence p_n ($n=1, 2, \ldots$) such that $p_n \epsilon E$, for $n=1, 2, \ldots$, and $p_n \rightarrow p$, and so, since $p \epsilon T$, $E \subset T$, and the function ϕ is continuous in T, it follows that $\phi(p_n) \rightarrow \phi(p)$. But, since $p_n \epsilon E$ ($n=1, 2, \ldots$), and from the property of the function ϕ, we get $\phi(p_n)=f(p_n)$, for $n=1, 2, \ldots$; since $E\, h_f\, F$, and since the function g is the inverse of the function f in E, we shall get, on putting $f(p_n)=q_n$, for $n=1, 2, \ldots$, $q_n \epsilon F \subset H$, and $p_n = g(q_n)$, for $n=1, 2, \ldots$, and so, from the property of the function ψ, we obtain $p_n = \psi(q_n)$, for $n=1, 2, \ldots$. This, on account of $p_n \rightarrow p$, gives $\psi(q_n) \rightarrow p$. But, since $q_n \epsilon H$, for $n=1, 2, \ldots$, and $q \epsilon H$, and the function ψ is continuous in H, we have $\psi(q_n) \rightarrow \psi(q)$, and this, on account of the relation $\psi(q_n) \rightarrow p$, formerly obtained, gives $\psi(q)=p$; since $q \epsilon H$, and $p \epsilon T$, it follows from the definition of the set N that $q \epsilon N$. Hence, relations (9) are established.

We have, therefore, proved that $M\, h_\phi\, N$.

From $E\, h_f\, F$, $E \subset T$, $F \subset H$, $\phi(p)=f(p)$, for $p \epsilon E$, and from the definition of the set M, we have $E \subset M$. Similarly, from $F\, h_g\, E$, $\psi(q)=g(q)$, for $q \epsilon F$, and from the definition of the set N, we get $F \subset N$. It will be shown that M and N are sets G_δ. Owing to the symmetry of the relations, it will be sufficient to prove that one of them, M say, is a set G_δ. To that end, we shall first prove the following:

LEMMA. *If a function $\phi(p)$ be continuous in a set T, which is a set G_δ, and if V be an open set, then the set S of all elements p of T for which $\phi(p) \epsilon V$, is a G_δ.*

Proof. Let p be an element of the set S. Since ϕ is continuous in T and V is open, and since $\phi(p) \epsilon V$, there exists an open set $U(p)$ such that $p \epsilon U(p)$, and

$$\phi(q) \epsilon V \text{ whenever } q \epsilon T.U(p).$$

Denote by U the sum of all the sets $U(p)$, where p ranges over all the elements of S. Obviously, $S=T.U$, where U is open. But, since T is a G_δ, the set $S=T.U$ is a G_δ.

COROLLARY. *If a function $\phi(p)$ is continuous in a set T, which is a G_δ, and if the set H is a G_δ, then the set M of all elements of T for which $\phi(p) \epsilon H$, is a G_δ.*

Proof. Since the set H is a G_δ, we may write $H = V_1 . V_2 . V_3 \ldots$, where $V_n (n = 1, 2, \ldots)$ is an open set. Denote by S_n the set of all elements p of T, for which $\phi(p) \epsilon V_n$; the set S_n is a G_δ by the above lemma. But, obviously $M = S_1 . S_2 . S_3 \ldots$ (For, if $p \epsilon M$, then $\phi(p) \epsilon H \subset V_n$, which gives $p \epsilon S_n$, for $n = 1, 2, \ldots$; and if $p \epsilon S_n$, for $n = 1, 2, \ldots$, then $\phi(p) \epsilon V_n$, for $n = 1, 2, \ldots$, and so $\phi(p) \epsilon H$ and $p \epsilon M$.) Since each set $S_n (n = 1, 2, \ldots)$ is a G_δ, the set M is also a G_δ.

Collecting the results obtained in this article we may state the following:

Theorem 62 (Lavrentieff).[2] *If $E\ h_f\ F$, then there exist sets M and N, each of which is a G_δ, such that $M\ h_\phi\ N$, where $E \subset M \subset \overline{E}$, $F \subset N \subset \overline{F}$, and $\phi(p) = f(p)$, for $p \epsilon E$.*

In other words, *the homeomorphism between two sets can always be extended to two sets G_δ, which contain the corresponding sets and are contained in their enclosures.*

Furthermore, it can be shown that the extension of the homeomorphism between the sets E and F to apply to the sets M and N, obtained above, is the best possible.[3]

59. Let E denote a given set G_δ; hence $E = E_1 . E_2 . E_3 \ldots$, where $E_n (n = 1, 2, \ldots)$ is an open set. Let T be a set homeomorphic with the set E. Consider the sets M and N which satisfy the conditions of Theorem 62. Put $Q_n = M . E_n$ $(n = 1, 2, \ldots)$. Since E_n is open, the set $M - E_n = M . CE_n$ is closed with respect to M, and so, by Corollary 1 to Theorem 22 (§ 12), is transformed by the homeomorphism between M and N into a set closed with respect to N; it may, therefore, be written in the form $N . F_n$, where F_n is closed. But, since from $M\ h_\phi\ N$, we have $(M - E_n)\ h_\phi\ N . F_n$, it follows that $M . E_n\ h_\phi\ (N - F_n)$, or, in other words, $Q_n\ h_\phi\ N . U_n$, where $U_n = CF_n$ is open. But, from $E \subset M$, $E = E_1 . E_2 . E_3 \ldots$ and $Q_n = M . E_n$, we have

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[2] *Fund. Math.*, vol. VI, p. 149.

[3] See Sierpinski, *Comptes Rendus*, vol. CLXXVIII, p. 545.

$E = M.E = Q_1.Q_2.Q_3 \ldots$, while the relation $Q_n\, h_\phi\, N.U_n$ gives (on account of $M\, h_\phi\, N$, and $Q_n \subset M$, for $n = 1, 2, \ldots$) $Q_1.Q_2.Q_3 \ldots h_\phi\, N.U_1.U_2.U_3 \ldots$, *i.e.* $\phi(E) = N.U_1.U_2.U_3 \ldots$, and so $T = N.U_1.U_2.U_3 \ldots$ (since $\phi(E) = T$). Since the set N is a G_δ and the sets U_n ($n = 1, 2, \ldots$) are open, it follows that T is a set G_δ. We have thus proved

Theorem 63.[4] *A homeomorphic transform of a set G_δ is a G_δ.*

A family F of sets will be spoken of as a *topological invariant* if every set homeomorphic with a set of the family F also belongs to F.

Theorem 64. *If a family F of sets is a topological invariant, then the family of sets, which are sums of a countable aggregate of sets belonging to F, is a topological invariant.*

For, let $E = E_1 + E_2 + E_3 + \ldots$, where $E_n \epsilon F$, for $n = 1, 2, \ldots$, and suppose that $E\, h_f\, T$. From $E_n \subset E$, we get $E_n\, h_f\, T_n$, where T_n is a certain subset of T ($n = 1, 2, \ldots$), and so also $(E_1 + E_2 + \ldots)\, h_f\, (T_1 + T_2 + \ldots)$, or $E\, h_f\, (T_1 + T_2 + \ldots)$ (since $E\, h_f\, T$). But, from $E_n \epsilon F$ and $E_n\, h_f\, T_n$, we get $T_n \epsilon F$ (since F is a topological invariant). Theorem 64 is, therefore, proved.

Theorem 65. *If a family F of sets is a topological invariant, and if the product of a set belonging to F and a set G_δ belongs to F, then the family of all products of countable aggregates of sets belonging to F is a topological invariant.*

Proof. Suppose that $E = E_1.E_2.E_3 \ldots$, where $E_n \epsilon F$, and assume that $E\, h_f\, T$. There exist, by Theorem 62, two sets M and N, each a G_δ, and for which $M\, h_\phi\, N$, $E \subset M$, $T \subset N$, and $\phi(p) = f(p)$, for $p \epsilon E$. It follows from the property of the family F that the sets $M.E_n$ ($n = 1, 2, \ldots$) belong to F and so also do the sets $T_n = \phi(M.E_n)$, since they are homeomorphic with the sets $M.E_n$. But, from $E = M.E_1.M.E_2 \ldots$, and the fact that ϕ is biuniform in M (§ 9), we have $T = \phi(E) = \phi(M.E_1).\phi(M.E_2) \ldots$, *i.e.* T is a product of a countable aggregate of sets belonging to F, as required.

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[4]This theorem was first proved by Mazurkiewicz in 1916 (*Biuletyn Ak. Um.*, 1916, pp. 490-496). Another proof was given by the author in *Fund. Math.*, vol. VIII, p. 135.

Theorem 66. *If a family F is a topological invariant, and if the product of a set belonging to F and a set G_δ, and the sum of a set belonging to F and a set F_σ always belong to F, then the family of all complements of sets belonging to F is a topological invariant.*

Proof. Suppose that $E=CX$, where $X \epsilon F$, and assume that $E\, h_f\, T$. There exist, by Theorem 62, two sets M and N, each a G_δ, and such that $Mh_\phi N$, $E\subset M$, $T\subset N$, and $\phi(p)=f(p)$, for $p \epsilon E$. From $X \epsilon F$ and the property of the family F, it follows that $M.X \epsilon F$, and $Q=\phi(M.X) \epsilon F$. But from $E=CX\subset M$, we have $E=M-X$, and so $T=\phi(E)=\phi(M-M.X)=\phi(M)-\phi(M.X)=N-Q$ (since ϕ is biuniform in M). Hence $CT=Q+CN$, and so, since $Q \epsilon F$ and CN is a set F_σ, it follows from the property of F that $CT \epsilon F$. Theorem 66 is, therefore, proved.

Theorem 67. *If a family F of sets is a topological invariant, and if the product of a set belonging to F and a G_δ again belongs to F, then the family of all differences of two sets belonging to F is a topological invariant.*

Proof. Suppose that $E_1 \epsilon F$, $E_2 \epsilon F$, $E=E_1-E_2$, $E\,h\,T$. There exist, by Theorem 62, two sets M and N, each a G_δ, and a function ϕ defined in M such that $E\subset M$, $T\subset N$, $M\,h_\phi N$, $E\,h_\phi T$. From $E=E_1-E_2$, and $E\subset M$, we have $E=M.E_1-M.E_2$, where $M.E_1$ and $M.E_2$ belong to F, owing to the property of F, and so the sets T_1 and T_2, which are such that $M.E_1\,h_\phi\,T_1$ and $M.E_2\,h_\phi\,T_2$, belong to F. But the last two sets give $(M.E_1-M.E_2)\,h_\phi\,(T_1-T_2)$ (since ϕ is biuniform in M), and so (since $E=M.E_1-M.E_2$) $\phi(E)=T_1-T_2$; but (from $E\,h_\phi\,T$) $\phi(E)=T$, therefore, $T=T_1-T_2$, where $T_1 \epsilon F$ and $T_2 \epsilon F$. Theorem 67 is, therefore, proved.

60. Following Hausdorff's[5] notation, denote all open sets by P^1, all closed sets by Q^1 and employ transfinite induction to define the sets P^α and Q^α for $1<\alpha<\Omega$ as sums and products respectively of a countable aggregate of sets $E_1, E_2, E_3, \ldots$, where E_n is a set Q^{ξ_n} and a set P^{ξ_n} respectively, and where $\xi_n<\alpha$, for $n=1, 2, \ldots$.

Hence, sets P^2 are sums of a countable aggregate of sets Q^1, and so sets P^2 are sets F_σ (and conversely). Sets Q^2 are products of a countable aggregate of sets P^1; hence, sets Q^2 are sets G_δ (and con-

[5] *Math. Zeitschrift*, vol. V (1919), p. 307.

versely). Sets P^3 are sums of a countable aggregate of sets Q^2, *i.e.* of sets G_δ; hence, sets P^3 are so-called sets $G_{\delta\sigma}$ (and conversely). Sets Q^3 are products of a countable aggregate of sets F_σ, *i.e.* they are so-called sets $F_{\sigma\delta}$.

We shall now deduce several properties of the sets P^α and Q^α.

PROPERTY 1. *Every set P^α is at the same time a set P^β for $\beta > \alpha$; every set Q^α is at the same time a set Q^β for $\beta > \alpha$.*

Proof. For $\alpha > 1$ property 1 follows directly from the definition of the sets P^α and Q^α. For $\alpha = 1$ it will be sufficient to show that every set P^1 is a P^2 and every set Q^1 is a Q^2. The first follows from Theorem 52a, since sets P^1 are open and sets P^2 are F_σ; the second part follows from Theorem 52, since sets Q^1 are closed and sets Q^2 are sets G_δ. Property 1 may, therefore, be considered to be proved.

PROPERTY 2. *The sum of a finite or countable aggregate of sets P^α is a set P^α. The product of a finite or countable aggregate of sets Q^α is a set Q^α.*

Proof. For $\alpha = 1$ the above property follows from axiom (*iii*) (§ 1) and Theorem 3, and for $\alpha > 1$ it follows directly from the definition of the sets P^α and Q^α.

PROPERTY 3. *The complement of a set P^α is a set Q^α; the complement of a set Q^α is a set P^α.*

Proof. From the definition of the sets P^1 and Q^1, it follows that property 3 is true for $\alpha = 1$. Let now β be an ordinal number such that $1 < \beta < \Omega$, and suppose that property 3 is true for all ordinal numbers $\alpha < \beta$. Let E be a set P^β. By the definition of the sets P^α, we may write $E = E_1 + E_2 + E_3 + \ldots$, where E_n is a set Q^{ξ_n}, and $\xi_n < \beta$, for $n = 1, 2, \ldots$. Since property 3 is assumed to hold for $\alpha < \beta$, it follows that CE_n is a set P^{ξ_n}, and since from $E = E_1 + E_2 + \ldots$ we have $CE = CE_1 . CE_2 \ldots$ and since $\xi_n < \beta$, for $n = 1, 2, \ldots$ CE is, therefore, a set Q^β.

On the other hand, let E denote a set Q^β. It follows from the definition of the sets Q^α that $E = E_1 . E_2 . E_3 \ldots$, where E_n is a set P^{ξ_n}, and $\xi_n < \beta$, for $n = 1, 2, \ldots$ Since property 3 is supposed to hold for $\alpha < \beta$, it follows that CE_n is a set Q^{ξ_n}, and since $E = E_1 . E_2 . E_3$

. . . , we have $CE = CE_1 + CE_2 + \ldots$, where $\xi_n < \beta$, for $n = 1, 2, \ldots$, and so CE is a set P^β.

Property 3 is thus established by transfinite induction.

PROPERTY 4. *The product of a finite number of sets P^α is a set P^α. The sum of a finite number of sets Q^α is a set Q^α.*

Proof. It is obviously sufficient to prove property 4 for two factors and two terms respectively; the rest then follows by ordinary induction.

Hence, let α be an ordinal number $< \Omega$, E and T two sets P^α. We may suppose that $\alpha > 1$, since for $\alpha = 1$ property 4 is true (by Theorem 26). It follows from the definition of P^α that $E = E_1 + E_2 + \ldots$, $T = T_1 + T_2 + \ldots$, where E_n is a set Q^{ξ_n}, and where T_n is a set Q^{η_n}, with $\xi_n < \alpha$ and $\eta_n < \alpha$, for $n = 1, 2, \ldots$; hence,

$$E.T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_m.T_n.$$

Denote by $\xi_{m,n}$ the greater of the numbers ξ_m and η_n (or their common value if they are equal); from $\xi_m < \alpha$, for $m = 1, 2, \ldots$ and $\eta_n < \alpha$, for $n = 1, 2, \ldots$, we have evidently $\xi_{m,n} < \alpha$, for all integers m and n. The set E_m is a set Q^{ξ_m}, and so by property 1, E_m is also a set $Q^{\xi_{m,n}}$, since $\xi_m \leq \xi_{m,n}$ by the definition of the number $\xi_{m,n}$. Similarly, T_n is a $Q^{\xi_{m,n}}$, since $\eta_n \leq \xi_{m,n}$. The set $E_m.T_n$ is a product of two sets $Q^{\xi_{m,n}}$, and so a set $Q^{\xi_{m,n}}$ by property 2.

It follows, therefore, from $E.T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_m.T_n$, since $\xi_{m,n} < \alpha$, for $m = 1, 2, \ldots$, $n = 1, 2, \ldots$, that $E.T$ is a set P^α.

The first part of property 4 is, therefore, proved. Let now E and T be two sets Q^α. The sets CE and CT are sets P^α, and so from the first part of property 4, the set $C(E+T) = CE.CT$ is a P^α; hence, $E+T$ is a Q^α by property 3. The second part of property 4 is, therefore, established.

PROPERTY 5. *Every set P^α is a set $Q^{\alpha+1}$. Every set Q^α is a set $P^{\alpha+1}$.*

Proof. If E be a set P^α it is sufficient to write E in the form $E = E.E.E \ldots$, and recalling the definition of sets $Q^{\alpha+1}$, it is seen at once that E is a set $Q^{\alpha+1}$. Similarly, if E be a set Q^{α} it is sufficient to write it in the form $E=E+E+E+\ldots$, and referring to the definition of sets $P^{\alpha+1}$, it is seen that E is a set $P^{\alpha+1}$.

From properties 5 and 2, we get at once

PROPERTY 6. *The sum of a countable aggregate of sets P^{α} is a set $Q^{\alpha+1}$. The product of a countable aggregate of sets Q^{α} is a set $P^{\alpha+1}$.*

From properties 3, 5, 1, and 2 and the relation $E_1-E_2=E_1.CE_2$, we obtain at once

PROPERTY 7. *The difference of two sets P^{α}, or of two sets Q^{α} is both a set $P^{\alpha+1}$ and a set $Q^{\alpha+1}$.*

We shall deduce one more property found by Lusin, which will be made use of later:

PROPERTY 8. *If $\alpha \geq 3$, then every set P^{α} is the sum of a countable aggregate of mutually exclusive sets $E_1, E_2, E_3, \ldots$, where E_n is a set Q^{ξ_n}, and $\xi_n<\alpha$, for $n=1, 2, \ldots$*

Proof. Let E be a set P^{α}, where $\alpha \geq 3$. It follows from the definition of sets P^{α} and property 1 that we may write $E=T_1+T_2+T_3+\ldots$, where T_n is a set Q^{η_n}, and $2 \leq \eta_n<\alpha$, for $n=1, 2, \ldots$. Put $S_n=T_1+T_2+\ldots+T_n$, and denote by ξ_n the greatest of the numbers $\eta_1, \eta_2, \ldots, \eta_n$; since $2 \leq \eta_n<\alpha$, we have obviously $2 \leq \xi_n<\alpha$, for $n=1, 2, \ldots$, and from properties 1 and 4 it follows that S_n is a set Q^{ξ_n}, for $n=1, 2, \ldots$

Put further, $R_1=S_1$ and $R_{n+1}=S_{n+1}-S_n$, for $n=1, 2, \ldots$; we shall have $R_{n+1}=S_{n+1}.CS_n$, for $n=1, 2, \ldots$ But CS_n is, by property 3, a set P^{ξ_n}; we may, therefore, write $CS_n=T_{n,1}+T_{n,2}+\ldots$, where $T_{n,k}$ is a set $Q^{\xi_{n,k}}$, and $\xi_{n,k}<\xi_n$, for $k=1, 2, \ldots$.[6] Denote by $\zeta_{n,k}$ the greatest of the numbers $\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,k}$; hence, $\zeta_{n,k}<\xi_n$, for $k=1, 2, \ldots$, and so from properties 1 and 4 the set $S_{n,k}=T_{n,1}+T_{n,2}+\ldots+T_{n,k}$ is a set $Q^{\zeta_{n,k}}$, for $k=1, 2, \ldots$. Put $R_{n,1}=S_{n,1}$, and $R_{n,k}=S_{n,k}-S_{n,k-1}$, for $k=2, 3, \ldots$; from the definition of the numbers $\zeta_{n,k}$, it follows that $\zeta_{n,k-1} \leq \zeta_{n,k}$, for $k=2, 3, \ldots$; hence, from property 7, $R_{n,k}$ is a set $Q^{\zeta_{n,k}+1}$, for $k=1, 2, \ldots$, and so, since $\zeta_{n,k}<\xi_n$, and $\zeta_{n,k}+1<\xi_n$, is a set Q^{ξ_n}.

[6]This could not be ascertained if we had not $\xi_n \geq 2$. We note that for $\alpha=2$ property 8 is not true.

Since the set S_{n+1} is a set $Q^{\xi_{n+1}}$ and $\xi_n \leq \xi_{n+1}$, it follows form properties 1 and 2 that $S_{n+1}.R_{n,k}$ is a set $Q^{\xi_{n+1}}$, for $k=1, 2, \ldots$ But, as is easily seen, E is the sum of mutually exclusive sets

$$E = S_1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} S_{n+1}.R_{n,k},$$

where S_1 is a set Q^{ξ_1} and $S_{n+1}.R_{n,k}$ a set $Q^{\xi_{n+1}}$, for $n=1, 2, \ldots,$ $k=1, 2, \ldots,$ and where $\xi_n < \alpha$, for $n=1, 2, \ldots$ Hence, this proves property 8.

61. From the fact that a set G_δ is a set Q^2 and from properties 5, 1, and 4, it follows that the product of a set P^α and a set G_δ is a set P^α, for $\alpha \geq 3$. Similarly, from the fact that a set F_σ is a set P^2 and from properties 1 and 2, it follows that the sum of a set P^α and a set F_σ is a set P^α, for $\alpha \geq 2$.

Theorem 68. *The sets P^α and Q^α of Hausdorff are a topological invariant for $\alpha \geq 2$.*

Proof. Theorem 68 follows at once from Theorem 60 for sets P^2, *i.e.* for sets F_σ and from Theorem 63 for sets Q^2, *i.e.* for sets G_δ.

Let now β denote an ordinal number such that $3 \leq \beta < \Omega$, and suppose Theorem 68 to be true for every ordinal number α such that $2 \leq \alpha < \beta$. Let E be a given set P^β; we may, therefore, write $E = E_1 + E_2 + E_3 + \ldots$, where E_n is a set Q^{ξ_n}, and $\xi_n < \beta$, for $n=1, 2, \ldots$, and where, since $\beta \geq 3$, we may suppose from property 1 that $\xi_n \geq 2$, for $n=1, 2, \ldots$ Let T be a set such that $E\, h_f\, T$. Hence, $E_n\, h_f\, T_n$, where $T_n \subset T$, and where $T = T_1 + T_2 + \ldots$ But, since E_n is a set Q^{ξ_n}, where $2 \leq \xi_n < \beta$, and from the supposition that Theorem 68 is true for numbers α, such that $2 \leq \alpha < \beta$, we conclude from $E_n\, h_f\, T_n$ that T_n is a set Q^{ξ_n}. Thus from $T = T_1 + T_2 + \ldots$, it follows that T is a set P^β.

Hence, the family of sets P^β is a topological invariant. But, since $\beta \geq 3$ and from the remark made at the beginning of this article, it follows that the product of a set P^β and a set G_δ and the sum of a set P^β and a set F_σ are sets P^β. The family of sets P^β satisfies, therefore, the conditions of Theorem 66; hence, the family of the complements of the sets P^β, *i.e.* the family of sets Q^β (by property 3) is a topological invariant.

Theorem 68 is, therefore, proved by transfinite induction. Theorem 68 is not true in case of sets P^1 and Q^1 unless special assumptions are made with respect to the space considered.

From Theorem 68 we get at once

COROLLARY 1. *A family of sets, each of which is both a set P^α and a set Q^α, is a topological invariant.*

Sets $P^{\alpha+1}$, which are not sets $P^{\xi+1}$ for any $\xi<\alpha$, are Lebesgue's sets O of class α; similarly, sets $Q^{\alpha+1}$, which are not $Q^{\xi+1}$ for any $\xi<\alpha$, are Lebesgue's sets F of class α. (Lebesgue's definition of sets O and F of class α is different but yet equivalent to the above.) From Theorem 68, we get immediately

COROLLARY 2. *Lebesgue's sets O and F of class α $(1\leq\alpha<\Omega)$ are topological invariants.*

62. Denote by **B** the family of all sets P^α and Q^α, for $1\leq\alpha<\Omega$. Sets which belong to that family are called sets measurable in the *Borel sense* or simply sets B. It follows from the properties of the sets P^α and Q^α, deduced in § 60, that the family **B** satisfies the following conditions:

1) *Every closed set belongs to* **B**;

2) *The sum of a countable aggregate of sets belonging to* **B**, *belongs to* **B**;

3) *The product of a countable aggregate of sets belonging to* **B**, *belongs to* **B**.

Condition 1) follows from the definition of the sets Q^1 and the family **B**.

Let now $E=E_1+E_2+E_3+\ldots$, where E_n $(n=1, 2, \ldots)$ belongs to **B**. We may always suppose, by property 5 (§ 60), that E_n is a set Q^{ξ_n}, where ξ_n is some ordinal number $<\Omega$. It is known that, to an infinite sequence of ordinal numbers $\xi_n (n=1, 2, \ldots)$ which are less than Ω, there exists an ordinal number $\alpha<\Omega$, such that $\xi_n<\alpha$, for $n=1, 2, \ldots$ The expression $E=E_1+E_2+\ldots$ proves, therefore, that E is a P^α and so belongs to **B**. Hence, the family **B** satisfies condition 2). Similarly (referring to property 5 and the definition of sets Q^α), it may be proved that the family **B** satisfies condition 3).

We shall now show that the family of all sets P^α and Q^α ($1 \leq \alpha < \Omega$) is the *smallest* family **B**, which satisfies conditions 1), 2), and 3). In other words, we shall show that if a certain family **B** satisfies conditions 1), 2), and 3) then every set P^α and every set Q^α belong to **B**, for $1 \leq \alpha < \Omega$.

Hence, let **B** denote any family of sets (contained in the metric space considered) which satisfies conditions 1), 2), and 3). The sets Q^1 belong to **B** by condition 1). It follows at once from conditions 1) and 2) that the sets F_σ, *i.e.* the sets P^2 belong to **B**; from this and property 1 (§ 60), it follows that the sets P^1 belong to **B**.

Let now α denote a given ordinal number such that $1 < \alpha < \Omega$, and suppose that all sets P^ξ and Q^ξ belong to **B**, where $1 \leq \xi < \alpha$. Let E be a given set P^α; since $\alpha > 1$, we may write $E = E_1 + E_2 + \ldots$, where E_n is a set Q^{ξ_n}, and $\xi_n < \alpha$; hence, by hypothesis, the sets $E_n (n = 1, 2, \ldots)$ belong to **B**, and so, by condition 2), the set E belongs to **B**. Similarly, if E be a set Q^α, we may write $E = E_1 . E_2 . E_3 \ldots$, where E_n is a set P^{ξ_n} and $\xi_n < \alpha$; by hypothesis, the sets $E_n (n = 1, 2, \ldots)$ belong to **B**, and so, by condition 3), the set E belongs to **B**.

We have proved, therefore, by transfinite induction, that all sets P^α and Q^α belong to **B**, where $1 \leq \alpha < \Omega$. We have thus proved that

The family of all Borel sets is the smallest family **B** *which satisfies conditions* 1), 2), *and* 3).

Borel sets may, therefore, be defined without the aid of ordinal numbers (and transfinite induction) as sets belonging to the smallest (or, if we like, every) family **B** of sets satisfying conditions 1), 2), and 3).

It follows from properties 3 and 7 (§ 60) that the family **B** of all Borel sets satisfies also the following conditions:

4) *The complement of a set belonging to* **B**, *belongs to* **B**;

5) *The difference of two sets belonging to* **B**, *belongs to* **B**.

The definition of Borel sets and Theorem 68 lead at once to

Theorem 69. *A set, which is homeomorphic with a Borel set, is a Borel set.*[7]

[7]W. Sierpinski, *Comptes Rendus*, vol. CLXXI (1920), p. 24.

We shall also prove the following property of Borel sets:

If a family F of sets satisfies the following three conditions:

1] *Every open set belongs to F;*

2] *The sum of a countable aggregate of mutually exclusive sets belonging to F, belongs to F;*

3] *The product of a countable aggregate of sets belonging to F, belongs to F;*

then every Borel set belongs to F.

Proof. It follows from conditions 1] and 3] that every set G_δ, *i.e.* every set Q^2 belongs to F; hence, by properties 1 and 5 of sets P^α and Q^α (§ 60), the sets Q^1 and P^1 belong to F; from condition 2] and property 8 (§ 60), it follows that sets P^3 belong to F, and so by property 1, the sets P^2 belong to F.

Let now α denote a given ordinal number such that $3 \leq \alpha < \Omega$, and suppose that all sets P^ξ and Q^ξ belong to F, where $\xi < \alpha$. It will follow from condition 2] and property 8, that every set P^α belongs to F and from property 3] and the definition of sets Q^α, that every set Q^α belongs to F. We have thus proved by transfinite induction that all sets P^α and Q^α, $1 \leq \alpha < \Omega$, *i.e.* all Borel sets, belong to F.

We note further that as a result of the property proved above, we find immediately that *the family of all Borel sets is the smallest family F which satisfies conditions* 1], 2], *and* 3].

63. We shall consider now a generalization of Borel sets. To arrive at this generalization in a natural way, we consider sets $F_{\sigma\delta}$, *i.e.* products of a countable aggregate of sets F_σ. Let E be a given set $F_{\sigma\delta}$. Since every set F_σ is the sum of a countable aggregate of closed and bounded sets (§ 56), we may write the set E in the form

$$(10) \qquad E = \prod_{k=1}^{\infty} (E_1^k + E_2^k + E_3^k + \ldots) = \prod_{k=1}^{\infty} \sum_{n=1}^{\infty} E_n^k,$$

where E_n^k ($k = 1, 2, \ldots; n = 1, 2, \ldots$) are closed and bounded sets. (10) gives evidently

$$(11) \qquad E = \sum_{(n_1, n_2, \ldots)} E_{n_1}^1 . E_{n_2}^2 . E_{n_3}^3 \ldots,$$

where the summation ranges over all infinite sequences of positive integers $n_1, n_2, n_3, \ldots$

Put (for every finite combination of indices $n_1, n_2, \ldots, n_k$)

$$E^1_{n_1} \cdot E^2_{n_2} \ldots E^k_{n_k} = E_{n_1, n_2, \ldots, n_k}; \tag{12}$$

these sets will obviously be closed and bounded (or may be null). On account of (12), (11) may be written in the form

$$E = \sum_{(n_1, n_2, \ldots)} E_{n_1} \cdot E_{n_1, n_2} \cdot E_{n_1, n_2, n_3} \ldots \tag{13}$$

Every set E of the form (13), where $E_{n_1, n_2, \ldots, nk}$ is closed and bounded is called an *analytical* set, briefly a set (A), or the nucleus of the system $S[E_{n_1, \ldots, n_k}]$. The system $S[E_{n_1, \ldots, n_k}]$ is known if, corresponding to every finite combination of indices $n_1, \ldots, n_k$, an associated set $E_{n_1, \ldots, n_k}$ is known.

We have, therefore, proved in this article that every set $F_{\sigma\delta}$ is a set (A).

64. Suppose now that to every finite combination of indices $n_1, n_2, \ldots, n_k$, there is assigned a certain set $E_{n_1, n_2, \ldots, nk}$ whose elements may be any objects (not necessarily elements of some metric space). We then say that a system $S[E_{n_1, n_2, \ldots, n_k}]$ of sets is given, and the set

$$E = \sum_{(n_1, n_2, \ldots)} E_{n_1} \cdot E_{n_1, n_2} \cdot E_{n_1, n_2, n_3} \ldots,$$

where the summation ranges over all infinite sequences of the positive integers $n_1, n_2, n_3, \ldots$ is said to be the nucleus of the system S.

If all the sets $E_{n_1, n_2, \ldots, n_k}$ belong to a certain family F of sets, then the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$ is called *the result of the operation A* performed on the sets of the family F.

LUSIN'S SIEVE. Denote by W the set of all rational numbers between 0 and 1. If to every number w of the set W there be assigned a certain set E_w (of any elements) we then obtain a *sieve* $[E_w]$. A set *sifted through a sieve* $[E_w]$ is a set of all elements p for which there exists a certain (dependent on p) infinite decreasing sequence of numbers of the set W, *i.e.*

$$w_1 > w_2 > w_3 > \ldots,$$

such that

$$p \in E_{w_n}, \text{ for } n = 1, 2, \ldots$$

We shall show that a set sifted through a sieve $[E_w]$ may be considered to be the result of the operation A performed on the sets constituting the sieve.

Hence, denote by

(R) $$r_1, r_2, r_3, \ldots$$

the infinite sequence consisting of all (different) numbers of the set W.

Put $E_n = E_{r_n}$, for $n = 1, 2, \ldots$

Let now k be a given integer and suppose that we have already defined all sets $E_{n_1, n_2, \ldots, n_k}$, where $n_1, n_2, \ldots, n_k$ is any combinaton of k integers and where the sets $E_{n_1, n_2, \ldots, n_k}$ belong to the sieve $[E_w]$. Let $n_1, n_2, \ldots, n_k, n_{k+1}$ denote a combination of $k+1$ integers. Hence, by hypothesis $E_{n_1, n_2, \ldots, n_k} = E_{r_s}$, where s is a certain positive integer. Put $E_{n_1, n_2, \ldots, n_k, n_{k+1}} = E_{r_q}$, where r_q is the n_{k+1}th term of the sequence (R) satisfying the inequality $r_q < r_s$ (such a term exists on account of the property of the set W).

The sets $E_{n_1, n_2, \ldots, n_k}$ are thus defined by induction. We shall show that the set P sifted through the sieve $[E_w]$ is the nucleus Q of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

Suppose that $p \in P$. Hence, there exists an infinite sequence of indices $m_1, m_2, m_3, \ldots$ such that

(†) $$r_{m_1} > r_{m_2} > r_{m_3} > \ldots,$$

and

(††) $$p \in E_{r_{m_i}}, \text{ for } i = 1, 2, \ldots$$

Put $n_1 = m_1$; from the definition of the sets E_n (for n an integer) we get $E_{r_{m_1}} = E_{n_1}$. From $r_{m_2} < r_{m_1}$ and the definition of the sets E_{n_1, n_2}, we have $E_{r_{m_2}} = E_{n_1, n_2}$, for a certain integer n_2. Furthermore, since $r_{m_3} < r_{m_2}$, there exists an integer n_3 such that $E_{r_{m_3}} = E_{n_1, n_2, n_3}$. Continuing this argument, we obtain, on account of (††), an infinite sequence of integers $n_1, n_2, n_3, \ldots$ such that $p \in E_{n_1, n_2, \ldots, n_k}$, for $k = 1, 2, \ldots$, and so $p \in Q$.

On the other hand, suppose that $p \in Q$. There exists, therefore, an infinite sequence of integers $n_1, n_2, n_3, \ldots$, such that $p \in E_{n_1, n_2, \ldots, n_k}$, for $k = 1, 2, \ldots$ It follows easily from the definition of the sets $E_{n_1, n_2, \ldots, n_k}$ that there exists an infinite sequence (†) of numbers of the set W such that $E_{n_1, n_2, \ldots, n_k} = E_{r_{m_k}}$, for $k = 1, 2, \ldots$, and so, from (††) and the definition of the set P, it follows that $p \in P$. We have, therefore, $P = Q$, and this proves our theorem.

We note that conversely, it may be proved that the result of the operation A on sets of a family F may be considered as a set sifted through a sieve $[E_w]$, where E_w (for $w \in W$) is a set of the family F.[8] The investigation of Lusin's sieve is, therefore, equivalent to the investigation of the operation A.

Theorem 70. *If every one of the sets $E^{r_1, r_2, \ldots, r_s}$ is the result of the operation A performed on the sets of a family F, then the nucleus of*

[8]See *Fund. Math.*, vol. XI, p. 16.

the system $S[E^{r_1, r_2, \ldots, r_s}]$ *is also the result of the operation A performed on the sets of the family F.*

Proof. All different pairs (p, q) of integers may be arranged, as is well known, as an infinite sequence

$$(p_1, q_1), (p_2, q_2), (p_3, q_3), \ldots \tag{14}$$

Put

$$\phi(k) = p_k, \ \psi(k) = q_k, \text{ for } k = 1, 2, \ldots$$

Every pair (p, q) of integers occurs, and that once only, in the sequence (14); hence, to each pair corresponds a definite index k such that $p = p_k$ and $q = q_k$; put $k = \nu(p, q)$.

We have, therefore, as is easily seen,

$$\nu(\phi(k), \psi(k)) = k, \text{ for } k = 1, 2, \ldots, \tag{15}$$

and

$$\phi(\nu(p, q)) = p, \ \psi(\nu(p, q)) = q, \text{ for } p = 1, 2, \ldots; \ q = 1, 2, \ldots \tag{16}$$

By hypothesis, every set $E^{r_1, r_2, \ldots, r_s}$ is the result of the operation A performed on the sets of the family F. Hence, corresponding to every combination $r_1, r_2, \ldots, r_s$ of indices, there exists a system $S^{r_1, r_2, \ldots, r_s}[E^{r_1, r_2, \ldots, r_s}_{n_1, n_2, \ldots, n_k}]$ of sets of the family F, whose nucleus is the set $E^{r_1, r_2, \ldots, r_s}$. For every finite combination of indices $n_1, n_2, \ldots, n_k$ put

$$E_{n_1, n_2, \ldots, n_k} = E^{\phi(n_1), \phi(n_2), \ldots, \phi(n_{\psi(k)})}_{\psi(n_{\nu(1, \psi(k))}), \ \psi(n_{\nu(2, \psi(k))}), \ldots, \ \psi(n_{\nu(\phi(k), \psi(k))})}; \tag{17}$$

the sets so defined will, therefore, belong to the family F.

It will be shown that the nucleus of the system $S[E^{r_1, r_2, \ldots, r_s}]$ is the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

In fact, let x be a given element of the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$. Hence, there exists an infinite sequence of indices $n_1, n_2, n_3, \ldots$ such that

$$x \in E_{n_1, n_2, \ldots, n_k}, \text{ for } k = 1, 2, \ldots \tag{18}$$

Put

$$r_s = \phi(n_s), \text{ for } s = 1, 2, \ldots, \tag{19}$$

and let s be a given integer. Put

$$j_h = \psi(n_{\nu(h, s)}), \text{ for } h = 1, 2, \ldots \tag{20}$$

From (17), (19), and (20), we obtain

$$E_{n_1, n_2, \ldots, n_{\nu(h,s)}} = E^{r_1, r_2, \ldots, r_s}_{j_1, j_2, \ldots, j_h}, \text{ for } h = 1, 2, \ldots$$

(since from (16) $\psi(\nu(h, s)) = s$, and $\phi(\nu(h, s)) = h$) and so, on account of (18),

$$x \in E^{r_1, r_2, \ldots, r_s}_{j_1, j_2, \ldots, j_h}, \text{ for } h = 1, 2, \ldots,$$

i.e. x belongs to the nucleus $E^{r_1, r_2, \ldots, r_s}$ of the system $S^{r_1, r_2, \ldots, r_s}$. We have, therefore,

(21) $$x \in E^{r_1, r_2, \ldots, r_s}, \text{ for } s = 1, 2, \ldots,$$

from which it follows that x belongs to the nucleus of the system $S[E^{r_1, r_2, \ldots, r_s}]$.

Suppose now that x belongs to the nucleus of the system $S[E^{r_1, r_2, \ldots, r_s}]$.

Hence, there exists an infinite sequence of integers $r_1, r_2, r_3, \ldots$ such that (21) is true. But $E^{r_1, r_2, \ldots, r_s}$ is the nucleus of the system $S[E^{r_1, r_2, \ldots, r_s}_{n_1, n_2, \ldots, n_k}]$; corresponding to every integer s there exists, therefore, an infinite set of indices $m_1^{(s)}, m_2^{(s)}, m_3^{(s)}, \ldots$ such that

(22) $$x \in E^{r_1, r_2, \ldots, r_s}_{m_1^{(s)}, m_2^{(s)}, \ldots, m_k^{(s)}}, \text{ for } k = 1, 2, \ldots; s = 1, 2, \ldots$$

Put

(23) $$n_h = \nu(r_h, m^{(\psi(h))}_{\phi(h)}), \text{ for } h = 1, 2, \ldots$$

From (23) and (16) we obtain

$$\phi(n_h) = r_h, \ \psi(n_h) = m^{(\psi(h))}_{\phi(h)}, \text{ for } h = 1, 2, \ldots,$$

and so for $h = \nu(i, \psi(k))$ we have

$$\psi(n_{\nu(i, \psi(k))}) = m_i^{(\psi(k))}, \text{ for } i = 1, 2, \ldots; k = 1, 2, \ldots,$$

since, on account of (16), $\phi(\nu(i, \psi(k))) = i$, and $\psi(\nu(i, \psi(k))) = \psi(k)$; hence, it follows from (17) and (23) that

$$E_{n_1, n_2, \ldots, n_k} = E^{r_1, r_2, \ldots, r_{\psi(k)}}_{m_1^{(\psi(k))}, m_2^{(\psi(k))}, \ldots, m^{(\psi(k))}_{\phi(k)}}, \text{ for } k = 1, 2, \ldots,$$

and so from (22)

$$x \in E_{n_1, n_2, \ldots, n_k}, \text{ for } k = 1, 2, \ldots;$$

this proves that x belongs to the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

We have thus proved that the systems

$$S[E^{r_1, r_2, \ldots, r_s}] \text{ and } S[E_{n_1, n_2, \ldots, n_k}]$$

have the same nucleus.

But the sets $E_{n_1, n_2, \ldots, n_k}$ belong to the family F by (17); Theorem 70 is, therefore, established.

COROLLARY. *The sum and product respectively of a countable aggregate of sets which belong to a family F are results of the operation A performed on the sets of F.*

Proof. If $E = T_1 + T_2 + T_3 + \ldots$, where $T_n \epsilon F$, for $n = 1, 2, \ldots$, then for every finite combination of indices $n_1, n_2, \ldots, n_k$, put

$$E_{n_1, n_2, \ldots, n_k} = T_{n_1}.$$

It is easily seen that E is the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

If $E = T_1.T_2.T_3 \ldots$, then put

$$E_{n_1, n_2, \ldots, n_k} = T_k,$$

for every finite combination of indices $n_1, n_2, \ldots, n_k$.

It follows again that E is the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

Let F be a given family of sets and let $A(F)$ denote the family of all sets which are results of the operation A performed on the sets of F. Theorem 70 may clearly be expressed by the relation

(24) $$A(A(F)) = A(F)$$

(for every family F of sets).

If now we denote by $S(F)$ and $P(F)$ the families of all sets which are sums and products respectively of a countable aggregate of sets which belong to F, then the corollary to Theorem 70 can be expressed, as is easily seen, in the form

(25) $$S(F) \subset A(F) \text{ and } P(F) \subset A(F).$$

Relations (24) and (25) hold for every family F of sets.

From these relations we get immediately

(26) $$F \subset A(F)$$

for every family F of sets.

We shall next prove the following property of the operation $A(F)$.

If a family F is a topological invariant and if the product of every set belonging to F and a G_δ belongs to F, then the family $A(F)$ is a topological invariant.

Let E denote a set belonging to the family $A(F)$.

Hence

(26a) $$E = \sum E_{n_1} . E_{n_1, n_2} . E_{n_1, n_2, n_3} \dots ,$$

where the sets $E_{n_1, n_2, \dots, n_k}$ belong to F and the summation ranges over all infinite sequences of the indices $n_1, n_2, n_3, \dots$

Further, let T be a set such that $E h T$. By the theorem of Lavrentieff, there exist two sets P and Q, each of which is a set G_δ, and a function f defined in P such that $E \subset P$, $T \subset Q$, $P h_f Q$, and $f(E) = T$.

Put $P.E_{n_1, n_2, \dots, n_k} = Y_{n_1, n_2, \dots, n_k}$; these sets will belong to F (since they are products of a G_δ and sets which belong to F), and from (26a) and $E \subset P$, we get

$$E = \sum Y_{n_1} Y_{n_1, n_2} Y_{n_1, n_2, n_3} \dots$$

and so, since f is biuniform in $E \subset P$,

$$T = f(E) = \sum f(Y_{n_1}) . f(Y_{n_1, n_2}) . f(Y_{n_1, n_2, n_3}) \dots ;$$

this proves that T belongs to the family $A(F)$, since the sets $f(Y_{n_1, n_2, \dots, n_k})$ belong to F (F being a topological invariant). The theorem is, therefore, proved.

65. Let now C denote the family of all closed and bounded sets (in the metric space under consideration); it follows from the definitions of analytical sets (§ 63) and the operation $A(F)$ (§ 64), that $A(C)$ is the family of all analytical sets. From (24) we obtain

$$A(A(C)) = A(C);$$

hence, we have

Theorem 71. *The result of the operation A performed on analytical sets is an analytical set.*

From (25) and (24) we get

$$S(A(C)) \subset A(A(C)) = A(C), \text{ and } P(A(C)) \subset A(C),$$

and so *the sum and the product of a countable aggregate of analytical sets are themselves analytical sets.*

Finally, from (26)

$$C \subset A(C),$$

i.e. every closed and bounded set is an analytical set.

The family **A** of all analytical sets is, therefore, one of the families **B** which satisfy the conditions 1), 2), 3) of § 62, and since every one of the families **B** contains the family of Borel sets (§ 62), we have

Theorem 71a. *Every Borel set is an analytical set.*

66. A system $S[E_{n_1, n_2, \ldots, n_k}]$, where $E_{n_1, n_2, \ldots, n_k}$ is a closed set satisfying the conditions:

$$\delta(E_{n_1, n_2, \ldots, n_k}) < \frac{1}{k}, \tag{27}$$

$$E_{n_1, n_2, \ldots, n_{k+1}} \subset E_{n_1, n_2, \ldots, n_k}, \tag{28}$$

$$E_{n_1, n_2, \ldots, n_k} \neq 0 \tag{29}$$

for every finite combination of the indices $n_1, n_2, \ldots, n_{k+1}$, will be called *regular.* We shall prove

Theorem 72. *Every analytical set, which is not null, is the nucleus of a certain regular system.*

Let E be a given analytical set, not null. It follows from the definition of analytical sets (§ 63) that E is the nucleus of a certain system $S[F_{n_1, n_2, \ldots, n_k}]$, where $F_{n_1, n_2, \ldots, n_k}$ are closed and bounded sets.

It was shown in § 47 that every compact set can be divided into a finite number of sets of arbitrarily small diameters. Hence, if Φ be a closed and bounded set and so by condition (W) (§ 55) also compact, then corresponding to every $\epsilon > 0$, we can write $\Phi = \Phi_1 + \Phi_2 + \ldots + \Phi_m$, where $\delta(\Phi_i) < \epsilon$, for $i = 1, 2, \ldots, m$; but Φ is closed and, therefore, $\Phi = \overline{\Phi}_1 + \overline{\Phi}_2 + \ldots + \overline{\Phi}_m$, where $\delta(\overline{\Phi}_i) < \epsilon$, since $\delta(\Phi) < \epsilon$(§ 47). Hence, every closed and bounded set is the sum of a finite number of closed sets of arbitrarily small diameters and so certainly the sum of an infinite series of such sets (since the missing terms may be replaced by null sets). We may, therefore, write (for every combination of indices $n_1, n_2, \ldots, n_k$)

(30) $$F_{n_1, n_2, \ldots, n_k} = F^{(1)}_{n_1, n_2, \ldots, n_k} + F^{(2)}_{n_1, n_2, \ldots, n_k} + \ldots,$$

where $F^{(i)}_{n_1, n_2, \ldots, n_k}$ are closed sets, and where

(31) $$\delta(F^{(i)}_{n_1, n_2, \ldots, n_k}) < \frac{1}{2k+2},$$

for all integers $n_1, n_2, \ldots, n_k$ and i.

The metric space M under consideration is, as we know (§ 55) the sum of a countable aggregate of bounded sets. If we consider their enclosures, it follows at once that M is the sum of a countable aggregate of closed and bounded sets and so (from the property of the latter), M is the sum of a countable aggregate of closed sets of arbitrarily small diameters. We may, therefore, write

(32) $$M = E_1 + E_2 + E_3 + \ldots$$

where $E_n (n = 1, 2, \ldots)$ are closed sets, and where

(33) $$\delta(E_n) < \tfrac{1}{2}, \text{ for } n = 1, 2, \ldots$$

Put for all positive integers n_1 and n_2

(34) $$E_{n_1, n_2} = E_{n_1}$$

and for $k > 1$ and every combination of positive integers $n_1, n_2, \ldots, n_{2k}$

(35) $$E_{n_1, n_2, \ldots, n_{2k-1}} = E_{n_1, n_2, \ldots, n_{2k}} = F^{(n_{2k-1})}_{n_2, n_4, \ldots, n_{2k-2}}.$$

From (33), (34), (35), and (31) it follows that condition (27) will be satisfied (for every finite combination of indices $n_1, n_2, \ldots, n_k$).

It will be shown that the set E is the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

In fact, suppose that $x \in E$. Since E is the nucleus of the system $S[F_{n_1, n_2, \ldots, n_k}]$, there exists for every element x a certain infinite sequence of indices $m_1, m_2, m_3, \ldots$ such that

(36) $$x \in F_{m_1, m_2, \ldots, m_k}, \text{ for } k = 1, 2, \ldots$$

It follows from (30) and (36) that there exists for every integer k an index i_k such that

(37) $$x \in F^{(i_k)}_{m_1, m_2, \ldots, m_k}, \text{ for } k = 1, 2, \ldots$$

Finally, from (32) (and the fact that x is an element of the space M), we conclude the existence of an index i_0 such that

(38) $$x \epsilon E_{i_0}.$$

Denote by $n_1, n_2, n_3, \ldots$ the successive terms of the infinite sequence

$$i_0, m_1, i_1, m_2, i_2, m_3, i_3, \ldots;$$

we shall have from (38), (33), (34), (37), and (35), as is easily seen,

(39) $$x \epsilon E_{n_1, n_2, \ldots, n_k}, \text{ for } k=1, 2, \ldots,$$

and so x belongs to the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$. Suppose, on the other hand, that x is an element of the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$. Hence, there exists an infinite sequence of indices $n_1, n_2, n_3, \ldots$, for which (39) holds. We have, therefore, from (39), (35), and (30)

$$x \epsilon F_{n_2, n_4, \ldots, n_{2k-2}}, \text{ for } k=2, 3, \ldots,$$

from which it follows that x belongs to the nucleus of the system $S[F_{n_1, n_2, \ldots, n_k}]$ and so to the set E.

We have thus proved that the set E is the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$. We have, therefore, shown thus far that every analytical set E is the nucleus of a certain system $S[E_{n_1, n_2, \ldots, n_k}]$, where $E_{n_1, n_2, \ldots, n_k}$ are closed sets which satisfy condition (27).

If now we put for every finite combination of indices $n_1, n_2, \ldots, n_k$

$$X_{n_1, n_2, \ldots, n_k} = E_{n_1} . E_{n_1, n_2} \ldots E_{n_1, n_2, \ldots, n_k},$$

then the sets $X_{n_1, n_2, \ldots, n_k}$ will be closed,

(40) $$\delta(X_{n_1, n_2, \ldots, n_k}) < \frac{1}{k}$$

(from (27)), and

(41) $$X_{n_1, n_2, \ldots, n_k, n_{k+1}} \subset X_{n_1, n_2, \ldots, n_k};$$

the set E will then obviously be the nucleus of the system $S[X_{n_1, n_2, \ldots, n_k}]$.

If now E is not a null set, there exists an element x_0 of E. Corresponding to a given finite combination of indices $r_1, r_2, \ldots, r_s$,

put

$$(42)\quad X^{r_1, r_2, \ldots, r_s} = \sum_{(n_1, n_2, \ldots)} X_{r_1, r_2, \ldots, r_s, n_1} \cdot X_{r_1, r_2, \ldots, r_s, n_1, n_2} \cdot X_{r_1, r_2, \ldots, r_s, n_1, n_2, n_3} \cdots$$

where the summation ranges over all infinite sequences of integers $n_1, n_2, n_3, \ldots$

If for a given set of indices $r_1, r_2, \ldots, r_s$, the set (42) is not null, then denote one of its elements by $x_{r_1, r_2, \ldots, r_s}$; on account of (41) and (42) this will be an element of E.

We shall define now for every finite combination of indices $r_1, r_2, \ldots, r_s$ sets $Y_{r_1, r_2, \ldots, r_s}$, as follows:

If $X^{r_1, r_2, \ldots, r_s} \neq 0$ then $Y_{r_1, r_2, \ldots, r_s} = X^{r_1, r_2, \ldots, r_s}$.

If $X^{r_1, r_2, \ldots, r_s} = 0$ and $X^{r_1} = 0$, then $Y_{r_1, r_2, \ldots, r_s} = (x_0)$ (where (x_0) is the set consisting of the single element x_0).

If $X^{r_1, r_2, \ldots, r_s} = 0$ and $X^{r_1} \neq 0$, and if $p+1$ is the smallest index such that $X^{r_1, r_2, \ldots, r_p, r_{p+1}} = 0$ (hence $0 < p < s$, and $X^{r_1, r_2, \ldots, r_p} \neq 0$), then put $Y_{r_1, r_2, \ldots, r_s} = (x_{r_1, r_2, \ldots, r_s})$.

It follows from (42), (41), and (40) that the sets $Y_{r_1, r_2, \ldots, r_s}$ all satisfy the conditions

$$\delta(Y_{n_1, n_2, \ldots, n_k}) < \frac{1}{k},$$

$$Y_{n_1, n_2, \ldots, n_k, n_{k+1}} \subset Y_{n_1, n_2, \ldots, n_k},$$

and

$$Y_{n_1, n_2, \ldots, n_k} \neq 0$$

for every finite combination $n_1, n_2, \ldots, n_k, n_{k+1}$. Furthermore, it follows easily from the definition of the sets $Y_{r_1, r_2, \ldots, r_s}$, from (42), (41), and from the fact that E is the nucleus of the system $S[X_{n_1, n_2, \ldots, n_k}]$, that E is the nucleus of the system $S[Y_{n_1, n_2, \ldots, n_k}]$.

Theorem 72 may, therefore, be considered as proved.

67. **Theorem 73.** *In order that a non-null set E be an analytical set, it is necessary and sufficient that E be the set of values of a certain function f(x) of a real variable, defined and continuous in the set of all irrational numbers.*

Proof. Let E be a given analytical set not null. Hence, by Theorem 72, E is the nucleus of a certain regular system

$S[E_{n_1, n_2, \ldots, n_k}]$, where $E_{n_1, n_2, \ldots, n_k}$ are closed sets, which satisfy conditions (27), (28), and (29).

Let x be a given irrational number, $[x]$ the greatest integer $\leq x$, and

(43) $$x = [x] + \frac{1}{n_1 +} \frac{1}{n_2 +} \frac{1}{n_3 +} \cdots,$$

the development of x as an (infinite) continued fraction. Put

(44) $$F(x) = E_{n_1} \cdot E_{n_1, n_2} \cdot E_{n_1, n_2, n_3} \cdots$$

It follows from (27), (28), and (29) that the set (44) is a product of an infinite decreasing sequence of closed and bounded (and, therefore, by condition (W), § 55, also compact) sets which are not null. Hence, by Theorem 27, the set (44) is not a null set. On the other hand, (44) gives $F(x) \subset E_{n_1, n_2, \ldots, n_k}$, for $k = 1, 2, \ldots\ldots$, and, therefore, from (27) $\delta(F(x)) < \frac{1}{k}$, for $k = 1, 2, \ldots$, and so $\delta(F(x)) = 0$; this proves that $F(x)$ consists of one element only (since $F(x) \neq 0$). Denote this element by $f(x)$. The function $f(x)$ will thus be defined for every irrational x and, as follows immediately from the definition of the function $f(x)$ (and the set E), its values will be elements of the set E. On the other hand, it is easily seen that every element of E is one of the values of the function $f(x)$ for an irrational x. For if p be an element of E, there exists an infinite sequence of indices $n_1, n_2, n_3, \ldots$ such that $p \in E_{n_1, n_2, \ldots, n_k}$, for $k = 1, 2, \ldots\ldots$, and so, if x be a real number defined by (43), we conclude (on account of (44)) that $p \in F(x)$; but from the definition of $f(x)$, since $F(x)$ consists of one element only, it follows that $p = f(x)$.

We have, therefore, proved that E is the set of all values of a function $f(x)$ for an irrational x.

We shall now show that $f(x)$ is continuous in the set of the irrational numbers.

Let

(45) $$x_0 = [x_0] + \frac{1}{n_1^0 +} \frac{1}{n_2^0 +} \frac{1}{n_3^0 +} \cdots.$$

denote a given irrational number, and ϵ a given positive number. Let k be an integer such that

(46) $$\frac{1}{k} < \epsilon.$$

It follows, as is well known, from the properties of continued fractions that, corresponding to the numbers x_0 and k, there exists a positive number η such that every irrational number x which satisfies the inequality

(47) $$|x - x_0| < \eta$$

may be expressed as the continued fraction (43) such that

$$n_i = n_i^0, \text{ for } i = 1, 2, \ldots, k,$$

and, therefore,

(48) $$E_{n_1, n_2, \ldots, n_k} = E_{n_1^0, n_2^0, \ldots, n_k^0}.$$

But, from the definition of $f(x)$ and from (43) and (45), we have

$$f(x) \in E_{n_1, n_2, \ldots, n_k}, \ f(x_0) \in E_{n_1^0, n_2^0, \ldots, n_k^0};$$

hence, from (48), (27), and (46), we obtain

(49) $$\rho(f(x), f(x_0)) < \epsilon.$$

We have thus shown that, corresponding to every irrational number x_0 and every positive number ϵ, there exists a positive number η such that the inequality (47) implies the inequality (49); this proves that the function $f(x)$ is continuous in the set of all irrational numbers.

The condition of Theorem 73 is, therefore, proved to be necessary.

Let now $f(x)$ be a function of a real variable defined and continuous in the set of all irrational numbers with its values the elements of a metric space satisfying condition (W) (§ 55). We shall show that the set of all values of $f(x)$ for x irrational (a set which is obviously not a null set) is an analytical set.

Since, as was shown in § 65, the sum of a countable aggregate of analytical sets is an analytical set, it will be sufficient to show that the set E of all values of the function $f(x)$ is an analytical set for all irrational values of x in the interval $(k, k+1)$, or in the interval $(0, 1)$ say.

Let $n_1, n_2, \ldots, n_k$ denote any finite sequence of positive integers. Denote by $X_{n_1, n_2, \ldots, n_k}$ the set of all irrational numbers x in the interval $(0, 1)$ whose developments as continued fractions have

$$\frac{1}{n_1+}\,\frac{1}{n_2+}\cdots\frac{1}{n_{k-1}+}\,\frac{1}{n_k}$$

for their k^{th} convergent, and put

$$E_{n_1, n_2, \ldots, n_k}=\overline{f(X_{n_1, n_2, \ldots, n_k})}; \tag{50}$$

these will be closed sets (not necessarily bounded). It follows that E is the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

In fact, let $p \in E$; then by the definition of the set E there exists an irrational x in the interval $(0, 1)$ such that $f(x)=p$; let

$$x=\frac{1}{n_1+}\,\frac{1}{n_2+}\,\frac{1}{n_3+}\cdots$$

be the development of x as a continued fraction. It follows, from the definition of the sets $X_{n_1, n_2, \ldots, n_k}$, that

$$x \in X_{n_1, n_2, \ldots, n_k}, \text{ for } k=1, 2, \ldots$$

and so, from (50), certainly

$$f(x) \in E_{n_1, n_2, \ldots, n_k}, \text{ for } k=1, 2, \ldots;$$

hence, the element $p=f(x)$ belongs to the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

On the other hand, let p denote an element of the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$.

There exists, therefore, an infinite sequence of indices n_1^0, n_2^0 $n_3^0, \ldots$ such that

$$p \in E_{n_1^0, n_2^0, \ldots, n_k^0}, \text{ for } k=1, 2, \ldots \tag{51}$$

Put

$$x_0=\frac{1}{n_1^0+}\,\frac{1}{n_2^0+}\,\frac{1}{n_3^0+}\cdots; \tag{52}$$

this will be an irrational number of the interval (0, 1). Let ϵ denote a given positive number. Since the function $f(x)$ is continuous in the set of irrational numbers, there exists a positive number η, depending on x_0 and ϵ, such that the inequality

$$|x-x_0|<\eta \tag{53}$$

implies the inequality

$$\rho(f(x), f(x_0)) < \epsilon, \tag{54}$$

for an irrational x of the interval (0, 1).

Moreover, it follows from the properties of continued fractions that, corresponding to every x_0 and η, there exists an integer k such that every irrational x whose k^{th} convergent is the same as the k^{th} convergent of (52), *i.e.* every number of the set $X_{n_1^0, n_2^0, \dots, n_k^0}$, satisfies the inequality (53) and, therefore. also the inequality (54). Hence

$$\delta(f(X_{n_1^0, n_2^0, \dots, n_k^0})) \leq \epsilon,$$

and, therefore, from (50) also

$$\delta(E_{n_1^0, n_2^0, \dots, n_k^0}) \leq \epsilon. \tag{55}$$

But from (52) and the definition of the sets (50), we have evidently

$$f(x_0) \epsilon E_{n_1^0, n_2^0, \dots, n_k^0},$$

and so, from (51) and (55),

$$\rho(p, f(x_0)) < \epsilon,$$

and, since ϵ is an arbitrary number, this gives $p=f(x_0)$; hence, $p \epsilon E$. We have, therefore, proved that E is the nucleus of the system $S[E_{n_1, n_2, \dots, n_k}]$, where $E_{n_1, n_2, \dots, n_k}$ are closed sets. Hence, E is a result of the operation A performed on closed sets and so, by Theorems 72 and 71, is an analytical set.

The condition of Theorem 73 is thus seen to be sufficient.

68. Theorem 74. *A continuous transform of an analytical set is an analytical set.*

Proof. Let E be an analytical set, and T its continuous transform. Hence, there exists a function $f(p)$ defined and continuous in E such that $T=f(E)$. But, by Theorem 73, there exists a function

$\phi(x)$, defined and continuous in the set of all irrational numbers N such that $E=\phi(N)$. For an irrational x put $\psi(x)=f(\phi(x))$; by Theorem 19, $\psi(x)$ is continuous in N, and obviously $T=\psi(N)$. Hence, by Theorem 73, T is an analytical set.

Theorems 72 and 74 lead immediately to

COROLLARY 1. *A continuous transform of a Borel set is an analytical set.*

From Theorem 74 we get further

COROLLARY 2. *The property of being analytical of a set is a topological invariant.*

Let now **A** denote the family of all analytical sets (contained in the metric space under consideration). The sets F_σ and G_δ, being Borel sets, belong to the family **A** (by Theorem 71a). Furthermore, since the sum and product respectively of two analytical sets are themselves analytical sets (§ 65), and on account of Corollary 2, the family **A** satisfies the conditions of Theorem 66. It follows, therefore, from Theorem 66, that the family of all complements of the sets which belong to **A** is a topological invariant. We have thus

Theorem 75.[9] *A set which is homeomorphic with the complement of an analytical set is itself the complement of an analytical set.*

We note, however, that a continuous transform (even when biuniform)[10] of the complement of an analytical set may not itself be the complement of an analytical set.

69. Theorem 76. *If $f(x)$ be a function of a real variable x, the values of which are elements of a metric space, then the set of all values of x for which the function is continuous on one side only, is countable at most.*[11]

Proof. Let x_0 denote a real number for which $f(x)$ is continuous only on the left. Since the function $f(x)$ is not continuous on the right for the value x_0, then it is not true that, corresponding to each positive ϵ, there exists a positive number η such that

$$\rho(f(x), f(x_0)) < \epsilon, \text{ for } x_0 < x < x_0 + \eta.$$

[9] P. Alexandroff, *Fund. Math.*, vol. V, p. 164; M. Lavrentieff, *Fund. Math.*, vol. VI, p. 154.

[10] Mazurkiewicz, *Fund. Math.*, vol. X, p. 172.

[11] Sierpinski, *Functions representable analytically* (in Polish), Lwow, 1925, p. 13 (Th. 8).

It follows, then, that there exists a positive rational number u such that the inequality

(56) $$\rho(f(x), f(x_0)) < 2u, \text{ for } x_0 < x < x_0 + \eta$$

is not satisfied for any positive η.

On the other hand, since $f(x)$ is continuous on the left at x_0, there exists a rational number $v < x_0$ such that

(57) $$\rho(f(x), f(x_0)) < u, \text{ for } v < x < x_0.$$

The numbers u and v may be taken to be the first terms of a certain infinite sequence of all rational numbers such that u does not satisfy (56) for any positive η, and v (after u has been selected) satisfies (57). In this manner, to every element of the set E of all real numbers for which the function $f(x)$ is continuous only on the left, there will be assigned a certain pair of rational numbers (u, v). It will be shown that to different elements of E there will always be assigned different pairs.

To prove this, suppose that the same pair is assigned to the number $x_1 > x_0$ and x_0. We then have (57) and

(58) $$\rho(f(x), f(x_1)) < u, \text{ for } v < x < x_1,$$

where $v < x_1$. Since $v < x_0 < x_1$, we may assume $x = x_0$ in (58) which gives $\rho(f(x_0), f(x_1)) < u$, and so, from (58),

$$\rho(f(x), f(x_0)) \leq \rho(f(x), f(x_1)) + \rho(f(x_1), f(x_0)) < 2u,$$

for $v < x < x_1$, and certainly for $x_0 < x < x_1$. If we put $\eta = x_1 - x_0$, we would have a positive η and (56) satisfied, contrary to the definition of the number u. Hence to different elements of E correspond different pairs (u, v) of rational numbers, and (since the set of all pairs of rational numbers is countable) the set E is countable at most. Similarly, it could be proved that the set of all values of x at which a function f is continuous on the right only, is countable at most. Theorem 76 may, therefore, be considered as proved.

Theorem 76 will obviously remain true if the function $f(x)$ be defined in a subset of the set of all real numbers.

Theorem 77. *In order that a non-null set E be an analytical set, it is necessary and sufficient that E be the set of values of a function of* *a real variable, which is continuous on the left in the whole set of real numbers.*[12]

Proof. Let E denote a given analytical set not null. Hence, by Theorem 72, E is the nucleus of a certain regular system $S[E_{n_1, n_2, \ldots, n_k}]$, where $E_{n_1, n_2, \ldots, n_k}$ are closed sets satisfying conditions (27), (28), and (29).

Let x be a given real number. Corresponding to every real number x there exists, as is well known, a definite infinite sequence $n_1, n_2, n_3, \ldots$ of positive integers such that

$$(59) \qquad x = [x] + \frac{1}{2^{n_1}} + \frac{1}{2^{n_1+n_2}} + \frac{1}{2^{n_1+n_2+n_3}} + \cdots.$$

Put

$$F(x) = E_{n_1} \cdot E_{n_1, n_2} \cdot E_{n_1, n_2, n_3} \cdot \cdots.$$

As in Theorem 73, we show that $F(x)$ consists of one element only, and if this element be denoted by $f(x)$, then E is the set of all values of the function $f(x)$ for x real.

We shall now prove that the function $f(x)$ is continuous on the left in the set of all real numbers.

Let

$$(60) \qquad x_0 = [x_0] + \frac{1}{2^{n_1^0}} + \frac{1}{2^{n_1^0+n_2^0}} + \frac{1}{2^{n_1^0+n_2^0+n_3^0}} + \cdots$$

denote a given real number, and ϵ an arbitrary positive number. Denote by k a positive integer such that

$$(61) \qquad \frac{1}{k} < \epsilon,$$

and put

$$(62) \qquad x_1 = [x_0] + \frac{1}{2^{n_1^0}} + \frac{1}{2^{n_1^0+n_2^0}} + \cdots + \frac{1}{2^{n_1^0+n_2^0+\cdots+n_k^0}};$$

obviously, on account of (60), $x_1 < x_0$.

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[12] *Cf.* N. Lusin, *Fund. Math.*, vol. X, pp. 12-15; W. Sierpinski, *Fund. Math.*, vol. X, pp. 169-171.

Let now x be a real number such that

(63) $$x_1 < x < x_0,$$

and let (59) be the development of x (as a binary fraction, actually infinite).

It follows easily from (59), (60), (62), and (63) that

$$n_i = n_i^0, \text{ for } i = 1, 2, \ldots, k,$$

whence

(64) $$E_{n_1, n_2, \ldots, n_k} = E_{n_1^0, n_2^0, \ldots, n_k^0}.$$

But from the definition of the function $f(x)$ and from (59) and (60), we have

$$f(x) \in E_{n_1, n_2, \ldots, n_k}, \; f(x_0) \in E_{n_1^0, n_2^0, \ldots, n_k^0},$$

and so, from (64), (27), and (61),

(65) $$\rho(f(x), f(x_0)) < \epsilon.$$

We have thus proved that, corresponding to every real number x_0 and every positive number ϵ, there exists a number $x_1 < x_0$ such that the inequality (63) implies the inequality (65), and this proves that the function $f(x)$ is continuous on the left in the set of all real numbers.

The condition of Theorem 77 is, therefore, necessary.

Let now $f(x)$ denote a function of a real variable, continuous on the left for every real value of x, and such that its values are elements of a metric space satisfying condition (W) (§ 55). Denote by E the set of all values of the function $f(x)$ for x real; we shall show that E is an analytical set.

Let M be the set of all values of x for which f is continuous (on both sides), and N the set of all remaining values of x, hence of those, for which f is continuous on the left only. The set N is countable at most by Theorem 76; it is, therefore, in any case, a set F_σ, and so M is a set G_δ and, therefore, a Borel set. The function f is obviously continuous in M; hence, $f(M)$ is a continuous transform of a Borel set, and so, by Corollary 1 to Theorem 74, an analytical set. The set $f(N)$ is countable at most (since N is countable at most); it is, therefore, in any case, a set F_σ and so an analytical set. The set $E=f(M+N)=f(M)+f(N)$ is, therefore, the sum of two analytical sets and hence an analytical set (§ 65).

The condition of Theorem 77 is, therefore, sufficient. This proves Theorem 77.

70. LEMMA. *If E be a given set, and K an open sphere such that the set $E.K$ is not countable, then there exist open spheres K_0 and K_1 of arbitrarily small radii and such that $K_0 \subset K$, $K_1 \subset K$, $\overline{K}_0.\overline{K}_1=0$, and the sets $E.K_0$, $E.K_1$ are not countable.*

Proof. The set $E.K$, not being countable, contains, by the corollary to Theorem 29 (§ 22), a non-countable aggregate of elements of condensation; let p_0 and p_1 be two of them. Since $p_0 \in K$ and $p_1 \in K$, and since K is open, we shall have for r_0 and r_1 sufficiently small (§ 44) $K_0=K(p_0, r_0) \subset K$, and $K_1=K(p_1, r_1) \subset K$, where it may be supposed that $r_0+r_1<\rho(p_0, p_1)$, which results in $\overline{K}_0.\overline{K}_1=0$. Finally, since K_0 and K_1 are open sets, and $p_0 \in K_0$, $p_1 \in K_1$, we conclude from the fact that p_0 and p_1 are elements of condensation of the set $E.K$, that the sets $E.K_0$ and $E.K_1$ are non-countable. The lemma is, therefore, proved.

Let now E be a given non-countable, analytical set, *i.e.* the nucleus of a regular system $S[E_{n_1, n_2, \ldots, n_k}]$.

For every finite combination of indices $r_1, r_2, \ldots, r_s$, put

$$(66)\quad E^{r_1, r_2, \ldots, r_s}=\sum E_{r_1} \,.\, E_{r_1, r_2} \ldots E_{r_1, r_2, \ldots, r_s} \,.\, E_{r_1, r_2, \ldots, r_s, n_1} \,.\, E_{r_1, r_2, \ldots, r_s, n_1, n_2} \ldots,$$

where the summation ranges over all infinite sequences of positive integers $n_1, n_2, n_3, \ldots$

It follows at once from (66) that

$$(67)\qquad E=E^1+E^2+E^3+\ldots,$$

and for every finite combination of indices $r_1, r_2, \ldots, r_s$ we get

$$(68)\quad E^{r_1, r_2, \ldots, r_s}=E^{r_1, r_2, \ldots, r_s, 1}+E^{r_1, r_2, \ldots, r_s, 2}+E^{r_1, r_2, \ldots, r_s, 3}+\ldots$$

With every finite combination $\alpha_1, \alpha_2, \ldots, \alpha_k$ consisting of the numbers 0 and 1, let there be correlated a sphere $K_{\alpha_1, \alpha_2, \ldots, \alpha_k}$ and an integer $m_{\alpha_1, \alpha_2, \ldots, \alpha_k}$ in such a manner that the following conditions are satisfied:

(69) $$\delta(K_{a_1, a_2, \ldots, a_k}) < \frac{1}{k},$$

(70) $$K_{a_1, a_2, \ldots, a_k} \subset K_{a_1, a_2, \ldots, a_{k-1}},^{13}$$

(71) $$\overline{K}_{a_1, a_2, \ldots, a_{k-1}, 0} \cdot \overline{K}_{a_1, a_2, \ldots, a_{k-1}, 1} = 0,$$

(72) the set $E^{m_{a_1}, m_{a_1, a_2}, \ldots, m_{a_1, a_2, \ldots, a_k}} . K_{a_1, a_2, \ldots, a_k}$ is non-countable.

We shall show that such a correlation is possible.

Since E is non-countable, it contains an element of condensation p. Put $K = K(p, 1)$; the set $E.K$ will be obviously non-countable; we may, therefore, apply to it our lemma. Hence, there exist spheres K_0 and K_1 such that $\overline{K}_0.\overline{K}_1 = 0$, $\delta(K_0) < 1$, $\delta(K_1) < 1$, and the sets $E.K_0$ and $E.K_1$ are non-countable. But from (67)

$$E.K_0 = E^1.K_0 + E^2.K_0 + E^3.K_0 + \ldots,$$

and, since $E.K_0$ is non-countable, there exists an index m_0 such that $E^{m_0}.K_0$ is non-countable. Similarly, we deduce the existence of m_1 such that the set $E^{m_1}.K_1$ is non-countable.

Let now k be a given positive integer, and suppose that we have already defined all spheres $K_{a_1, a_2, \ldots, a_k}$, and all integers $m_{a_1, a_2, \ldots, a_k}$ (where $a_1, a_2, \ldots, a_k$ is any combination of k numbers, every one of which is either 0 or 1) so as to have conditions (69), (70), (71), and (72) satisfied. Let $a_1, a_2, \ldots, a_k$ be any combination of k numbers consisting of 0's and 1's. It follows from (72) and our lemma, that there exist two spheres $K_{a_1, a_2, \ldots, a_k, 0}$ and $K_{a_1, a_2, \ldots, a_k, 1}$ such that

$$K_{a_1, a_2, \ldots, a_k, 0} \subset K_{a_1, a_2, \ldots, a_k}, \quad K_{a_1, a_2, \ldots, a_k, 1} \subset K_{a_1, a_2, \ldots, a_k},$$

$$\overline{K}_{a_1, a_2, \ldots, a_k, 0} \cdot \overline{K}_{a_1, a_2, \ldots, a_k, 1} = 0,$$

$$\delta(K_{a_1, a_2, \ldots, a_k, 0}) < \frac{1}{k+1}, \quad \delta(K_{a_1, a_2, \ldots, a_k, 1}) < \frac{1}{k+1},$$

and the sets

$$E^{m_{a_1}, m_{a_1, a_2}, \ldots, m_{a_1, a_2, \ldots, a_k}} . K_{a_1, a_2, \ldots, a_k, 0}$$

and

$$E^{m_{a_1}, m_{a_1, a_2}, \ldots, m_{a_1, a_2, \ldots, a_k}} . K_{a_1, a_2, \ldots, a_k, 1}$$

[13] For $k = 1$, condition (70) does not come under consideration.

are non-countable. From this last property and from (68), we deduce easily the existence of indices $m_{\alpha_1, \alpha_2, \ldots, \alpha_k, 0}$ and $m_{\alpha_1, \alpha_2, \ldots, \alpha_k, 1}$ such that the sets

$$E^{m_{\alpha_1}, m_{\alpha_1, \alpha_2}, \ldots, m_{\alpha_1, \alpha_2, \ldots, \alpha_k, 0}} . K_{\alpha_1, \alpha_2, \ldots, \alpha_k, 0}$$

and

$$E^{m_{\alpha_1}, m_{\alpha_1, \alpha_2}, \ldots, m_{\alpha_1, \alpha_2, \ldots, \alpha_k, 1}} . K_{\alpha_1, \alpha_2, \ldots, \alpha_k, 1}$$

are non-countable.

The spheres $K_{\alpha_1, \alpha_2, \ldots, \alpha_k}$ and the integers $m_{\alpha_1, \ldots, \alpha_k}$, which satisfy conditions (69), (70), (71), and (72) are thus defined by induction for every finite combination of indices $\alpha_1, \alpha_2, \ldots, \alpha_k$ consisting of the numbers 0 and 1.

Denote now by S_k, for every integer k, the set

$$(73) \qquad S_k = \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_k)} E^{m_{\alpha_1}, m_{\alpha_1, \alpha_2}, \ldots, m_{\alpha_1, \alpha_2, \ldots, \alpha_k}} . \overline{K}_{\alpha_1, \alpha_2, \ldots, \alpha_k},$$

where the summation ranges over all combinations of k numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, each of which is 0 or 1. (Hence, the sum (73) consists of 2^k terms.) The sets S_k are obviously closed and bounded (since they are sums of a finite number of closed and bounded sets), and from (68) and (70) it follows easily that

$$(74) \qquad S_{k+1} \subset S_k, \text{ for } k = 1, 2, \ldots,$$

whereas from (72)

$$(75) \qquad S_k \neq 0, \text{ for } k = 1, 2, \ldots$$

The sequence $S_1, S_2, S_3, \ldots$ is, therefore, a decreasing sequence of bounded, closed, and non-null sets; hence, by Theorem 27 (and condition (W) of § 55), the set

$$(76) \qquad S = S_1 . S_2 . S_3 \ldots$$

is not null and is closed by Theorem 3.

Let p denote a given element of the set S. From (76) $p \in S_1$ and from (73)

$$S_1 = E^{m_0} . \overline{K}_0 + E^{m_1} . \overline{K}_1,$$

where $\overline{K}_0 . \overline{K}_1 = 0$ by (71). Since $p \in S_1$, we have either $p \in E^{m_0} . \overline{K}_0$

or $p \in E^{m_1}.\overline{K}_1$; in the first case put $\beta_1=0$, in the second $\beta_1=1$. Hence,

(77) $$p \in E^{m_{\beta_1}}.\overline{K}_{\beta_1}.$$

Furthermore, $p \in S_2$ from (76), and from (73) we have

$$S_2=E^{m_0,\, m_{0,0}}.\overline{K}_{0,\,0}+E^{m_0,\, m_{0,1}}.\overline{K}_{0,\,1}+E^{m_1,\, m_{1,0}}.\overline{K}_{1,\,0}+E^{m_1,\, m_{1,1}}.\overline{K}_{1,\,1}$$

where, on account of (71) and (70), the terms of the sum S_2 are mutually exclusive. The element $p \in S_2$ and so belongs to only one of the four terms, and, by means of (77) and (70), it may be easily deduced that this term has the form $E^{m_{\beta_1},\, m_{\beta_1,\beta_2}}.\overline{K}_{\beta_1,\beta_2}$, where β_2 is one of the two numbers 0 and 1.

Again from (76) we have $p \in S_3$, whence, arguing as before, we deduce from (73), (71), and (70) that, for a certain β_3 which is either 0 or 1, we have $p \in E^{m_{\beta_1},\, m_{\beta_1,\beta_2},\, m_{\beta_1,\beta_2,\beta_3}}.\overline{K}_{\beta_1,\beta_2,\beta_3}$.

Continuing in this manner we obtain an infinite sequence

$$\beta_1,\ \beta_2,\ \beta_3,\ \ldots$$

the terms of which are the numbers 0 or 1, and which is such that

(78) $$p \in E^{m_{\beta_1},\, m_{\beta_1,\beta_2},\ldots,m_{\beta_1,\beta_2,\ldots,\beta_k}}.\overline{K}_{\beta_1,\beta_2,\ldots,\beta_k},\ \text{for } k=1,\ 2,\ \ldots$$

Let now ϵ denote an arbitrary positive number. Denote by s an integer such that

(79) $$\frac{1}{s}<\epsilon,$$

and put $\beta'_{s+1}=1-\beta_{s+1}$; this will be one of the numbers 0 or 1, and $\beta'_{s+1}\neq\beta_{s+1}$, and so, from (71),

(80) $$\overline{K}_{\beta_1,\beta_2,\ldots,\beta_s,\,\beta_{s+1}}\cdot\overline{K}_{\beta_1,\beta_2,\ldots,\beta_s,\,\beta'_{s+1}}=0.$$

Put

(81) $$\beta_i'=\beta_i,\ \text{for } i\leq s,\ \beta_i'=0,\ \text{for } i\geq s+2.$$

From (68), (69), (70), and Theorem 27 we conclude that the product

(82) $$P=\prod_{k=1}^{\infty} E^{m_{\beta_1'},\, m_{\beta_1',\beta_2'},\ldots,m_{\beta_1',\beta_2',\ldots,\beta_k'}}.\overline{K}_{\beta_1',\beta_2',\ldots,\beta_k'}$$

is not null and from (69) that $\delta(P)<\dfrac{1}{k}$, for $k=1,\ 2,\ \ldots$, and so

$\delta(P)=0$; this proves that the product P consists of one element only, which we shall denote by q. From (82) and (73) we find $q \in S_k$, for $k=1, 2, \ldots$ (since $q \in P$), and so $q \in S$ from (76).

From (78), (80), (81), (82), and the fact that $\beta'_{s+1} \neq \beta_{s+1}$, we find that $p \neq q$.

Finally, on account of (78), (81), and (82) (since $q \in P$) we have

$$p \in \overline{K}_{\beta_1, \beta_2, \ldots, \beta_s} \text{ and } q \in \overline{K}_{\beta_1, \beta_2, \ldots, \beta_s},$$

and so, from (69) and (79) (since $\delta(\overline{Q})=\delta(Q)$ for every set Q),

$$\rho(p, q) < \epsilon.$$

We have thus proved that, corresponding to every element p of the set S and every positive number ϵ, there exists an element q of S, different from p and such that $\rho(p, q) < \epsilon$. Hence, p is a limit element of the set S (§ 43). Every element of the set S is, therefore, a limit element of S; the set S is, therefore, dense-in-itself and being closed, it is perfect.

Clearly, from (77), (73), (68), and (67), the set $S \subset E$. The set S is, therefore, a perfect, compact, and non-null subset of E. We have thus proved

Theorem 78. *Every analytical, non-countable set contains a perfect (and compact) subset, which is not null.*[14]

From Theorems 78 and 71a we get at once

Theorem 79. *Every non-countable Borel set contains a perfect subset, which is not null.*[15]

From Theorems 78 and 45 and the fact that the metric space under consideration has potency equal to or less than that of the continuum (Corollary 3 to axiom (*vi*), § 21) we get at once

COROLLARY 1. *Every non-countable analytical set has potency* $\mathfrak{c}$.

From Theorem 72 we get

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[14]This theorem was obtained by Souslin in 1916; see N. Lusin, *Comptes Rendus*, note from Jan. 8th, 1917 and *Fund. Math.*, vol. X, p. 25.

[15]F. Hausdorff, *Math. Annalen*, vol. LXXVII, p. 430. See also *Fund. Math.*, vol. V, p. 166.

COROLLARY 2. *Every non-countable Borel set has potency* $\mathfrak{c}$.[16]

71. Two sets P and Q are said to be *exclusive* B, if there exist two Borel sets M and N such that

(1) $$P \subset M,\ Q \subset N, \text{ and } M.N=0.$$

LEMMA. *If*

(2) $$P=P_1+P_2+P_3+\ldots,\ Q=Q_1+Q_2+Q_3+\ldots,$$

and if P *and* Q *are not exclusive* B, *there exist indices* p *and* q *such that the sets* P_p *and* Q_q *are not exclusive* B.

Proof. Suppose, contrary to the lemma, that the sets P_p and Q_q are exclusive B for all integers p and q. Hence, for every pair of integers p and q there exist two Borel sets $M_{p,q}$ and $N_{p,q}$ such that

(3) $$P_p \subset M_{p,q},\ Q_q \subset N_{p,q}, \text{ and } M_{p,q}.N_{p,q}=0.$$

Put

(4) $$M=\sum_{p=1}^{\infty}\prod_{q=1}^{\infty} M_{p,q},\ N=\sum_{q=1}^{\infty}\prod_{p=1}^{\infty} N_{p,q}.$$

It follows from the properties of Borel sets (§ 62) that M and N are Borel sets, and from (2), (3), and (4) we conclude readily that (1) is satisfied, contrary to the hypothesis of the lemma that P and Q are not exclusive B. The lemma is, therefore, proved.

Theorem 80. *Two mutually exclusive analytical sets are always exclusive* B.

Proof.[17] Let E and T be two analytical sets and $S[E_{n_1, n_2, \ldots, n_k}]$ and $S[T_{n_1, n_2, \ldots, n_k}]$ the corresponding regular systems generating these sets. Let the sets $E^{r_1, r_2, \ldots, r_s}$ be defined by (66) (§ 70) for every finite combination of indices $r_1, r_2, \ldots, r_s$ and the sets $T^{r_1, r_2, \ldots, r_s}$ in an analogous manner (*i.e* by the set obtained from (66) on substituting T for E). We shall have (67) and (68) and analogous relations for the sets T and $T^{r_1, r_2, \ldots, r_s}$.

Suppose that E and T are mutually exclusive but not exclusive B. Hence, it follows from (67), from the analogous relation for T,

[16] P. Alexandroff, *Comptes Rendus*, vol. CLXII, note of Feb. 22nd. 1916.

[17] This proof is due to Lusin.

and from our lemma, that there exist indices p_1 and q_1 such that the sets E^{p_1} and T^{q_1} are not exclusive B.

From (68) and the analogous relation for $T^{r_1, r_2, \dots, r_s}$, we have

$$E^{p_1} = E^{p_1, 1} + E^{p_1, 2} + E^{p_1, 3} + \dots,$$

$$T^{q_1} = T^{q_1, 1} + T^{q_1, 2} + T^{q_1, 3} + \dots,$$

from which, since E^{p_1} and T^{q_1} are not exclusive B, we conclude, from our lemma, the existence of indices p_2 and q_2, such that E^{p_1, p_2} and T^{q_1, q_2} are not exclusive B.

Proceeding thus indefinitely we obtain two infinite sequences of indices

$$p_1, p_2, p_3, \dots \text{ and } q_1, q_2, q_3, \dots$$

such that the sets

$$E^{p_1, p_2, \dots, p_k} \text{ and } T^{q_1, q_2, \dots, q_k} \quad (k = 1, 2, \dots)$$

are not exclusive B.

But from (66)

(5) $$E^{p_1, p_2, \dots, p_k} \subset E_{p_1, p_2, \dots, p_k}$$

and analogously

(6) $$T^{q_1, q_2, \dots, q_k} \subset T_{q_1, q_2, \dots, q_k}.$$

If the sets on the right of (5) and (6) were mutually exclusive, then, since they are closed and so Borel sets, it would follow that the sets on the left of (5) and (6) are exclusive B, contrary to the conclusion we have arrived at above.

Hence

(7) $$P_k = E_{p_1, p_2, \dots, p_k} \cdot T_{q_1, q_2, \dots, q_k} \neq 0, \text{ for } k = 1, 2, \dots$$

But the systems $S[E_{n_1, n_2, \dots, n_k}]$ and $S[T_{n_1, n_2, \dots, n_k}]$ are regular, and, therefore, we have (§ 66, (28))

$$E_{p_1, p_2, \dots, p_k} \supset E_{p_1, p_2, \dots, p_k, p_{k+1}} \text{ and } T_{q_1, q_2, \dots, q_k} \supset T_{q_1, q_2, \dots, q_k, q_{k+1}},$$

for $k = 1, 2, \dots$, and so $P_k \supset P_{k+1}$ from (7); moreover, the sets P_k are closed and bounded on account of (7); hence, they are compact by condition (W) of § 55.

By Theorem 27, the set $P_1.P_2.P_3 \ldots$ is not null. Hence there exists an element $x \in P_k$, for $k=1, 2, \ldots$, *i.e.* from (7)

$$x \in E_{p_1, p_2, \ldots, p_k} \text{ and } x \in T_{q_1, q_2, \ldots, q_k}, \text{ for } k=1, 2, \ldots,$$

and so x belongs to the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$ and to the nucleus of the system $S[T_{n_1, n_2, \ldots, n_k}]$; it is, therefore, a common element of the sets E and T, contrary to the supposition that $E.T=0$.

The assumption that Theorem 80 is not true leads to a contradiction. Theorem 80 is, therefore, proved.

Suppose now that E is an analytical set and that the complement of E, *i.e.* the set $T=CE$, is also an analytical set. Since, obviously, $E.T=0$, we may apply Theorem 80 to the sets E and T. According to this theorem, there exist two Borel sets M and N such that $E \subset M$, $T \subset N$, and $M.N=0$, and so certainly $M.T=0$, *i.e.* $M \subset CT=E$; but $E \subset M$ and $M \subset E$ give $E=M$, and this proves that E is a Borel set. We have thus proved that, if the complement of an analytical set E is an analytical set, then E is a Borel set. On the other hand, if E be a Borel set, then, by property 4 (§ 62) of Borel sets, the complement of E is also a Borel set, and so, by Theorem 72, the sets E and CE are analytical sets.

We have thus proved

Theorem 81 (Souslin). *In order that a set E be a Borel set, it is necessary and sufficient that the set E and its complement be analytical sets.*

From Theorem 81 we get the immediate

COROLLARY. *In order that an analytical set be a Borel set, it is necessary and sufficient that its complement be an analytical set.*

We note that Theorem 80 may be easily generalized as follows:

Theorem 80a. *If $P_1, P_2, P_3, \ldots$ be an infinite sequence of mutually exclusive analytical sets, there exist Borel sets $M_1, M_2, M_3, \ldots$ such that*

(8) $$P_k \subset M_k, \textit{ for } k=1, 2, \ldots,$$

and

(9) $$M_p.M_q=0, \textit{ for } p \neq q.$$

Proof. Let p and q be two different positive integers. Since the analytical sets P_p and P_q are mutually exclusive, there exist, by Theorem 80, Borel sets $M_{p,q}$ and $M_{q,p}$ such that

$$(10) \qquad P_p \subset M_{p,q}, \; P_q \subset M_{q,p}$$

and

$$(11) \qquad M_{p,q} \cdot M_{q,p} = 0.$$

But (10) and (11) hold for every pair of different integers p and q.

Put (for every integer k)

$$(12) \qquad M_k = \prod_{n \neq k} M_{k,n},$$

where the product $\prod$ ranges over all positive integers n different from k. It follows from the properties of Borel sets (§ 62) that the sets (12) are Borel sets.

From (10) and (12) we easily obtain (8), and from (11) and (12) we get (9) (since from (12) for $p \neq q$ we have $M_p \subset M_{p,q}$ and $M_q \subset M_{q,p}$).

Theorem 80a is, therefore, proved.

72. We have shown in § 67 that a continuous transform of the set of all irrational numbers is always an analytical set. We shall prove now

Theorem 82. *A biuniform and continuous transform of the set of all irrational numbers is a Borel set.*

Proof. Since the set of all irrational numbers is homeomorphic with the set E of all irrational numbers in the interval (0, 1), it will be sufficient to show that, if f be a function defined in E, taking different values for different elements of E (these values being elements of a metric space satisfying condition (W) of § 55), and continuous in E, then $T = f(E)$ is a Borel set.

For every finite combination $n_1, n_2, \ldots, n_k$ of positive integers denote by $E_{n_1, n_2, \ldots, n_k}$ the set of all irrational numbers in the interval (0, 1) whose k^{th} convergent is

$$\frac{1}{n_1 +} \, \frac{1}{n_2 +} \cdots \frac{1}{n_{k-1} +} \, \frac{1}{n_k}.$$

Each of the sets $E_{n_1, n_2, \ldots, n_k}$ is, as can be easily seen, homeomorphic with the set of all irrational numbers; hence, by Theorem 73, the sets

$$(13) \qquad T_{n_1, n_2, \ldots, n_k} = f(E_{n_1, n_2, \ldots, n_k})$$

are analytical.

Let k denote a given integer. It follows readily from the definition of the sets $E_{n_1, n_2, \ldots, n_k}$ that, if $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$ are two different sets of k integers, then

$$(14) \qquad E_{p_1, p_2, \ldots, p_k} \cdot E_{q_1, q_2, \ldots, q_k} = 0,$$

and so from (13), since f is biuniform in T,

$$(15) \qquad T_{p_1, p_2, \ldots, p_k} \cdot T_{q_1, q_2, \ldots, q_k} = 0.$$

Furthermore,

$$(16) \qquad E = \sum_{(n_1, n_2, \ldots, n_k)} E_{n_1, n_2, \ldots, n_k},$$

where the summation ranges over all combinations of k integers.

Finally, for every finite combination $n_1, n_2, \ldots, n_{k+1}$ of positive integers we have

$$E_{n_1, n_2, \ldots, n_k, n_{k+1}} \subset E_{n_1, n_2, \ldots, n_k};$$

hence, from (13)

$$(17) \qquad T_{n_1, n_2, \ldots, n_k, n_{k+1}} \subset T_{n_1, n_2, \ldots, n_k}.$$

With every finite combination $n_1, n_2, \ldots, n_k$ of integers, let there be correlated a certain Borel set $M_{n_1, n_2, \ldots, n_k}$ such that the following conditions are satisfied:

$$(18) \qquad T_{n_1, n_2, \ldots, n_k} \subset M_{n_1, n_2, \ldots, n_k} \subset \overline{T}_{n_1, n_2, \ldots, n_k},$$

$$(19) \qquad M_{n_1, n_2, \ldots, n_k} \subset M_{n_1, n_2, \ldots, n_{k-1}},$$ [18]

$$(20) \qquad M_{p_1, p_2, \ldots, p_k} \cdot M_{q_1, q_2, \ldots, q_k} = 0$$

for different combinations $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$ of k integers ($k = 1, 2, \ldots$).

[18] For $k = 1$, condition (19) does not come under consideration.

We shall show that such a correlation is possible.

The sets T_1, T_2, T_3, . . . are mutually exclusive by (15), and since they are analytical, there exists, by Theorem 80a, a Borel set M_n, for every integer n such that

$$T_n \subset M_n, \text{ for } n=1, 2, \ldots,$$

and

$$M_p \cdot M_q = 0, \text{ for } p \neq q,$$

where it may be supposed that

$$M_n \subset \overline{T}_n,$$

for otherwise it would be sufficient to replace M_n by the set $M_n . \overline{T}_n$, which is also a Borel set.

Relations (18) and (20) are, therefore, true for $k=1$.

Let now k be a given integer, and suppose that all sets $M_{n_1, n_2, \ldots, n_k}$ (where $n_1, n_2, \ldots, n_k$ is any combination of k positive integers) are already defined and so that conditions (18), (19), and (20) are satisfied.

Since the aggregate of all sets of $k+1$ integers is countable, from (15), and from the fact that the sets (13) are analytical, we conclude, by Theorem 80a, the existence of a Borel set $N_{n_1, n_2, \ldots, n_k, n_{k+1}}$ for every combination of $k+1$ integers such that

$$T_{n_1, n_2, \ldots, n_{k+1}} \subset N_{n_1, n_2, \ldots, n_{k+1}} \tag{21}$$

and

$$N_{p_1, p_2, \ldots, p_{k+1}} \cdot N_{q_1, q_2, \ldots, q_{k+1}} = 0 \tag{22}$$

for different combinations $p_1, p_2, \ldots, p_{k+1}$ and $q_1, q_2, \ldots, q_{k+1}$ of $k+1$ integers. Put

$$M_{n_1, n_2, \ldots, n_{k+1}} = M_{n_1, n_2, \ldots, n_k} \cdot \overline{T}_{n_1, n_2, \ldots, n_{k+1}} \cdot N_{n_1, n_2, \ldots, n_k, n_{k+1}}; \tag{23}$$

these will be Borel sets (being products of three Borel sets), and from (17), (18), (21), and (23), we find

$$T_{n_1, n_2, \ldots, n_{k+1}} \subset M_{n_1, n_2, \ldots, n_{k+1}} \subset \overline{T}_{n_1, n_2, \ldots, n_{k+1}}.$$

Moreover, (23) gives immediately

$$M_{n_1, n_2, \ldots, n_{k+1}} \subset M_{n_1, n_2, \ldots, n_k};$$

finally, from (23) and (22), we get

$$M_{p_1, p_2, \ldots, p_{k+1}} \cdot M_{q_1, q_2, \ldots, q_{k+1}} = 0,$$

for two different combinations $p_1, p_2, \ldots, p_{k+1}$ and $q_1, q_2, \ldots, q_{k+1}$ of $k+1$ integers.

Hence (18), (19), and (20) remain true if we replace in them k by $k+1$. The Borel sets $M_{n_1, n_2, \ldots, n_k}$ which satisfy (18), (19), and (20), are thus defined by induction (for every finite combination $n_1, n_2, \ldots, n_k$ of positive integers).

Put

$$S_k = \sum_{(n_1, n_2, \ldots, n_k)} M_{n_1, n_2, \ldots, n_k}, \tag{24}$$

where the summation ranges over all combinations $n_1, n_2, \ldots, n_k$ of k positive integers. The sets (24) are obviously Borel sets, for $k=1, 2, \ldots$ (since they are sums of a countable aggregate of Borel sets, § 62), and so the set

$$S = S_1 . S_2 . S_3 \ldots \tag{25}$$

is a Borel set. It will be shown that $T = f(E) = S$.

From (16), (13), (18), and (24) we get easily $f(E) \subset S_k$ for $k=1, 2, \ldots$ and so, from (25), $f(E) \subset S$. It will, therefore, be sufficient to show that $S \subset f(E)$.

Hence, let y denote an element of the set S. From (25) $y \in S_1$ and so, from (24), y is an element of the sum $M_1 + M_2 + \ldots$, *i.e.* $y \in M_{m_1}$, for some integer m_1.

Similarly, $y \in S_2$ from (25), whence we conclude by means of (24) that $y \in M_{m_1', m_2}$, for certain indices m_1' and m_2. But, $M_{m_1', m_2} \subset M_{m_1'}$ from (19); hence, $y \in M_{m_1'}$, and, since $y \in M_{m_1}$, we conclude from (20) that $m_1' = m_1$.

Similarly, starting with $y \in S_3$, we might conclude the existence of an index m_3 such that $y \in M_{m_1, m_2, m_3}$. Proceeding thus indefinitely we obtain an infinite sequence of indices $m_1, m_2, m_3, \ldots$ such that

$$y \in M_{m_1, m_2, \ldots, m_k}, \text{ for } k = 1, 2, \ldots \tag{26}$$

Put

$$x = \frac{1}{m_1 +} \, \frac{1}{m_2 +} \, \frac{1}{m_3 +} \cdots; \tag{27}$$

this will be a number of the set E.

Let ϵ be an arbitrarily given positive number. Since the function f is continuous in E, there exists, corresponding to ϵ, a number $\eta > 0$ such that the inequality

$$|x - x'| < \eta \tag{28}$$

implies the inequality

$$\rho(f(x), f(x')) < \epsilon, \tag{29}$$

for all numbers x' of E.

From (27) and the properties of continued fractions, it follows that, corresponding to the number η, there exists an index k such that every number x' of E whose k^{th} convergent is the same as the k^{th} convergent of the number (27), *i.e.* every number of the set $E_{m_1, m_2, \ldots, m_k}$, satisfies the inequality (28) and, therefore, also the inequality (29).

On account of (13) we may say that every number t of the set $T_{m_1, m_2, \ldots, m_k}$ satisfies the inequality

$$\rho(f(x), t) < \epsilon,$$

and so every number t of the set $\overline{T}_{m_1, m_2, \ldots, m_k}$ satisfies the inequality

$$\rho(f(x), t) \leq \epsilon. \tag{30}$$

But, from (26) and (18), we have $y \in \overline{T}_{m_1, m_2, \ldots, m_k}$; we may, therefore, put $t = y$ in (30), which gives

$$\rho(f(x), y) \leq \epsilon.$$

Since ϵ is an arbitrary number it follows that $\rho(f(x), y) = 0$, *i.e.* $y = f(x)$. We have, therefore, proved that $S \subset f(E)$.

Theorem 82 is, therefore, proved. We shall deduce some important results from this theorem in one of the following articles.

73. Theorem 83. *Every closed and compact set not null* (consisting of elements of a metric space) *is a continuous transform of a certain closed and bounded set of real numbers.*

Proof. Let T be a given closed and compact set. Hence, T is, by Theorem 48 (§ 47), the sum of a finite number of sets of arbitrarily small diameters; we may, therefore, write $T=M_1+M_2+\ldots+M_{s_1}$, where $\delta(M_i)\leq 1$, for $i=1, 2, \ldots, s_1$, and where M_i may be supposed to be not null. Put $T_i=\overline{M}_i$; T_i is closed and compact (being a subset of the compact set T), for $i=1, 2, \ldots, s_1$ and, as is easily seen,

(1) $$T=T_1+T_2+\ldots+T_{s_1},$$

(2) $$T_i\neq 0, \text{ for } i=1, 2, \ldots, s_1,$$

(3) $$\delta(T_i)\leq 1, \text{ for } i=1, 2, \ldots, s_1.$$

Similarly, each of the sets $T_i (i=1, 2, \ldots, s_1)$ may be represented as the sum of a finite number of closed and compact sets

$$T_i=T_{i,1}+T_{i,2}+\ldots+T_{i,s_{2,i}},$$

where

$$T_{i,j}\neq 0, \text{ for } j=1, 2, \ldots, s_{2,i},$$

and

$$\delta(T_{i,j})\leq \tfrac{1}{2}, \text{ for } j=1, 2, \ldots, s_{2,i}.$$

Denote by s_2 the greatest of the numbers $s_{2,1}, s_{2,2}, \ldots, s_{2,s_1}$; letting $T_{i,j}=T_{i,1}$, for $s_{2,i}<j<s_2$, we shall have, for $i=1, 2, \ldots, s_1$,

$$T_i=T_{i,1}+T_{i,2}+\ldots+T_{i,s_2},$$

$$T_{i,j}\neq 0, \text{ for } j=1, 2, \ldots, s_2,$$

and

$$\delta(T_{i,j})\leq \tfrac{1}{2}, \text{ for } j=1, 2, \ldots, s_2.$$

Carrying on this argument indefinitely we obtain an infinite sequence $s_1, s_2, s_3, \ldots$ of positive integers (where $s_k>1$, for $k>1$), and for every finite combination $n_1, n_2, \ldots, n_k$ of indices such that

(4) $$n_i\leq s_i, \text{ for } i=1, 2, \ldots, k$$

we have a closed and compact set

(5) $$T_{n_1, n_2, \ldots, n_k} \neq 0,$$

where

(6) $$\delta(T_{n_1, n_2, \ldots, n_k}) \leq \frac{1}{k}$$

and

(7) $$T_{n_1, n_2, \ldots, n_k} = T_{n_1, n_2, \ldots, n_k, 1} + T_{n_1, n_2, \ldots, n_k, 2} + \ldots + T_{n_1, n_2, \ldots, n_k, s_{k+1}}.$$

We shall define now for every finite combination $n_1, n_2, \ldots, n_k$ of indices satisfying condition (4), an interval $E_{n_1, n_2, \ldots, n_k}$, as follows:

Divide the interval (0, 1) into $2s_1$ equal closed intervals. Denote successively by $E_1, E_2, \ldots, E_{s_1}$ every second of these intervals. In general, having obtained the interval $E_{n_1, n_2, \ldots, n_k}$, where $n_1, n_2, \ldots, n_k$ is a combination of indices satisfying (4), divide it into $2s_{k+1}$ equal closed intervals and denote successively every second of these intervals by

$$E_{n_1, n_2, \ldots, n_k, 1}, E_{n_1, n_2, \ldots, n_k, 2}, \ldots, E_{n_1, n_2, \ldots, n_k, s_{k+1}}.$$

Put

(8) $$S_k = \sum_{(n_1, n_2, \ldots, n_k)} E_{n_1, n_2, \ldots, n_k},$$

for $k = 1, 2, \ldots$, where the summation ranges over all combinations $n_1, n_2, \ldots, n_k$ of indices which satisfy (4). The sets (8) will be closed and compact (being sums of a finite number of closed and compact sets) and obviously not null; also $S_{k+1} \subset S_k$, for $k = 1, 2, \ldots$, since, from the definition of the intervals $E_{n_1, n_2, \ldots, n_k}$, we obtain at once

(9) $$E_{n_1, n_2, \ldots, n_k, n} \subset E_{n_1, n_2, \ldots, n_k}, \text{ for } n = 1, 2, \ldots, s_{k+1}.$$

Hence, by Theorem 27, the set

(10) $$E = S_1 . S_2 . S_3 \ldots$$

is closed, compact, and not null. (From the fact that $s_k > 1$ for $k > 1$, it can be easily deduced that E is a perfect, nowhere dense[19] set of numbers of the interval (0, 1).)

[19] A set E is said to be *nowhere dense* if there exists no open set in which it is everywhere dense.

Let now x be a given element of the set E. Hence $x \in S_1$ by (10), and so $x \in E_1 + E_2 + \ldots + E_{s_1}$ by (8). It follows, however, from the definition of the sets $E_n (n = 1, 2, \ldots, s_1)$ that the terms of the above sum are mutually exclusive; there exists, therefore, a perfectly definite index $n_1 \leq s_1$ such that $x \in E_{n_1}$. On account of the exclusiveness of the terms of the sum (8) and on account of (9), we conclude from the above the existence of a definite index $n_2 \leq s_2$ such that $x \in E_{n_1, n_2}$.

Proceeding in this manner, we obtain for the number x of the set E an infinite sequence of indices $n_1, n_2, n_3, \ldots$ completely determined (by the number x) such that

(11) $$n_i \leq s_i, \text{ for } i = 1, 2, \ldots$$

and

(12) $$x \in E_{n_1, n_2, \ldots, n_k}, \text{ for } k = 1, 2, \ldots$$

Put

(13) $$F(x) = T_{n_1} \cdot T_{n_1, n_2} \cdot T_{n_1, n_2, n_3} \cdots;$$

this will be a subset of T, determined completely by the number x of the set E.

The set (13) is, by (5) and (7), the product of a decreasing sequence of closed, compact, non-null sets; hence, $F(x)$ is not null by Theorem 27. But, from (13) and (6), we have $\delta(F(x)) \leq \frac{1}{k}$, for $k = 1, 2, \ldots$, and so $\delta(F(x)) = 0$; the set $F(x)$ consists, therefore, of one element only, which we shall denote by $f(x)$. From (13) and (1) we get $f(x) \in F(x) \subset T_{n_1} \subset T$; hence $f(x)$ is an element of T.

We have thus correlated with every number x of E a definite element $f(x)$ of T.

It will be shown that $f(E) = T$. From the definition of the function f, we get at once $f(E) \subset T$; it will, therefore, be sufficient to show that $T \subset f(E)$.

Let, therefore, y denote a given element of T. There exists, by (1), at least one index $n_1 \leq s_1$ such that $y \in T_{n_1}$. Similarly, by (7) (for $k = 1$), there exists at least one index $n_2 \leq s_2$ such that $y \in T_{n_1, n_2}$. Proceeding in this manner indefinitely, we obtain an infinite sequence of indices $n_1, n_2, n_3, \ldots$, satisfying (11) and such that

(14) $$y \in T_{n_1, n_2, \ldots, n_k}, \text{ for } k = 1, 2, \ldots .$$

Put

(15) $$\Phi(y) = E_{n_1} \cdot E_{n_1, n_2} \cdot E_{n_1, n_2, n_3} \cdot \ldots .$$

It follows at once from the definition of the sets $E_{n_1, n_2, \ldots, n_k}$ that the set (15) is the product of an infinite decreasing sequence of closed, compact, non-null sets, where

$$\delta(E_{n_1, n_2, \ldots, n_k}) \leq \frac{1}{2^k}.$$

Hence, by Theorem 27, the set (15) consists of one element only, *i.e.* of $x = \phi(y)$. From (10) and (8) we get $x \in E$, and, moreover, from (14), (13), and the definition of the function $f(x)$, we find immediately that $y = f(x)$ and so (from $x \in E$), $y \in f(E)$. We have thus proved that $T \subset f(E)$.

To complete the proof of Theorem 83 it will be sufficient to show that the function $f(x)$ is continuous in the whole set E.

Let x denote a given element of E, and ϵ a given positive number. Choose a positive integer p such that $\frac{1}{p} < \epsilon$. There exists, as we know, for the number $x \in E$ a completely determined infinite sequence $n_1, n_2, n_3, \ldots$ of indices satisfying conditions (11) and (12).

Put

(16) $$\eta = \delta(E_{n_1, n_2, \ldots, n_p});$$

this will be a definite positive number.

Let now x' be a number of the set E such that

(17) $$|x - x'| < \eta.$$

It follows readily from the definition of the intervals $E_{n_1, n_2, \ldots, n_k}$, on account of (16), (17), (10), and (8), that

$$x' \in E_{n_1, n_2, \ldots, n_p},$$

and so, from the definition of the function f, we find easily

(18) $$f(x') \in T_{n_1, n_2, \ldots, n_p};$$

also from (6) and the fact that $\frac{1}{p} < \epsilon$, we get

(19) $$\delta(T_{n_1, n_2, \ldots, n_p}) < \epsilon.$$

But also $f(x) \epsilon T_{n_1, n_2, \ldots, n_p}$; hence, (18) and (19) give

(20) $$\rho(f(x), f(x')) < \epsilon.$$

We have thus proved that, corresponding to every number x of the set E, and every number $\epsilon > 0$, there exists a number $\eta > 0$ such that the inequality (17) implies the inequality (20) for all numbers x' of the set E, which proves that the function f is continuous in E.

Theorem 83 may, therefore, be considered as proved.

Let now E denote a closed and bounded (and, therefore, compact) set of real numbers and $f(x)$ a function defined and continuous in E, whose values are elements of a certain metric space. Let y be an element of the set $T = f(E)$; there will be, as is easily seen, among the numbers x of E for which $f(x) = y$, a greatest. In fact, let $\phi(y)$ denote the upper bound of the set $P(y)$ of all numbers x belonging to E for which $f(x) = y$. Since the set (y) (*i.e.* consisting of the single element y) is closed, it follows from the corollary to Theorem 20 (§ 10) that the set $P(y)$ is closed, and so it contains its upper bound $\phi(y)$ (which is a finite number, since $P(y) \subset E$, and E is bounded). From $\phi(y) \epsilon P(y)$, we have $f(\phi(y)) = y$, and so, from the definition of the number $\phi(y)$, it follows that it is the greatest number x of the set E for which $f(x) = y$.

Denote by X the set of all numbers $\phi(y)$ for which $y \epsilon T$; obviously, $X \subset E$, $f(X) = T$, and the function f is continuous and biuniform in the set X. The set T is, therefore, a continuous and biuniform transform of the set X.

It will be shown that the set X is a G_δ. Since E is closed, it will obviously be sufficient to show that the set $E - X$ is an F_σ.

It follows readily from the definition of the set X that X is the set of all numbers x of E for which $f(x) \neq f(x')$ for all numbers x' belonging to E and greater than x. Consequently, if x is a number of the set $E - X$, there exists a number $x' > x$, which belongs to E

and such that $f(x)=f(x')$. Denote by F_n the set of all numbers x of E for which there exists a number x' of E such that $x' \geq x + \frac{1}{n}$ and $f(x)=f(x')$; clearly, $E-X=F_1+F_2+F_3+\ldots$. To prove that the set $E-X$ is an F_σ, it will be sufficient to show that the sets F_n $(n=1, 2, \ldots)$ are closed.

Let n denote a given integer and x_0 a limit element of the set F_n. There exists, therefore, an infinite sequence x_k $(k=1, 2, \ldots)$ of numbers of the set F_n such that $\lim_{k\to\infty} x_k=x_0$. On account of $x_k \in F_n$, for $k=1, 2, \ldots$, and the definition of F_n, there exists for every integer k a number x_k' of the set E such that $x_k' \geq x_k + \frac{1}{n}$, and $f(x_k)=f(x_k')$. The infinite sequence of the numbers x_k' $(k=1, 2, \ldots)$ is bounded (since the terms of the sequence belong to E, which is bounded); consequently, it contains a convergent subsequence x'_{k_j} $(j=1, 2, \ldots)$. Put $\lim_{j\to\infty} x'_{k_j}=x_0'$; since $x'_{k_j} \in E$, for $j=1, 2, \ldots$, and since E is closed, x_0' is an element of E. But, from $x_k' \geq x_k + \frac{1}{n}$, $f(x_k)=f(x_k')$, and the continuity of the function f in E, we shall have $x_0' \geq x_0+\frac{1}{n}$, and $f(x_0)=f(x_0')$ (since $\lim_{k\to\infty} x_k=x_0$, and $\lim_{j\to\infty} x'_{k_j}=x_0'$); hence, $x_0 \in F_n$. The set F_n is, therefore, closed.

We have thus proved that, if E be a closed and bounded set of real numbers, and $f(x)$ a function continuous in E, then the set $T=f(E)$ is a continuous and biuniform transform of a certain set G_δ contained in E. In connection with Theorem 83, this leads immediately to

Theorem 84. *Every closed and compact set* (of elements of a metric space) *is a continuous and biuniform transform of a certain set G_δ of real numbers.*

Let G denote an open and compact set contained in a given metric space. Put $E=\overline{G}$; this will be a closed and compact set (for, as shown at the close of §47, the enclosure of a compact set is compact). Hence, there exists, by Theorem 84, a set Γ of real numbers, which set is a G_δ, and a function f continuous and biuniform in Γ and for which $f(\Gamma)=E$. Denote by Γ_1 the set of all numbers x of Γ for which $f(x) \in G$; since $G \subset \overline{G}=E$, we shall evidently have $f(\Gamma_1)=G$, and since $\Gamma_1 \subset \Gamma$, the function f will be continuous and biuniform in the set Γ_1. But the set Γ is a G_δ; we may, therefore, apply to it the lemma of § 58, from which it follows that the set Γ_1 is a G_δ. We have thus proved

Theorem 84a. *Every open and compact set* (of elements of a metric space) *is a continuous and biuniform transform of a certain set G_δ of real numbers.*

74. **Theorem 85** (Mazurkiewicz). *Every set G_δ of real numbers is the sum of two sets, one of which is a null set or a set homeomorphic with the set of all irrational numbers, and the other is a set countable at most.*

We shall first prove the following

LEMMA. *If U be an open set of real numbers containing a non-countable set N, and η be a positive number, then there exists an infinite sequence of non-abutting open intervals $D_1, D_2, D_3, \ldots$, each D_n of length $<\eta$, each $\overline{D}_n \subset U$, and each $N.D_n$ non-countable, and where the set $N-(D_1+D_2+\ldots)$ is countable at most.*

Proof. Let U be an open set of real numbers containing a non-countable set N, and let η be a given positive number. The set N, since non-countable, contains an element of condensation x, and since $N \subset U$, we have $x \in U$. Since U is open, there exists a positive integer k such that the interval $P_k=\left(x-\frac{1}{k}, x+\frac{1}{k}\right)$ is contained in U, end-points included, where it may also be assumed that $\frac{2}{k}<\eta$. It is easily seen that the interval P_s differs from the sum of all the open intervals $Q_n=\left(x+\frac{1}{n+1}, x+\frac{1}{n}\right)$ and $R_n=\left(x-\frac{1}{n}, x-\frac{1}{n+1}\right)$ by a countable aggregate of points, where the summation ranges over all positive integers $n \geq s$. Again, the interval P_s contains a non-countable aggregate of elements of N, since it contains x, an element of condensation of N, as an interior point. Hence, we conclude that, for every integer s, there exists an integer $n \geq s$ such that at least one of the intervals Q_n and R_n contains a non-countable aggregate of elements of N. Consequently, an infinity of open intervals Q_n and R_n, for $n \geq k$, contains a non-countable aggregate of elements of N; let these be the intervals $H_1, H_2, H_3, \ldots$ The set $N.\overline{P}_k - (H_1 + H_2 + \ldots)$ is clearly countable at most. Furthermore, since $H_n \subset P_k$, for $n = 1, 2, \ldots$, and the interval P_k, end-points included, is contained in U, and the length of the interval P_k is $< \eta$, it follows that $\overline{H}_n \subset U$, for $n = 1, 2, \ldots$, also that the length of the interval H_n is $< \eta$, for $n = 1, 2, \ldots$, and finally that the sets $H_n.N$ are, by the definition of the intervals H_n, non-countable, for $n = 1, 2, \ldots$.

The set $U - \overline{P}_k$ is open, and so may be divided into a countable aggregate of open intervals of length $< \eta$ on removing a countable aggregate of points; let (a, b) denote one of the intervals thus obtained. Let $a_1, a_2, a_3, \ldots$ be an infinite decreasing sequence of numbers $< \frac{1}{2}(a+b)$ and approaching a, and let $b_1, b_2, \ldots$ be an infinite increasing sequence of numbers $> \frac{1}{2}(a+b)$ and approaching b. The interval (a, b) differs, as is easily seen, from the sum of the intervals (a_1, b_1), (a_{n+1}, a_n), and (b_n, b_{n+1}), for $n = 1, 2, \ldots$, by a countable aggregate of points; these intervals, end-points included, are contained in the open interval (a, b) and so also in U. It follows easily from this that, except for a countable set of points, the set $U - \overline{P}_k$ can be divided into a countable aggregate of open intervals of length $< \eta$, which, end-points included, are contained in U. Those of the intervals thus obtained, which contain a non-countable aggregate of points of N (if such intervals exist) will be denoted by $K_1, K_2, \ldots$ (This sequence may not exist, but existing, may be finite or infinite.) The set $N.(U - \overline{P}_k) - (K_1 + K_2 + \ldots)$ is clearly countable at most. The aggregate of all intervals $H_1, H_2, H_3, \ldots$ and $K_1, K_2, K_3, \ldots$ is obviously countable; it may, therefore, be ordered as an infinite sequence $D_1, D_2, D_3, \ldots$. It is easily seen that the intervals $D_n (n = 1, 2, \ldots)$ satisfy all the conditions of our lemma, which may, therefore, be considered as proved.

Let now E be a set G_δ of real numbers. Then there exists an infinite sequence of open sets G_n $(n = 1, 2, \ldots)$ such that $E =$

$G_1.G_2.G_3\ldots$ Suppose that E is not countable. Since $E\subset G_1$ and G_1 is open, we may apply our lemma on putting $U=G_1$, $N=E$, $\eta=1$. We thus obtain an infinite sequence of non-abutting open intervals $D_1, D_2, D_3, \ldots$, each D_n of length <1, each $\overline{D}_n\subset G_1$ and each $E.D_n$ non-countable, and where the set $E-(D_1+D_2+\ldots)$ is countable at most.

Let n_1 denote a positive integer. On account of the sets G_2 and D_{n_1} being open, the set $G_2.D_{n_1}$ is open, and since $E\subset G_2$ and $E.D_{n_1}$ is non-countable, the set $E.G_2.D_{n_1}$ is non-countable. We may, therefore, apply the lemma on putting $U=G_2.D_{n_1}$, $N=E.D_{n_1}$, $\eta=\frac{1}{2}$. We thus obtain an infinite sequence of non-abutting open intervals $D_{n_1,1}$, $D_{n_1,2}, \ldots$, each $D_{n_1,n}$ of length $<\frac{1}{2}$, each $D_{n_1,n}\subset G_2.D_{n_1}$, and each $E.D_{n_1,n}$ non-countable, and where the set $E.D_{n_1}-(D_{n_1,1}+D_{n_1,2}+\ldots)$ is countable at most.

Let, further, n_1 and n_2 be two positive integers. Since the sets G_3 and D_{n_1,n_2} are open and the set $E.G_3.D_{n_1,n_2}$ is non-countable, we may apply our lemma to $U=G_3.D_{n_1,n_2}$, $N=E.D_{n_1,n_2}$, $\eta=\frac{1}{3}$.

Repeating this argument, we obtain for every finite combination $n_1, n_2, \ldots, n_k$ of positive integers an open interval $D_{n_1,n_2,\ldots,n_k}$ such that

1, the length of the interval $D_{n_1,n_2,\ldots,n_k}$ is $<\dfrac{1}{k}$,

2, $D_{n_1,n_2,\ldots,n_{k-1},p}.D_{n_1,n_2,\ldots,n_{k-1},q}=0$, for $p\neq q$,

3, $\overline{D}_{n_1,n_2,\ldots,n_k}\subset G_k.D_{n_1,n_2,\ldots,n_{k-1}}$,

4, the set $E.D_{n_1,n_2,\ldots,n_k}$ is non-countable,

5, the set $E.D_{n_1,n_2,\ldots,n_{k-1}}-(D_{n_1,n_2,\ldots,n_{k-1},1}+D_{n_1,n_2,\ldots,n_{k-1},2}+\ldots)$ is countable at most.

Let now N denote the set of all irrational numbers in the interval $(0, 1)$ and x a given number of N, and let

$$x=\frac{1}{m_1+}\;\frac{1}{m_2+}\;\frac{1}{m_3+}\;\cdots \tag{21}$$

be the development of x as a continued fraction. Put

(22) $$F(x) = \overline{D}_{m_1} \cdot \overline{D}_{m_1, m_2} \cdot \overline{D}_{m_1, m_2, m_3} \cdot \cdot \cdot \cdot$$

The set (22) is, from 3 and 4, the product of a decreasing sequence of closed, non-null intervals and is, therefore, non-null itself; but, from 1, $F(x)$ is contained in an interval of length $< \frac{1}{k}$, for $k=1, 2, \ldots$; hence $F(x)$ consists of one element only, which we shall denote by $f(x)$. From $f(x) \in F(x)$, (22), and 3 we have $f(x) \in G_k$, for $k=1, 2. \ldots$, and so $f(x) \in E$. Hence, every number x of the set N is correlated with a number $f(x)$ of the set E. The set T of all the numbers $f(x)$ for $x \in N$, is, therefore, a subset of the set E. It will be seen that the set $E-T$ is countable at most.

To prove this, denote by R the set

(23) $$R = (E-S) + \sum_{(n_1, n_2, \ldots, n_k)} (E.D_{n_1, n_2, \ldots, n_k} - S_{n_1, n_2, \ldots, n_k}),$$

where the summation ranges over all finite combinations $n_1, n_2, \ldots, n_k$ of positive integers, and where $S = D_1 + D_2 + \ldots$, while

(24) $$S_{n_1, n_2, \ldots, n_k} = D_{n_1, n_2, \ldots, n_k, 1} + D_{n_1, n_2, \ldots, n_k, 2} + \ldots$$

It is evident, from (24) and 5, that the terms of the sum (23) are sets countable at most; consequently, the set R is countable at most.

Let now y denote a number of the set $E-R$. Hence $y \in E$, and $y \bar{\in} R$, and so, from (23), $y \bar{\in} (E-S)$; but $y \in E$; therefore, $y \in S$, and so, from $S = D_1 + D_2 + \ldots$, there exists an index m_1 such that $y \in D_{m_1}$. From $y \bar{\in} R$ and (23), we find further that $y \bar{\in} (E.D_{m_1} - S_{m_1})$; but since $y \in E.D_{m_1}$, we have $y \in S_{m_1}$, and so, from 24, there exists an index m_2, such that $y \in D_{m_1, m_2}$.

Continuing this argument, we obtain an infinite sequence of indices $m_1, m_2, m_3, \ldots$ such that

$$y \in D_{m_1, m_2, \ldots, m_k}, \text{ for } k = 1, 2, \ldots,$$

which, on account of (22), gives $y \in F(x)$, where x is a number defined by (21); but, according to the definition of the set T, this proves that $y \in T$. We have thus proved that $E-R \subset T$, which gives $E-T\subset R$, and, since R is countable at most, the set $E-T$ is countable at most.

To prove Theorem 85 it will, therefore, be sufficient to show that $N\, h_f\, T$.

It follows from the definition of the set T that $f(N)=T$. Let now x and x' be two different numbers of the set N. Suppose that the developments of the numbers x and x' as continued fractions differ first in the r^{th} terms, and that the denominator of this term for x' is $m_r'\neq m_r$. From (22) we have $F(x)\subset \overline{D}_{m_1, m_2, \ldots, m_{r+1}}$, and so, from 3, $F(x)\subset D_{m_1, m_2, \ldots, m_r}$; similarly, $F(x')\subset D_{m_1, m_2, \ldots, m_{r-1}, m_r'}$, and so, from $m_r\neq m_r'$ and 2, $F(x).F(x')=0$, and this proves that $f(x)\neq f(x')$, since $f(x)\in F(x)$ and $f(x')\in F(x')$. Hence f is a biuniform function in N.

We shall now show that the function f is continuous in N. Let x be a given element of the set N and η an arbitrary positive number. Select a positive integer k such that $\frac{1}{k}<\eta$. Corresponding to the numbers x and k there exists, as is well known, a positive number δ such that every number x' of N satisfying the inequality

$$(24a)\qquad |x-x'|<\delta$$

can be developed as a continued fraction with its first k terms the same as the first k terms of the corresponding development of x. On account of (21) and (22), we conclude that $F(x)\subset \overline{D}_{m_1, m_2, \ldots, m_k}$ and $F(x')\subset \overline{D}_{m_1, m_2, \ldots, m_k}$, and this, on account of $f(x)\in F(x)$, $f(x')\in F(x')$, $\frac{1}{k}<\eta$, and 1, gives

$$(25)\qquad |f(x)-f(x')|<\eta.$$

Hence, corresponding to every number x of N and every positive number η, there exists a number $\delta>0$ such that the inequality (24a) implies the inequality (25), for numbers $x'\in N$. This, however, establishes the continuity of the function f in the whole set N. Hence, to prove that $N\, h_f\, T$, it will be sufficient to show that the inverse of the function f is continuous in the set T.

Let $\phi(y)$ denote a function defined in T and inverse to the function f, and let y be a given number of T. There exists, therefore, a number x of N, completely determined, such that $f(x)=y$. If now (21) be the development of x as a continued fraction, then, from $f(x) \epsilon F(x)$, (22), and 3, we have

(26) $$y \,\epsilon\, D_{m_1, m_2, \ldots, m_k}, \text{ for } k=2, 3, 4, \ldots$$

Let δ denote a positive number. Corresponding to the numbers x and δ, there exists, as is well known, an integer $k>1$ such that every number x', which has the same first k terms in its development as a continued fraction as x has, satisfies the inequality

(27) $$|x-x'| < \delta.$$

Let now y' denote a number of the set T within the interval $D_{m_1, m_2, \ldots, m_k}$, and let

$$x' = \frac{1}{m_1'+}\;\frac{1}{m_2'+}\;\frac{1}{m_3'+}\;\cdots$$

be the development of the number x' as a continued fraction. Suppose that the developments of the numbers x and x' differ first in their r^{th} terms. We have, therefore, $m_i'=m_i$, for $i=1, 2, \ldots, r-1$, and $m_r' \neq m_r$. From (22) and 3, we have $y \,\epsilon\, D_{m_1, m_2, \ldots, m_{r-1}, m_r}$, and $y' \,\epsilon\, D_{m_1, m_2, \ldots, m_{r-1}, m_r'}$, and so, since $m_r \neq m_r'$, it follows from 2 that $y' \,\bar{\epsilon}\, D_{m_1, m_2, \ldots, m_r}$, and so certainly, from 3, $y' \,\bar{\epsilon}\, D_{m_1, m_2, \ldots, m_i}$, for $i \geq r$; but, by hypothesis, $y' \,\epsilon\, D_{m_1, m_2, \ldots, m_k}$; hence, we must have $r>k$, *i.e.* the number x' has the same first k terms in its development as a continued fraction as x has. Owing to the definition of the number k we obtain, therefore, the inequality (27). We have thus proved that, corresponding to every number y of T, there exists an open interval $D=D_{m_1, m_2, \ldots, m_k}$ containing y (from (26)) and such that every number y' of the set $T.D$ satisfies the inequality (27), where $x=\phi(y)$, $x'=\phi(y')$; this establishes the continuity of the function ϕ in the set T. The relation $N\, h_f\, T$ is, therefore, proved.

But the set N is homeomorphic with the set of all irrational numbers; hence, Theorem 85 is proved.

It follows readily from Theorem 85 that *two non-countable linear sets* G_δ *are homeomorphic, except for a countable aggregate of their elements.*

From Theorems 84a and 85 we get immediately the following

COROLLARY. *Every open and compact set* (consisting of elements of a metric space) *is the sum of two sets, one of which is a null set or a continuous and biuniform transform of the set of all irrational numbers and the other is countable at most.*

75. Denote by L the family of all sets E (contained in a metric space for which condition (W) of § 55 applies) satisfying the following condition:

The set E is the sum of two sets, one of which is a null set or a continuous and biuniform transform of the set of all irrational numbers, and the other is countable at most.

We shall show that the sum of a countable aggregate of mutually exclusive sets belonging to L, itself belongs to L.

Let E be a set which is the sum $E=E_1+E_2+\ldots$ of mutually exclusive sets, where $E_n \epsilon L$, for $n=1, 2, \ldots$. We can, therefore, write $E_n=P_n+Q_n$, where P_n is a null set or a continuous and biuniform transform of the set of all irrational numbers, and Q_n is a set countable at most.

But the set of all irrational numbers is, as is well known, homeomorphic with the set N_n of all irrational numbers of the interval $(n, n+1)$. Hence P_n is a null set or a continuous and biuniform transform of the set N_n. Denote by S the sum of all sets N_n, extending over the indices n, for which P_n is not null. The set S will obviously be a null set, or homeomorphic with the set of all irrational numbers, and the set $P=P_1+P_2+P_3+\ldots$ will be a continuous and biuniform transform of the set S. Moreover, since $E=P+Q$, where $Q=Q_1+Q_2+\ldots$ is a set countable at most, we have $E \epsilon L$.

We shall next show that the product of a countable aggregate of sets belonging to L, itself belongs to L.

Let E be a set such that $E=E_1.E_2.E_3\ldots$, where $E_n \epsilon L$, for $n=1, 2, \ldots$. We may, therefore, write $E_n=P_n+Q_n$, where the sets P_n and Q_n have the same meaning as above. If the set P_n were null for some n, then the set E_n and so also the set E would be countable at most and so would belong to the family L.

We may, therefore, put $P_n = f_n(N)$ for every integer n, where f_n is a continuous and biuniform function in the set N of all irrational numbers in the interval $(0, 1)$.

From $E = E_1.E_2.E_3 \ldots$, $E_n = P_n + Q_n$, and from the fact that the sets $Q_n (n = 1, 2, \ldots)$ are countable at most, it follows that we may write $E = P + R$, where $P = P_1.P_2.P_3 \ldots$, and R is countable at most, since it is contained in the set $Q_1 + Q_2 + \ldots$. Hence, to prove that $E \epsilon L$, it will be sufficient to show that $P \epsilon L$.

Let n denote a given positive integer, t a given irrational number of the interval $(0, 1)$, and

$$(28) \qquad t = \frac{1}{k_1 +} \; \frac{1}{k_2 +} \; \frac{1}{k_3 +} \cdots$$

its development as a continued fraction. Put

$$(29) \quad \phi_n(t) = \frac{1}{k_{2^{n-1}} +} \; \frac{1}{k_{3.2^{n-1}} +} \; \frac{1}{k_{5.2^{n-1}} +} \cdots \frac{1}{k_{(2m-1)2^{n-1}} +} \cdots$$

It follows from the properties of continued fractions that the functions $\phi_n(t)$ are continuous in the set N $(n = 1, 2, \ldots)$, and that $\phi_n(N) = N$. Put $F_n(t) = f_n(\phi_n(t))$, for $t \epsilon N$; hence, the functions $F_n(t)$ are continuous in the set N, and $P_n = F_n(N)$, for $n = 1, 2, \ldots$

It will be shown that $P = F_1(T)$, where T is the set of all numbers t of N for which

$$(30) \qquad F_n(t) = F_1(t), \; n = 1, 2, \ldots$$

Assume that $x \epsilon P$. Hence, for every integer n, $x \epsilon P_n = f_n(N)$, and so, for every integer n, there exists a number t_n such that $f_n(t_n) = x$. Let

$$(31) \qquad t_n = \frac{1}{k_{n,1} +} \; \frac{1}{k_{n,2} +} \; \frac{1}{k_{n,3} +} \cdots$$

be the development of the number t_n as a continued fraction.

Every positive integer p may be, as is well known, expressed in the form

(32) $$p=(2m_p-1)\cdot 2^{n_p-1}$$

and that in one way only, where m_p and n_p are positive integers. The numbers m_p and n_p are, therefore, defined completely by the number p. Put

(33) $$k_p=k_{n_p,\,2m_p-1},\text{ for } p=1,\,2,\,\ldots,$$

and let t be a number defined by (28).

In order to show that $x \in F_1(T)$, it will be sufficient to show, owing to the definition of T, that $x \in F_n(t)$, for $n=1, 2, \ldots$

From (32) and (33) we have for all integers m and n

$$k_{(2m-1)2^{n-1}}=k_{n,\,2m-1},$$

and so, from (29) and (31),

$$\phi_n(t)=t_n,\text{ for } n=1,\,2,\,\ldots,$$

whence $F_n(t)=f_n(\phi_n(t))=f_n(t_n)=x$, for $n=1, 2, \ldots$

We have, therefore, proved that the relation $x \in P$ implies the relation $x \in F_1(T)$; consequently, $P \subset F_1(T)$.

Let now x denote an element of $F_1(T)$. It follows from the definition of the set T that there exists a number t of the set N such that

$$x=F_n(t),\text{ for } n=1,\,2,\,\ldots$$

But $P_n=F_n(N)$; hence, $x \in P_n$, for $n=1, 2, \ldots$, and so $x \in P$. We have thus proved that $F_1(T) \subset P$, and since we have seen that $P \subset F_1(T)$, we have $P=F_1(T)$.

Since $T \subset N$ and the function $F_1(t)$ is continuous in N, $F_1(t)$ is also continuous in T. It will be shown that $F_1(t)$ is biuniform in T.

In fact, let t and t' be two different numbers of T. Let

(34) $$t'=\frac{1}{k_1'+}\,\frac{1}{k_2'+}\,\frac{1}{k_3'+}\,\cdots$$

be the development of t' as a continued fraction. Since $t \neq t'$, the developments (28) and (34) must first differ in some term, the p^{th} say. Hence $k_p \neq k_p'$. Thus, from (32) and (29), $\phi_{n_p}(t) \neq \phi_{n_p}(t')$, and so, since f_n is biuniform in N, we obtain $f_{n_p}(\phi_{n_p}(t)) \neq f_{n_p}(\phi_{n_p}(t'))$, *i.e.* $F_{n_p}(t) \neq F_{n_p}(t')$; on account of (30) (since $t \in T$ and $t' \in T$), this gives $F_1(t) \neq F_1(t')$, as required.

We have, therefore, proved that the set P is a continuous and biuniform transform of the set T. We shall next show that T is a G_δ. Since N is a G_δ and $T \subset N$, it will be sufficient to show that T is closed in the set N.

Let t_0 be an element of the set $T'.N$; we must show that $t_0 \epsilon T$. Since $t_0 \epsilon T'$, there exists an infinite sequence t_k of elements of T such that $\lim_{k \to \infty} t_k = t_0$. From $t_k \epsilon T$ and the definition of the set T, we have $F_n(t_k) = F_1(t_k)$, for $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$, and so, since $F_n(t)$ is continuous in N, and $t_0 \epsilon N$, and $\lim_{k \to \infty} t_k = t_0$, we have

$$F_n(t_0) = F_1(t_0), \text{ for } n = 1, 2, \ldots;$$

this proves (owing to the definition of the set T) that $t_0 \epsilon T$.

We have thus proved that the set T is a G_δ. We may, therefore, by Theorem 85, write $T = X + Y$, where X is a null set or a set homeomorphic with the set of all irrational numbers, and Y is countable at most. Hence, from $P = F_1(T)$, we get $P = F_1(X) + F_1(Y)$, where $F_1(Y)$ is evidently countable at most. If now X be a null set, then P is countable at most, and so $P \epsilon L$. If $X \neq 0$, we may write $X = \psi(N)$, where ψ is a continuous and biuniform function in the set N, and since $P = F_1(X)$, and F_1, as we have seen, is continuous and biuniform in T and so certainly in $X \subset T$, therefore, $P = F_1(\psi(N)) = \Phi(N)$, where $\Phi(t) = F_1(\psi(t))$ is a function continuous and biuniform in the set N. This leads at once to the conclusion that $P \epsilon L$.

We have, therefore, proved that the family $F = L$ satisfies conditions 2] and 3] of § 62.

It follows from the corollary at the end of § 74 that every open and compact set belongs to the family L. Let E be a compact set G_δ; we may, therefore, write $E = G_1.G_2.G_3 \ldots$, where the sets G_n $(n = 1, 2, \ldots)$ are open. Since E is compact, and, therefore, bounded, there exists a sphere K containing E. By condition (W) of § 55, spheres are compact sets; the open sets $\Gamma_n = G_n.K$ will, therefore, also be compact; since $E \subset K$ and $E = G_1.G_2 \ldots$, we have obviously $E = \Gamma_1.\Gamma_2.\Gamma_3 \ldots$. Each set Γ_n $(n = 1, 2, \ldots)$, being open and compact, belongs to L; hence, from property 3], the product of all these sets also belongs to L. Hence, every compact set G_δ belongs to L.

Let now U be an open set (not necessarily compact). Let p denote an element of the metric space M under consideration, and put $H_1=K(p, 1)$ and $H_n=K(p, n)-K(p, n-1)$, for $n=2, 3, \ldots$; we have obviously $M=H_1+H_2+H_3+\ldots$, where the sets $H_n (n=1, 2, \ldots)$ are compact and mutually exclusive sets G_δ. We have, therefore, $U=U.H_1+U.H_2+\ldots$, where the terms of the sum are compact and mutually exclusive sets G_δ. As proved above, compact sets G_δ belong to L; consequently, by property 2] of the family L, the set U belongs to L.

We have thus proved that every open set belongs to L, *i.e.* the family $F=L$ satisfies condition 1] of § 62. The family L satisfies the conditions 1], 2], and 3] of § 62, and, therefore, every Borel set belongs to the family L.

On the other hand, it follows, from Theorem 82 and the definition of the family L, that every set belonging to the family L is a Borel set. The family L is, therefore, identical with the family of all Borel sets. We may, therefore, state

Theorem 86. *In order that a set E be a Borel set, it is necessary and sufficient that it be the sum of two sets, one of which is a null set or a continuous and biuniform transform of the set of all irrational numbers and the other is countable at most.*

We note that Theorem 86 leads immediately to the result that every non-countable Borel set has the potency of the continuum (a result obtained along different lines towards the close of § 70; Corollary 2, Theorem 79).

Let now E denote a given Borel set, and f a function continuous and biuniform in E. If E is countable at most, then so also is $T=f(E)$, which is, therefore, a Borel set. If E is non-countable, then, by Theorem 86, we may write $E=\phi(N)+P$, where ϕ is a continuous and biuniform function in the set N of all irrational numbers, and P is countable at most. We shall then have $T=f(E)=f(\phi(N))+f(P)$. Put $F(t)=f(\phi(t))$, for $t \in N$; it follows at once from the properties of the functions f and ϕ that the function F is continuous and biuniform in N. Hence, the relation $T=F(N)+$

$f(P)$ proves that T is a Borel set by Theorem 86 (and since $f(P)$ is countable at most). We thus get

Theorem 87 (Lusin).[20] *A continuous and biuniform transform of a Borel set is a Borel set.*

This theorem may be considered as a generalization of Theorem 69.

76. Let E denote a non-countable Borel set. By Theorem 86, we have $E \supset \phi(N)$, where ϕ is a continuous and biuniform function in the set N of all irrational numbers. But, by Theorem 79, the set N contains a perfect subset $D \neq 0$, and it may be supposed that D is compact (for it would be sufficient to apply Theorem 79 to the set of all irrational numbers in the interval (0, 1) which is contained in N). From $D \subset N$ and $E \supset \phi(N)$, we have $E \supset \phi(D)$. But, by Theorem 42, the set $\phi(D)$ will not only be a continuous and biuniform transform of the (closed and compact) set D but its homeomorphic transform. Hence

Every non-countable Borel set contains a subset Q which is homeomorphic with a certain perfect and compact set $D \neq 0$ of real numbers. But, by Theorem 85 (since the set D, being perfect, is a set G_δ), we may write $D = E + P$, where $E \, h \, N$ and P is countable at most. Since $Q \, h \, D$, there corresponds to the subset P of D a subset R of Q (countable at most), where $(Q - R) \, h \, E$ and so, since $E \, h \, N$, we have $(Q - R) \, h \, N$. But, from $Q \, h \, D$, since D is closed and compact, it follows by Theorem 41 that Q is closed, and since R, being countable, is an F_σ, the set $H = Q - R = Q.CR$ is a G_δ. Hence

Every non-countable Borel set contains a subset which is a G_δ and which is homeomorphic with the set of all irrational numbers. The metric space under consideration (which satisfies condition (W) of § 55), being a closed set, is obviously a Borel set. We have, therefore,

Theorem 88. *In every metric space, in which bounded sets are compact, there exist sets G_δ which are homeomorphic with the set of all irrational numbers.*

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[20]See *Fund. Math.*, vol. X, p. 60.

It follows from the above theorem that there exist in metric spaces in which bounded sets are compact, all topological types which exist in the set of all irrational numbers. In particular, we may deduce the existence in such spaces of Lebesgue's sets O and F of any class $\alpha < \Omega$ (§ 61) and the existence of analytical sets, which are not Borel sets; for it is sufficient to refer to the existence of such sets in a linear space.[21]

We shall next deduce an important corollary from Theorem 88. There exists by this theorem, in the metric space considered, a set H which is a G_δ and homeomorphic with the set N of all irrational numbers. But, every analytical set is, by Theorem 73, a continuous transform of the set N. Consequently, every analytical set of the space considered is a continuous transform of the set H, and so of a certain set G_δ of this space. On the other hand, by Corollary 1 of § 68, every continuous transform of a G_δ is an analytical set. Hence, we have

Theorem 89. *Analytical sets* (in a metric space in which bounded sets are compact) *are continuous transforms of sets* G_δ (contained in that space), *and conversely.*

Furthermore, it may be easily deduced from Theorem 86 that *Borel sets are continuous and biuniform transforms of sets* G_δ, *and conversely.*

77. Let E denote an analytical set which is the nucleus of the system $S[E_{n_1, n_2, \ldots, n_k}]$, consisting of Borel sets but not necessarily regular.

For every finite combination of indices $n_1, n_2, \ldots, n_k$, put

$$(1) \qquad E^0_{n_1, n_2, \ldots, n_k} = E_{n_1, n_2, \ldots, n_k},$$

and further, for every ordinal number $\alpha < \Omega$,

$$(2) \qquad E^{\alpha+1}_{n_1, n_2, \ldots, n_k} = E^{\alpha}_{n_1, n_2, \ldots, n_k} \cdot \sum_{n=1}^{\infty} E^{\alpha}_{n_1, n_2, \ldots, n_k, n},$$

[21] See my book: *Functions representable analytically* (in Polish), Lwow, 1925, pp. 66 and 89.

and finally, for every ordinal number $\alpha < \Omega$ of the second kind,

$$(3) \qquad E^{\alpha}_{n_1, n_2, \ldots, n_k} = \prod_{\xi < \alpha} E^{\xi}_{n_1, n_2, \ldots, n_k},$$

where the product Π ranges over all the ordinal numbers $\xi < \alpha$. The sets $E^{\alpha}_{n_1, n_2, \ldots, n_k}$, defined above by transfinite induction, are obviously Borel sets (§ 62) for every finite combination of indices $n_1, n_2, \ldots, n_k$, and every ordinal number $\alpha < \Omega$.

From (2) and (3) it follows readily by transfinite induction that

$$(4) \qquad E^{\alpha}_{n_1, n_2, \ldots, n_k} \subset E^{\beta}_{n_1, n_2, \ldots, n_k}, \text{ for } \alpha \geq \beta.$$

Put

$$(5) \qquad S^{\alpha} = \sum_{n=1}^{\infty} E^{\alpha}_{n}$$

and

$$(6) \qquad T^{\alpha} = \sum_{(n_1, n_2, \ldots, n_k)} (E^{\alpha}_{n_1, n_2, \ldots, n_k} - E^{\alpha+1}_{n_1, n_2, \ldots, n_k}),$$

where the sum (6) ranges over all the finite combinations $n_1, n_2, \ldots, n_k$ of positive integers.

The sets (5) and (6) and their difference $S^{\alpha} - T^{\alpha}$ are evidently Borel sets (§ 62) for $\alpha < \Omega$.

We shall show that

$$(7) \qquad E = \sum_{\alpha < \Omega} (S^{\alpha} - T^{\alpha}) = \prod_{\alpha < \Omega} S^{\alpha},$$

where the summation and the multiplication range over all ordinal numbers $\alpha < \Omega$.

Let α be a given ordinal number $< \Omega$ and x an element of the set $S^{\alpha} - T^{\alpha}$. Hence,

$$(8) \qquad x \in S^{\alpha},$$

and

$$(9) \qquad x \bar{\in} T^{\alpha}.$$

From (8) and (5), we deduce the existence of a positive integer m_1 such that $x \in E^{\alpha}_{m_1}$. Moreover, from (9) and (6), it follows that $x \bar{\in} (E^{\alpha}_{m_1} - E^{\alpha+1}_{m_1})$ (since the set $E^{\alpha}_{m_1} - E^{\alpha+1}_{m_1}$ is one of the terms of the sum (6)); since $x \in E^{\alpha}_{m_1}$, we have $x \in E^{\alpha+1}_{m_1}$. But, from (2),

$$E^{\alpha+1}_{m_1} = E^{\alpha}_{m_1} \cdot \sum_{n=1}^{\infty} E^{\alpha}_{m_1, n},$$

and so, since $x \in E^{\alpha+1}_{m_1}$, there exists an index m_2 such that $x \in E^{\alpha}_{m_1, m_2}$. Furthermore, on account of (9) and (6), we have $x \bar{\in} (E^{\alpha}_{m_1, m_2} - E^{\alpha+1}_{m_1, m_2})$, and so, since $x \in E^{\alpha}_{m_1, m_2}$, we get $x \in E^{\alpha+1}_{m_1, m_2}$. But, from (2),

$$E^{\alpha+1}_{m_1, m_2} \subset \sum_{n=1}^{\infty} E^{\alpha}_{m_1, m_2, n},$$

and so, since $x \in E^{\alpha+1}_{m_1, m_2}$, there exists an index m_3 such that $x \in E^{\alpha}_{m_1, m_2, m_3}$. Continuing the argument in this manner, we obtain an infinite sequence of indices $m_1, m_2, m_3, \ldots$ such that

$$x \in E^{\alpha}_{m_1, m_2, \ldots, m_k}, \text{ for } k = 1, 2, \ldots$$

This implies, on account of (1) and (4) (for $\beta = 0$), that

$$x \in E_{m_1, m_2, \ldots, m_k}, \text{ for } k = 1, 2, \ldots,$$

and so (from the definition of the set E) $x \in E$.

We have thus proved that $(S^{\alpha} - T^{\alpha}) \subset E$ for every ordinal number $\alpha < \Omega$; it follows that

$$\sum_{\alpha < \Omega} (S^{\alpha} - T^{\alpha}) \subset E. \tag{10}$$

Let now x denote an element of the set E. Hence there exists an infinite set of indices $m_1, m_2, m_3, \ldots$ such that

$$x \in E_{m_1, m_2, \ldots, m_k}, \text{ for } k = 1, 2, \ldots \tag{11}$$

It will be shown that

$$x \in E^{\alpha}_{m_1, m_2, \ldots, m_k}, \text{ for } k = 1, 2, \ldots \tag{12}$$

for every ordinal number $\alpha < \Omega$. For $\alpha = 0$, (12) is true on account of (11) and (1). Let now β denote an ordinal number $< \Omega$, and suppose that (12) is true for every ordinal number $\alpha < \beta$. If β is a number of the second kind, it follows from (3) that (12) is true for the number β. If β is a number of the first kind, we may put $\beta = \alpha + 1$, where $\alpha < \beta$, and from (2) we then have (for every positive integer k)

$$E^{\beta}_{m_1, m_2, \ldots, m_k} = E^{\alpha}_{m_1, m_2, \ldots, m_k} \cdot \sum_{n=1}^{\infty} E^{\alpha}_{m_1, m_2, \ldots, m_k, n} \supset E^{\alpha}_{m_1, m_2, \ldots, m_k} \cdot E^{\alpha}_{m_1, m_2, \ldots, m_k, m_{k+1}},$$

and so $x \in E^{\beta}_{m_1, m_2, \ldots, m_k}$, since from (12), $x \in E^{\alpha}_{m_1, m_2, \ldots, m_k}$, and $x \in E^{\alpha}_{m_1, m_2, \ldots, m_k, m_{k+1}}$. Relation (12) is, therefore, proved by transfinite induction for every ordinal number $\alpha < \Omega$.

In particular, it follows from (12) that $x \in E^{\alpha}_{m_1}$, for $\alpha < \Omega$, and so from (5) $x \in S^{\alpha}$, for $\alpha < \Omega$. We have, therefore, proved that $E \subset S^{\alpha}$, for $\alpha < \Omega$, whence

(13) $$E \subset \prod_{\alpha < \Omega} S^{\alpha}.$$

Furthermore,

(14) $$\prod_{\alpha < \Omega} T^{\alpha} = 0.$$

For suppose that (14) is not true. Consequently, there exists an element x such that

(15) $$x \in T^{\alpha}, \text{ for } \alpha < \Omega.$$

It follows from (15) and (6) that corresponding to every ordinal number $\alpha < \Omega$ there exists at least one set of indices $n_1, n_2, \ldots, n_k$ (dependent on α) such that

$$x \in (E^{\alpha}_{n_1, n_2, \ldots, n_k} - E^{\alpha+1}_{n_1, n_2, \ldots, n_k}).$$

But the set of all finite combinations of indices $n_1, n_2, \ldots, n_k$ is countable, whereas the set of all ordinal numbers $\alpha < \Omega$ is non-countable. From this we deduce the existence of a set of indices $p_1, p_2, \ldots, p_r$ and also of two ordinal numbers $\xi < \Omega$ and $\eta < \xi$ such that

(16) $$x \in (E^{\xi}_{p_1, p_2, \ldots, p_r} - E^{\xi+1}_{p_1, p_2, \ldots, p_r}),$$

(17) $$x \in (E^{\eta}_{p_1, p_2, \ldots, p_r} - E^{\eta+1}_{p_1, p_2, \ldots, p_r}).$$

It follows from (16) and (17) that

$$x \in E^{\xi}_{p_1, p_2, \ldots, p_r} \text{ and } x \bar{\in} E^{\eta+1}_{p_1, p_2, \ldots, p_r},$$

contrary to (4), since (from $\eta < \xi$) we have $\eta + 1 \leq \xi$.

Hence, (14) is proved.

Let now x be an element of the set E. On account of (14), there exists an ordinal number $\alpha < \Omega$ such that $x \bar{\epsilon} T^\alpha$. But, from (13) and the fact that $x \epsilon E$, we have $x \epsilon S^\alpha$. Hence, $x \epsilon S^\alpha - T^\alpha$. We have thus shown that, if $x \epsilon E$, there exists an ordinal number $\alpha < \Omega$ such that $x \epsilon (S^\alpha - T^\alpha)$. This establishes the fact that

(18) $$E \subset \sum_{\alpha<\Omega} (S^\alpha - T^\alpha).$$

Relations (10) and (18) give the first part of (7). Let now x denote an element of the set

(19) $$P = \prod_{\alpha<\Omega} S^\alpha.$$

From (14), there exists an ordinal number $\alpha < \Omega$ such that $x \bar{\epsilon} T^\alpha$. But, since $x \epsilon P$, we have $x \epsilon S^\alpha$ from (19); consequently, $x \epsilon S^\alpha - T^\alpha$, and so, by (10), $x \epsilon E$. We have, therefore, $P \subset E$, and since from (13) and (19) $E \subset P$, we have $E = P$; this, on account of (19) and the first part of (7), gives the second part of (7).

Relation (7) is, therefore, proved completely.

It follows from (7) that *every analytical set is both the sum and the product of* $\aleph_1$ *Borel sets.* From this it follows at once (on passing to complements) that *the complement of an analytical set is both the product and the sum of* $\aleph_1$ *Borel sets.*

From (7) we get immediately

(20) $$CE = \sum_{\alpha<\Omega} CS^\alpha.$$

If none of the terms of the sum (20) is non-countable, then (since the sum (20) contains $\aleph_1$ terms) the set CE has potency $\aleph_1$ at least; if, however, there exists among the terms of (20) a non-countable set, then, being a Borel set, it must contain, by Theorem 79, a perfect subset not null, which on account of (20) is a subset of the set CE. Hence, we obtain

Theorem 90. *The complement of an analytical set not containing a perfect non-null subset has potency* $\aleph_1$ *at most.*

This theorem would be trivial if the potency of the continuum were equal to $\aleph_1$. As it is, we do not know (without the hypothesis that $2^{\aleph_0}=\aleph_1$) whether there exist analytical sets whose complements have potency $\aleph_1$.

Furthermore, we are not able to establish (even with the hypothesis that $2^{\aleph_0}=\aleph_1$) whether or not every non-countable complement of an analytical set contains a perfect, non-null subset.

It follows easily from Theorem 90 that the complement of an analytical set cannot have a potency intermediate between $\aleph_1$ and the potency of the continuum.

Let now E denote an analytical set and f a function continuous in the set $T=CE$, and put $Q=f(T)$; this will be a continuous transform of the complement of an analytical set. We shall then have from (20)

$$Q=f(T)=f(CE)=\sum_{\alpha<\Omega} f(CS^\alpha). \tag{21}$$

The sets CS^α are Borel sets; their continuous transforms $f(CS^\alpha)$ are, therefore, analytical sets (Corollary 1 to Theorem 74) and so, as shown above, sums of $\aleph_1$ Borel sets. Hence, it follows from (21) (since $\aleph_1 \cdot \aleph_1=\aleph_1$) that the set Q is the sum of $\aleph_1$ Borel sets. We have thus proved that a continuous transform of the complement of an analytical set is the sum of $\aleph_1$ Borel sets.[22] As before, from the above we may deduce for complements of analytical sets

Theorem 90a. *A continuous transform of the complement of an analytical set, not containing a perfect, non-null subset, has potency* $\aleph_1$ *at most.*

We do not know, however, whether an analogous theorem is true for complements of continuous transforms of complements of analytical sets.

78. *Projective* sets of a given metric space are said to be sets obtained from Borel sets on successive application of the following two operations: continuous mapping and taking the complement of a set in hand. Strictly speaking, the family **P** of all projective sets (of the metric space considered) is the smallest family F of sets to satisfy the following three conditions:

1. Every Borel set belongs to F.
2. A continuous transform of a set belonging to F belongs to F.
3. The complement of a set belonging to F belongs to F.

All projective sets may be divided into a countable aggregate of classes (not necessarily mutually exclusive) in the following

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[22]It is not known, however, whether or not every continuous transform of the complement of an analytical set is also the product of $\aleph_1$ Borel sets.

way. Denote by K_0 the class consisting of all Borel sets (of the metric space considered). Let now n be a positive integer, and suppose the class K_{n-1} to be already defined. If n be odd, the class K_n will be understood to be the class of all continuous transforms of sets belonging to the class K_{n-1}, and if n be even, then K_n will be the class of all complements of sets belonging to K_{n-1}. Evidently, the family

$$F=K_0+K_1+K_2+\dots$$

is the smallest family of sets, which satisfies conditions 1, 2, and 3; hence, $F=\mathbf{P}$.

It follows immediately from Theorem 89 that K_1 is the class of all analytical sets; hence, K_2 is the class of all complements of analytical sets, and K_3 the class of all continuous transforms of complements of analytical sets. As a consequence of Theorem 90, we find that non-countable sets belonging to the class K_3 have either potency $\aleph_1$ or that of the continuum. Nothing is known, however, about the potency of the class K_4. It can be shown that the family **P** contains a continuum of different sets. The properties of projective sets have been studied very little hitherto.[23]

Another generalization of analytical sets is represented by the smallest family F of sets to satisfy, in addition to conditions 1, 2, and 3, also condition

4. The sum of a countable aggregate of sets belonging to F, belongs to F.

It could be shown that such a family F of sets contains the family **P**, but not conversely.

79. A set E is said to be *dense on a set* T, if

$$(1)\qquad T\subset(\overline{T.E}).$$

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[23] In addition to the 5 papers by Lusin in *Comptes Rendus*, 1925 (of May 4, 25, June 15, July 13, Aug. 17), the following articles treat projective sets: W. Sierpiński, *Fund. Math.*, vol. VII, pp. 237-243, vol. XI, pp. 122, 126, vol. XII, pp. 1-3; *Comptes Rendus*, vol. CLXXXV, p. 833; N. Lusin, *Fund. Math.*, vol. X, pp. 89-94; *Comptes Rendus*, vol. CLXXXV, p. 835.

As is easily seen, *in order that a set E be dense on a set T, it is necessary and sufficient that every open set containing an element of the set T should contain an element of the set T.E.*

In fact, suppose that (1) is satisfied, and let U be an open set containing an element p of the set T. From (1), $p \in [T.E+(T.E)']$. Hence, if $p \bar{\epsilon}\, T.E$ then $p \in (T.E)'$, and since $p \in U$ and U is open, there exists an element q of the set $T.E$ such that $q \in U$. Hence, in any case, $U.T.E \neq 0$, and this proves that the condition is necessary.

Suppose, on the other hand, that (1) is not true. There exists, therefore, an element p of the set T such that $p \bar{\epsilon}\, (\overline{T.E})$, and so $p \in U$, where $U = C(\overline{T.E})$ is an open set. But, from the definition of U, we have $U.(\overline{T.E}) = 0$, and so certainly $U.T.E = 0$. In this case the open set U would contain an element p of T but would not contain any element of $T.E$, and so our condition would not be satisfied. This proves that the condition is sufficient.

A set E is said to be *nowhere dense on a set T, if*

$$T \subset \overline{(T-(\overline{T.E}))}. \tag{2}$$

In order that a set E be nowhere dense on the set T, it is necessary and sufficient that every open set containing an element of T, should contain an open subset which contains an element of the set T but does not contain any element of the set T.E.

In fact, suppose that (2) is satisfied, and let U be an open set containing an element p of T. Put $V = U.C(\overline{T.E})$; this will be an open set. From (2), $p \in \overline{(T-(\overline{T.E}))}$, and so, since $p \in U$, and U is open, there exists an element q such that $q \in U$ and $q \in (T-(\overline{T.E}))$; hence, from the definition of V, $q \in V$, and so $V \neq 0$. But, from the definition of V, we have $V.T.E = 0$; the above condition is, therefore, necessary. Again, suppose that (2) does not hold, and put $U = C\overline{(T-(\overline{T.E}))}$; this will be an open set. Since (2) is not true, there exists an element p of T such that $p \in U$. Let now V be any open set contained in U and containing an element of the set T, q say. From $V \subset U$ and the definition of the set U, we find that $V.(T-\overline{T.E})) = 0$; since $q \in V.T$, we must have $q \in V.(\overline{T.E})$, and so, since V is open, there exists an element r such that

$r \in V.T.E$. The open set U contains, therefore, an element p of the set T, and every open set V containing an element of T contains also an element of the set $T.E$; the condition is, therefore, not satisfied. We have thus proved that the condition is necessary.

It follows immediately from the above that the sum of two (and, therefore, of any finite number of) sets nowhere dense on the set T is nowhere dense on the set T.

Obviously, a subset of a set nowhere dense on T is a set nowhere dense on T.

A set which is the sum of a countable aggregate of sets nowhere dense on T is said to be *of the first category* (Baire) *on the set* T. A set which is not of the first category on the set T is said to be *of the second category* on the set T. Obviously, a subset of a set of the first category on T is a set of the first category on T. It is easily seen that the sum of a finite or countable aggregate of sets of the first category on T is a set of the first category on T.

Theorem 91. *A closed non-null set cannot be of the first category on itself.*

Proof. Let T be a closed set not null, and E a set of the first category on T; it will be sufficient to show that $T-E\neq 0$.

Since the set E is of the first category on T, we may write $E=E_1+E_2+E_3+\ldots$, where the sets $E_n (n=1, 2, \ldots)$ are nowhere dense on T.

Since $T\neq 0$, there exists an element p of T. Put $U=K(p, 1)$; since E_1 is nowhere dense on T, there exists an open set $V\subset U$ such that $V.T\neq 0$ but $V.T.E_1=0$. Since $V.T\neq 0$, there exists an element p_1 of the set $V.T$. Since $p_1 \in V$ and V is open, there exists a positive number $r_1<1$ such that $U_1=K(p_1, r_1)\subset \overline{U}_1\subset V$. Since $p_1 \in U_1.T$ and U_1 is open, and E_2 is nowhere dense on T, there exists an open set $V_1\subset U_1$ such that $V_1.T\neq 0$ and $V_1.T.E_2=0$. Since $V_1.T\neq 0$, there exists an element p_2 of the set $V_1.T$. Since $p_2 \in V_1$ and V_1 is open, there exists a positive number $r_2<\frac{1}{2}$ such that $U_2=K(p_2, r_2)\subset \overline{U}_2\subset V_1$. Proceeding thus indefinitely we obtain an infinite sequence $p_1, p_2, p_3, \ldots$ of elements of T, and an infinite sequence $U_1, U_2, U_3, \ldots$ of open sets such that $U_n\supset \overline{U}_{n+1}$, $p_n \in U_n$, and

$U_n.T.E_n=0$; also $\delta(U_n)=2r_n<\frac{2}{n}$, for $n=1, 2, \ldots$ This leads immediately to the conclusion that $\rho(p_{n+k}, p_n)<\frac{2}{n}$, for $n=1, 2, \ldots$, and $k=1, 2, \ldots$, and so, by Theorem 59, the sequence $p_1, p_2, \ldots$ has a limit $p=\lim_{n\to\infty} p_n$, which, on account of T being closed, is an element of T. But from $p_{n+1} \epsilon U_{n+1} \subset \overline{U}_{n+1}$, for $n=1, 2, \ldots$ and the fact that $\overline{U}_{n+1}$ is closed, we conclude that $p \epsilon \overline{U}_{n+1}$, and since $\overline{U}_{n+1} \subset U_n$ and $U_n.T.E_n=0$, we find that $p \bar{\epsilon} T.E_n$, for $n=1, 2, \ldots$; but, since $E=E_1+E_2+\ldots$, we have $p \bar{\epsilon} T.E$, and since $p \epsilon T$, this gives $p \epsilon (T-E)$, and so $T-E\neq 0$.

A set E (contained in a metric space) is said to be *of the first category at an element* p if there exists an (open) sphere K containing the element p and such that the set $K.E$ is of the first category.

Theorem 92 (Banach).[24] *A set* (contained in any metric space) *which is of the first category at every one of its elements is itself of the first category.*

Proof. Let E be a set of elements of a metric space M, and suppose that E is of the first category at every one of its elements. There exists, then, by the theorem of Zermelo,[25] a transfinite sequence

$$(3) \qquad K_1, K_2, \ldots, K_\omega, K_{\omega+1}, \ldots, K_\alpha, \ldots \quad (\alpha<\phi)$$

consisting of all open spheres K of the space M such that the set $K.E$ is of the first category.

Put

$$(4) \qquad Q_1=K_1.E, \text{ and } Q_\alpha=K_\alpha.E-\sum_{\xi<\alpha} K_\xi, \text{ for } 0<\alpha<\phi.$$

It follows from the definition of the sets (3) that each of the sets $K_\alpha.E$ is of the first category; hence, by reason of (4), the sets Q_α $(\alpha<\phi)$ possess the same property. We may, therefore, write for $\alpha<\phi$

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[24]*Fund. Math.*, vol. XVI, p. 395; see also S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 13.

[25]This theorem states that *every set may be well-ordered*. *Cf.* Sierpiński, *Leçons sur les nombres transfinis*, Paris, 1928, p. 231.

(5) $$Q_\alpha = Q_\alpha^1 + Q_\alpha^2 + Q_\alpha^3 + \dots,$$

where each set Q_α^n $(n=1, 2, \dots)$ is nowhere dense. Put

(6) $$E_n = \sum_{\alpha<\phi} Q_\alpha^n, \text{ for } n=1, 2, \dots$$

It is easily seen that

(7) $$E = E_1 + E_2 + E_3 + \dots$$

In fact, let p denote an element of the set E. It follows from the properties of the set E that there exists an open sphere K containing p and such that the set $K.E$ is of the first category. There exists, therefore, a set of the sequence (3) containing the element p; let K_α be the first term of the sequence (3) which is a set containing p. Since $p \epsilon E$, we conclude from (4) that $p \epsilon Q_\alpha$, and so, from (5) and (6) we deduce at once that $p \epsilon E_1+E_2+E_3+\dots$. We have, therefore,

$$E \subset E_1 + E_2 + E_3 + \dots;$$

on the other hand, it follows from (5) and (4) that

$$Q_\alpha^n \subset Q_\alpha \subset E, \text{ for } n=1, 2, \dots; \ \alpha<\phi,$$

whence, on account of (6), $E_n \subset E$, for $n=1, 2, \dots$, and this gives

$$E_1 + E_2 + E_3 + \dots \subset E.$$

Relation (7) is, therefore, proved.

We shall next show that each of the sets (6) is nowhere dense.

Suppose that, for some integer n, the set E_n is not nowhere dense. There exists, then, an open sphere K such that $K \subset \bar{E}_n$. Hence, on account of (6), (5), and (4), there exist ordinal numbers $\alpha<\phi$ such that $K.K_\alpha \neq 0$; let β be the smallest of such numbers α.

The set $K.K_\beta$ is, therefore, non-null and open (being the product of two spheres); hence, there exists a sphere K^* such that $K^* \subset K.K_\beta$.

It follows from the definition of the numbers α that

$$K.K_\xi = 0, \text{ for } \xi<\beta,$$

and so, since $K^* \subset K$, and $Q_\xi \subset K_\xi$ by (4), and $Q_\xi^n \subset Q_\xi$ by (5), we obtain

(8) $$K^*.Q_\xi^n = 0, \text{ for } \xi<\beta.$$

Again, from (4), we have $Q_\xi . K_\beta = 0$, for $\beta < \xi < \phi$, and so, since $K^* \subset K_\beta$, and $Q_\xi^n \subset Q_\xi$, we get

$$K^* . Q_\xi^n = 0, \text{ for } \beta < \xi < \phi. \tag{9}$$

From relations (6), (8), and (9) we obtain at once

$$K^* . E_n = K^* . Q_\beta^n ,$$

and so

$$\overline{K^* . E_n} = \overline{K^* . Q_\beta^n} \subset \overline{Q_\beta^n} . \tag{10}$$

Moreover, since $K^* \subset K \subset \bar{E}_n$ and K^* is open, we find immediately that

$$K^* \subset \overline{K^* . E_n} . \tag{11}$$

Relations (10) and (11) give

$$K^* \subset \overline{Q_\beta^n} ,$$

which is impossible, since $\overline{Q}_\beta^n$ being nowhere dense (since it is the enclosure of a nowhere dense set) cannot contain the open sphere K^*.

Thus the supposition that for some integer n the set E_n is not nowhere dense leads to a contradiction; the sets $E_n (n = 1, 2, \ldots)$ are, therefore, all nowhere dense, and so, from (7), the set E is of the first category. Theorem 92 is, therefore, proved.

COROLLARY 1. *For every set E of the metric space M, the set E_1 of all elements of E at which the set E is of the first category, is itself of the first category.*

For, since $E_1 \subset E$, if E is of the first category at an element p, then certainly so is E_1 also. The set E_1 is, therefore, of the first category at every one of its elements, and so, by Theorem 92, is itself of the first category.

COROLLARY 2. *If a set E* (contained in the metric space M) *is of the second category, there exists an open sphere K such that, at every one of the elements of K, the set E is of the second category* (*i.e.* not of the first category).

For, let E_1 denote the set of all elements of E at which E is of the first category. The set E_1 is of the first category by Corollary 1, and the set $E - E_1$ is of the second category (since E is of the second category). Hence, the set $E-E_1$ is not nowhere dense, and so (by the necessary and sufficient condition for a nowhere dense set given above) there exists an open sphere K contained in the enclosure of the set $E-E_1$. It is easily seen that the set E is of the second category at every element of the sphere K. In fact, let q be an element of the sphere K, and let K^* be any sphere containing q. Since $K \subset \overline{E-E_1}$, and $q \epsilon K$, there exists an element p of the set $E-E_1$ in the sphere K^*; hence, the set $K^*.E$ cannot be of the first category (since then the set E would be of the first category at the element p, which is impossible since $p \epsilon E-E_1$ and so $p \bar{\epsilon} E_1$). The set E is, therefore, of the second category at every element q of the sphere K. This proves Corollary 2.

A set E (contained in a metric space M) is said to be *of the first category with respect to a set* T (contained in M) *at an element* p, if there exists an (open) sphere K containing p, and such that the set $K.E$ is of the first category with respect to the set T.

The theorem of Banach may be, as is easily seen, expressed in its relativistic form (*i.e.*, relatively to any subset of the space considered) as follows:

A set E (contained in any metric space) *which is of the first category with respect to a set* T (contained in the space considered) *at every one of its elements, is likewise of the first category with respect to the set* T.

In order to prove this, it will be sufficient to apply Banach's theorem to the set $E.T$, taking the set T to be the metric space under consideration.

The two corollaries to Banach's theorem may be similarly expressed at once in their relativistic form.

A set E (contained in a metric space M) is said to *satisfy the Baire condition*, if every (non-null) perfect set P (contained in M) contains (at least one) element p such that one or other or both of the sets $P.E$ or $P-E$ is of the first category at such an element p with respect to the set P.

It follows immediately from this definition that *if a set* E *satisfies the Baire condition, then so also does the set* $CE(=M-E)$ (since $P.CE=P-E$ and $P-CE=P.E$).

Furthermore, it can be easily proved that *every closed set in a metric space satisfies the Baire condition* and also with the aid of Banach's theorem that, *if each set* E_n $(n=1, 2, \ldots)$ *satisfies the Baire condition, then the same is true of the set* $E=E_1+E_2+E_3+\ldots$.

It follows at once from these last three properties that *in a metric space every Borel set* (with respect to this space) *satisfies the Baire condition*.[26]

At one time (1905) Lebesgue assumed the converse of this theorem (in linear space) to be true. This assumption was, however, not justified, as has been shown by Lusin;[27] *a linear set may satisfy the Baire condition without being a Borel set.*

80. *A relative neighbourhood* (with respect to a set E) of an element p of E is every set $T\subset E$ such that $p\in T$ and $p\bar{\in}(E-T)'$.

As may be easily seen, in order that a set T be a relative neighbourhood of an element p of a set E, it is necessary and sufficient that there exists an open set U such that $p\in U$ and $U.E\subset T$. In fact, if $p\in T\subset E$ and $p\bar{\in}(E-T)'$, then putting $U=C(\overline{E-T})$, we shall have an open set U such that $U.E=E.C(\overline{E-T})\subset E.C(E-T)$ $=E.T=T$, and so $U.E\subset T$. On the other hand, if $p\in E$, $T\subset E$, and if there exists an open set U such that $p\in U$ and $U.E\subset T$, then $p\in T$, and $E-T\subset CU$, and so, since U is open and, therefore, CU closed, we find that $(E-T)'\subset(CU)'\subset CU$; but $p\in U$ and, therefore, $p\bar{\in}(E-T)'$.

It follows immediately, from the condition proved above and axiom (v) of § 15, that the product of two relative neighbourhoods of an element p of a set E is a relative neighbourhood of that element.

Furthermore, it follows immediately from the definition of a relative neighbourhood that in the homeomorphic mapping of a set E on the set $f(E)$ every relative neighbourhood (with respect

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[26]This theorem may be even extended to apply to analytical sets with respect to the given metric space; see E. Szpilrajn, "Omierzalności i warunku Baire'a", *Comptes-Rendus du I congrès des pays slaves*, Warsaw, 1929, p. 301.

[27]In 1914 with the aid of the hypothesis of the continuum in *Comptes Rendus*, vol. CLVIII, p. 1259, and in 1917 without the aid of this hypothesis in *Fund. Math.*, vol. II, p. 155.

to E) of an element p of E is mapped into a relative neighbourhood (with respect to $f(E)$) of the transform $f(p)$.

A set E is said to be *locally closed for an element* p if there exists a relative neighbourhood of the element p which is a closed set.

In order that a set E be locally closed for an element p, it is obviously necessary and sufficient that there exist a closed set T and an open set U such that $p \in U.E \subset T \subset E$.

We may clearly suppose that the set T is bounded. It follows then readily that, in a homeomorphic mapping of a set E on the set $f(E)$, an element p for which the set E is locally closed is transformed into the element $f(p)$ for which $f(E)$ is locally closed.

A set E which is locally closed for every one of its elements is said to be *locally closed.* A homeomorphic transform of a locally closed set is obviously a locally closed set.

Theorem 93. *In order that a set E be locally closed, it is necessary and sufficient that the set $E'-E$ be closed.*

Proof. Suppose that E is locally closed, and let p be a limit element of the set $E'-E$. We have, therefore, $p \in (E'-E)' \subset E'' \subset E'$; to prove that $p \in (E'-E)$ it will be sufficient to show that $p \bar{\in} E$. Suppose, on the contrary, that $p \in E$. Since E is locally closed, there exist a closed set T and an open set U such that $p \in U.E \subset T \subset E$. Since $p \in (E'-E)'$ and $p \in U.E \subset U$, where U is open, there exists an element q such that $q \in (E'-E)$ and $q \in U$.

Since $q \in E'-E \subset E'$, there exists an infinite sequence of elements q_n such that $\lim_{n\to\infty} q_n = q$, and since $q \in U$, which is open, we have $q_n \in U$ for $n > \mu$. Hence $q_n \in U.E$ for $n > \mu$, and so, since $U.E \subset T$ and T is closed, $q = \lim_{n\to\infty} q_n \in T \subset E$, contrary to the fact that $q \in E'-E$. The condition of our theorem is, therefore, necessary.

Suppose now that the set $E'-E$ is closed, and let p be an element of E. We have, then, $p \bar{\in} (E'-E)$, *i.e.* $p \in U = C(E'-E)$, where U is an open set. By axiom (*vii*) (§ 38) there exists an open set V such that $p \in V$ and $\overline{V} \subset U$. Put $T = \overline{V}.E$; from the definition of U we have $U.(E'-E) = 0$, which gives at once $T = \overline{V}.E$

$=\overline{V}.(E+E')=\overline{V}.\overline{E}$ (since $E+E'=E+(E'-E)$), and so T is closed. But V is open and $p \in V.E \subset \overline{V}.E = T \subset E$; the set E is, therefore, locally closed for the element p. Since p is any element of E, it follows that E is locally closed. The condition of our theorem is, therefore, sufficient.

Theorem 93 is thus proved.

COROLLARY. *In order that a set be locally closed, it is necessary and sufficient that it should be the difference of two closed sets.*

In fact, if E is locally closed then, by Theorem 93, the set $F=E'-E=(E+E')-E=\overline{E}-E$ is closed and so $E=\overline{E}-F$, which proves that E is the difference of two closed sets.

On the other hand, if $E=F_1-F_2$, where F_1 and F_2 are closed, then, since $E\subset F_1$ and F_1 is closed, we have $E'\subset F_1$ and so $E'=E'.F_1$, and $E'-E=E'.F_1-(F_1-F_2)=E'.F_1.F_2$; this proves that the set $E'-E$ is closed (being the product of three closed sets), and so, by Theorem 93, the set E is locally closed.

We shall next prove a lemma which will be made use of in the next article.

LEMMA. *Every non-null set which is both an F_σ and a G_δ contains an element for which it is locally closed.*

Proof. Let E be a non-null set which is both an F_σ and a G_δ. Put $T=\overline{E}$. The sets E and $T-E=\overline{E}.CE$ are F_σ's, and so we may write $E=F_1+F_2+F_3+\ldots$ and $T-E=H_1+H_2+H_3+\ldots$, where F_n and $H_n (n=1, 2, \ldots)$ are closed sets. From $T=E+(T-E)=F_1+H_1+F_2+H_2+\ldots$ and Theorem 91, we conclude that the sets $F_1, H_1, F_2, H_2, \ldots$ cannot all be nowhere dense on the set T. Hence, there exists an n such that F_n or H_n is not nowhere dense on T. Suppose that the set H_n is not nowhere dense on T. It follows, therefore, from the sufficient condition for a set to be nowhere dense on T(§ 79) that there exists an open set U such that $U.T\neq 0$, and also that every open set V such that $V\subset U$ and $V.T\neq 0$ contains an element of the set $T.H_n$. We conclude immediately from this that every open set G containing an element q of the set $U.T$ contains at least one element of the set H_n (since it would be sufficient to take $G=U.V$). Since H_n is closed, $q \in H_n$. We have, therefore, $U.T \subset H_n$, and so, since $H_n \subset T-E$, $U.E=U.T.E=0$. But $U.T \neq 0$ and $T=\bar{E}$; hence, since U is open, $U.E \neq 0$, contrary to the above.

The set H_n is, therefore, nowhere dense on T. Consequently, the set F_n is not nowhere dense on T, and so, as above, we deduce the existence of an open set U such that $U.T \neq 0$, and $U.T \subset F_n$. Since $U.T \neq 0$, there exists an element p of $U.T$. We have, therefore, $p \in U.T \subset F_n \subset T$, and, since F_n is closed and U is open, it follows that T is locally closed for p. The lemma is, therefore, proved.

81. The set

(1) $$R^1(E)=E.(E'-E)'$$

is said to be the first *residue* of the set E.

The set $E-R^1(E)=E-(E'-E)'$ is the set of all those elements of E for which E is locally closed.

To prove this we note that $E.(E'-E)=0$ and, therefore, (1) gives

(2) $$R^1(E)=E.\overline{(E'-E)}.$$

Put $V=C(E'-E)$; this will be an open set. Suppose that $p \in (E-R'(E))$; then, from (2) $p \in V$ and by axiom (*vii*) (§ 38), there exists an open set U such that $p \in U$ and $\bar{U} \subset V$, and so $T=(\overline{U.E}) \subset \bar{U}.\bar{E} \subset V.\bar{E}$. It follows, however, from the definition of the set V, that $V.(E'-E)=0$; therefore, $V.E' \subset V.E$ and $V.\bar{E} \subset V.E$. Hence, $p \in U.E \subset T \subset E$, where T is closed and U is open. This proves that E is locally closed for the element p (§ 80).

Suppose now that p is an element of the set E for which E is locally closed. Hence, there exist a closed set T and an open set U such that $p \in U.E \subset T \subset E$. Suppose that $p \in (E'-E)'$. Since $p \in U$ and U is open, there exists an element $q \in (E'-E).U$. We have, therefore, $q \in E'.U$ and so, since U is open, there is a sequence $q_n (n=1, 2, \ldots)$ such that $q_n \in E.U$ for $n=1, 2, \ldots$ and for which $\lim_{n\to\infty} q_n=q$. Since $E.U \subset T$ and T is closed, we conclude that $q \in T$, from which, since $T \subset E$, we deduce that $q \in E$, which is impossible since $q \epsilon E'-E$. We must, therefore, have $p \bar{\epsilon} (E'-E)'$ and so, from (1), $p \epsilon E-R^1(E)$.

We shall next prove the relation

$$E-R^1(E)=\bar{E}-\overline{(E^1-E)}. \tag{3}$$

We have obviously $\bar{E}=E+(E'-E)$, from which we get at once

$$\bar{E}-\overline{(E'-E)}=[E+(E'-E)]-\overline{(E'-E)}=E-\overline{(E'-E)}=E-E.\overline{(E'-E)},$$

which on account of (2) gives (3).

It follows from (3) that the set $E-R^1(E)$ is the difference of two closed sets and so, by the corollary to Theorem 93, a locally closed set.

From the properties of the set $E-R^1(E)$ deduced above and the fact that in a homeomorphic mapping an element for which a set E is locally closed is transformed into an element for which the transform of E is locally closed, it may be concluded at once that in the homeomorphic mapping of the set E on the set $f(E)$, the set $E-R^1(E)$ is transformed into the set $f(E)-R^1(f(E))$ and so the set $R^1(E)$ is transformed into the set $R^1(f(E))$. Hence

In a homeomorphic mapping the first residue of a set is transformed into the first residue of the transform of the set. The property of belonging to the first residue of a given set is, therefore, an invariant under all homeomorphic transformations of that set.

The following immediate corollaries may be deduced at once from the properties of the set $E-R^1(E)$:

COROLLARY 1. *In order that E be locally closed for an element $p \epsilon E$, it is necessary and sufficient that $p \bar{\epsilon} (E'-E)'$.*

COROLLARY 2. *In order that E be locally closed, it is necessary and sufficient that its first residue be a null set* (*i.e.* that $E.(E'-E)'=0$).

Let now α denote an ordinal number >1, and suppose that we have already defined all sets $R^\xi(E)$, where $\xi<\alpha$ (and where E is a given set). If α is a number of the first kind, *i.e.* $\alpha=\beta+1$, then put $R^\alpha(E)=R^1(R^\beta(E))$. If, however, α is a number of the second kind, then put $R^\alpha(E) = \prod_{\xi<\alpha} R^\xi(E)$. The sets $R^\alpha(E)$ are thus defined by transfinite induction for every ordinal number α. The set $R^\alpha(E)$ is said to be *the residue of order* α of the set E. By $R^0(E)$ we mean the set E itself.

It follows at once from (1) that the set $R^1(E)$ is closed in the set E. The set $R^{\alpha+1}(E)$ is, therefore, closed in the set $R^\alpha(E)$ for every ordinal number α. Furthermore, it follows from the definition of the sets $R^\alpha(E)$ that $R^\xi(E) \supset R^\eta(E)$, for $\xi<\eta$ and so, by the corollary to Theorem 37 (§ 26), there exists an ordinal number $\alpha<\Omega$ such that

(4) $$R^\xi(E) = R^\alpha(E), \text{ for } \alpha<\xi<\Omega.$$

We may also suppose that α is the smallest ordinal number for which (4) holds, so that

(5) $$R^\xi(E) \neq R^\alpha(E), \text{ for } \xi<\alpha.$$

The set $R^\alpha(E) = R^\Omega(E)$ is called *the last residue* of the set E. It will be shown that *the set* $E - R^\Omega(E)$ *is always both an* F_σ *and a* G_δ.

Let ξ be a given ordinal number ≥ 0. From (3) and the fact that $R^{\xi+1}(E) = R^1(R^\xi(E))$, we have

(6) $$R^\xi(E) - R^{\xi+1}(E) = \overline{R^\xi(E)} - \overline{\{[R^\xi(E)]' - R^\xi(E)\}}.$$

Put

(7) $$P^\xi(E) = \overline{R^\xi(E)},\ Q^\xi(E) = \overline{R^\xi(E)} \,.\, \overline{\{[R^\xi(E)]' - R^\xi(E)\}};$$

we shall then have from (6)

(8) $$R^\xi(E) - R^{(\xi+1)}(E) = P^\xi(E) - Q^\xi(E)$$

where, on account of (7), and since from (4)

$R^{\xi+1}(E) = R^\xi(E) \,.\, \overline{\{[R^\xi(E)]' - R^\xi(E)\}}$, we shall have

(9) $$P^\xi(E) \supset Q^\xi(E) \supset P^{\xi+1}(E),$$

whereas, from (7) and the fact that $R^\xi(E) \supset R^\eta(E)$ for $\xi<\eta$, we get

(10) $$P^\xi(E) \supset P^\eta(E), \text{ for } \xi<\eta.$$

Let now p be an element of the set E. If $p \in R^\xi(E)$ for $0 \leq \xi < \Omega$, then $p \in R^\Omega(E)$. Hence, if $p \in (E - R^\Omega(E))$, there exists an ordinal number $\lambda<\Omega$ such that $p \bar{\in} R^\lambda(E)$, where it may be supposed that

λ is the smallest of such numbers, *i.e.* that $p \in R^{\xi}(E)$, for $\xi < \lambda$. If λ were a number of the second kind, we would have from the definition, $R^{\lambda}(E) = \prod_{\xi<\lambda} R^{\xi}(E)$ and so $p \in R^{\lambda}(E)$, contrary to the definition of λ. λ is, therefore, a number of the first kind, and so we may write $\lambda = \xi + 1$.

Thus, corresponding to every element p of the set $E - R^{\Omega}(E)$, there exists an ordinal number $\xi \geq 0$, such that $p \in [R^{\xi}(E) - R^{\xi+1}(E)]$; if now α be an ordinal number which satisfies (4), then it must be that $\xi < \alpha$ (since for $\xi \geq \alpha$ we have, from (4), $R^{\xi}(E) - R^{\xi+1}(E) = 0$). This and the fact that for $0 \leq \xi < \alpha$ we have $R^{\xi}(E) \subset E$ and $R^{\alpha}(E) \subset R^{\xi+1}(E)$ (and so $R^{\xi}(E) - R^{\xi+1}(E) \subset E - R^{\alpha}(E) = E - R^{\Omega}(E)$) lead to the relation

$$E - R^{\Omega}(E) = \sum_{0 \leq \xi < \alpha} [R^{\xi}(E) - R^{\xi+1}(E)];$$

hence, on account of (8)

$$E - R^{\Omega}(E) = \sum_{0 \leq \xi < \alpha} [P^{\xi}(E) - Q^{\xi}(E)]. \tag{11}$$

The terms of the sum (11) are sets F_{σ}, since they are differences of two closed sets; and since (11) contains at most a countable aggregate of terms (since $\alpha < \Omega$), $E - R^{\Omega}(E)$ is a set F_{σ}.

Again, (11) gives

$$C(E - R^{\Omega}(E)) = \prod_{0 \leq \xi < \alpha} [Q^{\xi}(E) + CP^{\xi}(E)]. \tag{12}$$

But (9) and (10) give for $\eta \leq \xi$

$Q^{\xi}(E) \subset P^{\xi}(E) \subset P^{\eta}(E)$, and so $Q^{\xi}(E) \, . \, C(P^{\eta}(E)) = 0$, for $\eta \leq \xi$;

(12) gives, therefore, at once

$$C(E - R^{\Omega}(E)) = \prod_{0 \leq \eta < \alpha} Q^{\eta}(E) + C(P^{0}(E)) + \sum_{0 < \xi < \alpha} C(P^{\xi}(E)) \prod_{0 \leq \eta < \xi} Q^{\eta}(E). \tag{13}$$

The products $\prod_{0 \leq \eta < \alpha} Q^{\eta}(E)$ and $\prod_{0 \leq \eta < \xi} Q^{\eta}(E)$ are closed sets, since the sets Q^{η} are closed, and since the sets (7) are closed, we conclude that the sets $C(P^{\xi}(E))$ are F_{σ}'s (for $0 \leq \xi < \alpha$). The terms of the sum (13) are, therefore, sets F_{σ}, and so (13) is an F_{σ}. The set $E - R^{\Omega}(E)$ is, therefore, a G_{δ}.

We have thus proved that *the difference of a given set and its last residue is both an* F_σ *and a* G_δ.

The set E for which $R^\Omega(E)=0$ is said to be *reducible* (Hausdorff). From the theorem deduced above, it follows that a reducible set is both an F_σ and a G_δ.

Let now E denote a set which is both an F_σ and a G_δ. Since the set $T=E-R^\Omega(E)$ is both an F_σ and a G_δ, therefore the set $X=R^\Omega(E)=E.CT$ is both an F_σ and a G_δ. Furthermore, if X is not a null set, then, by the lemma proved toward the close of § 80, it contains an element p for which it is locally closed, and so it follows from the property of the set $X-R^1(X)$ that $p \epsilon [X-R^1(X)]$ and thus $X-R^1(X)\neq 0$, *i.e.* $X\neq R^1(X)$. But, as we know, there exists an α for which (4) is satisfied; hence, $X=R^\Omega(E)=R^\alpha(E)= R^{\alpha+1}(E)=R^1(R^\alpha(E))=R^1(X)$, which is a contradiction. Hence $X=R^\Omega(E)=0$, and, therefore, E is reducible. We have thus proved

Theorem 94. *In order that a set E be reducible, it is necessary and sufficient that it be both an* F_σ *and a* G_δ.

Concerning the residues of a set, we note further that it follows immediately from their definition and the properties of the first residue that in a homeomorphic mapping of a set, its residue of order λ is transformed into the residue of order λ of the transform (for every ordinal number $\lambda<\Omega$). In particular, the last residue of a set is transformed in a homeomorphic mapping into the last residue of the transform. Hence, a homeomorphic transform of a reducible set is a reducible set, which follows otherwise from Theorem 94 and other known properties of the sets F_σ and G_δ (§ 61).

82. The set E is said to be 0-*dimensional* (Menger) if, corresponding to every element p of E and every open set U containing p, there exists an open set V contained in U and such that its frontier does not contain any elements of E.

We shall prove that the property of being 0-dimensional is a topological invariant. In order to do so, we shall adopt a different definition of a 0-dimensional set, which is, however, equivalent to the definition of Menger.

The set $E.\bar{E}_1.(\overline{E-E_1})$ is said to be *the relative frontier* with respect to E of the set $E_1 \subset E$; hence, it is closed in E.

The relative frontier of a set $E_1 \subset E$ with respect to the set E is transformed in a homeomorphic mapping into the relative frontier of the transform of E_1 with respect to the transform of E. In order to prove this, it is sufficient to refer to Theorem 22, and the fact that if $E\, h_f T$ and $E_1 \subset E$, then $f(E-E_1) = f(E) - f(E_1)$.

We shall now prove that the Menger definition of a 0-dimensional set is equivalent to the following:

A set E is said to be 0-dimensional if, corresponding to every element p of E and every relative neighbourhood T of that element (with respect to E), there exists a relative neighbourhood S of the element p which is contained in T and whose relative frontier (with respect to E) is null.

Suppose that the set E is 0-dimensional according to the Menger definition, and let T denote a relative neighbourhood of a given element p of E. There exists, therefore, an open set U (§ 80) such that $p \epsilon U$ and $U.E \subset T$. Furthermore, since E is 0-dimensional in the Menger sense and from the definition of the frontier of a set (§ 5), it follows that there exists an open set V such that $p \epsilon V \subset U$ and $E.\bar{V}.\overline{CV} = 0$. Put $E_1 = E.V$; then $E_1 \subset E$, $\bar{E}_1 \subset \bar{V}$, $E - E_1 = E - E.V = E - V = E.CV \subset CV$; hence, $\overline{E-E_1} \subset \overline{CV}$, and so $E_1.\bar{E}_1.(\overline{E-E_1}) \subset E.\bar{V}.\overline{CV} = 0$. On the other hand, $p \epsilon E.V = E_1$. The set E_1 is, therefore, a relative neighbourhood of the element p (with respect to E) whose relative frontier is null, and since $V \subset U$, and $U.E \subset T$, we have $E_1 = E.V \subset U.E \subset T$. The set E is, therefore, 0-dimensional according to our definition.

Suppose now that the set E is 0-dimensional according to our definition, and let U be an open set containing an element p of E. Put $T = U.E$; the set T is a relative neighbourhood of p (with respect to E), and so since E is 0-dimensional according to our definition, there exists a relative neighbourhood S of p which is contained in T and such that $E.S.(\overline{E-S}) = 0$. Since $S \subset E$, it follows that $S \subset E.S$, and so $S.(\overline{E-S}) \subset E.\bar{S}.(\overline{E-S}) = 0$. Furthermore, we evidently have $(E-S).S \subset E.(\overline{E-S}).\bar{S} = 0$. The sets S and $E-S$ are, therefore, mutually exclusive, and neither of them contains limit elements of the other; hence, there exist (§§ 39, 46) open sets P and Q such that $S \subset P$, $E - S \subset Q$, $P.Q = 0$. Put $V = P.U$; this is an open set, contained in U and containing p (since $p \in U$ and $p \in S \subset P$). We have further $V = P.U \subset P$, and so $\overline{V} \subset \overline{P}$; but $P.Q = 0$; therefore, $P \subset CQ$, and so, since Q is open (and, therefore, CQ closed), we have that $\overline{P} \subset CQ$. Hence, $\overline{V} \subset CQ$. Since V is open, we have $\overline{CV} = CV$; hence, $E.\overline{CV} = E.CV = E - P.U = E - P.U.E$, and so, since $S \subset P$, and $S \subset T = U.E$, it follows that $E.\overline{CV} \subset E - S$. Hence, $E.\overline{V}.\overline{CV} \subset (E - S).CQ$, and so $E.\overline{V}.\overline{CV} = 0$ since $E - S \subset Q$. The set V is, therefore, open; it contains p and is contained in U, and the frontier of V does not contain any elements of the set E. The set E is, therefore, 0-dimensional in the Menger sense.

The two definitions of 0-dimensional sets are thus shown to be equivalent.

It follows immediately from our definition of 0-dimensional sets and from the properties of relative neighbourhoods (§ 80) and relative frontiers in homeomorphic mapping that *a homeomorphic transform of a* 0-*dimensional set is a* 0-*dimensional set.*

We note further that changing slightly the proof of Theorem 56 given in § 53, we can prove the following more general theorem: *Every* 0-*dimensional metric space* (in the Menger sense) *which contains a countable subset everywhere dense is homeomorphic with a certain set of irrational numbers.*

The set of all irrational numbers is itself, as is easily seen, a 0-dimensional metric space which contains an everywhere dense countable subset. Hence, *the set of all irrational numbers has the greatest dimensional type of all the* 0-*dimensional sets which have a countable subset everywhere dense.*

As regards n-dimensional sets (n a positive integer), they may be defined by induction as follows:

A set E is said to be n-dimensional at most, if corresponding to every element p of E and every relative neighbourhood T of that element (with respect to E), *there exists a relative neighbourhood S of p contained in T whose relative frontier* (with respect to E) *is a* $(n\text{-}1)$-*dimensional set at most.*

A set E is said to be n-dimensional if n is the smallest positive integer such that, corresponding to every element p of E and every *relative neighbourhood T of p* (with respect to E), *there exists a relative neighbourhood S of p contained in T, whose relative frontier* (with respect to E) *is a* $(n\text{-}1)$*-dimensional set at most.*

(This definition also holds for $n=1$ if a (-1)-dimensional set is taken to mean a null set.)

It follows easily by induction from this definition of n-dimensional sets and from the properties of relative neighbourhoods and frontiers that *a homeomorphic transform of a n-dimensional set is a n-dimensional set.* The (Menger) dimension of a set is, therefore, a topological invariant.[28]

Menger calls a 1-dimensional continuum a *curve.* Since continua (§ 34) and 1-dimensional sets are topological invariants, it follows that *a homeomorphic transform of a curve is a curve.*

A homeomorphic transform of the closed interval $0 \leq x \leq 1$ is called *a simple arc* (Janiszewski). It follows at once from this definition that *a homeomorphic transform of a simple arc is a simple arc* and that *a simple arc is a curve in the Menger sense.*

A continuous transform of the closed interval $0 \leq x \leq 1$ is called a *Jordan curve.* It follows at once from this definition that *a continuous transform of a Jordan curve is a Jordan curve* and that *a simple arc is a Jordan curve.* A Jordan curve need not, however, be a Menger curve, nor is it necessary for a Menger curve to be a Jordan curve.

[28]An exposition of the Menger Dimension Theory may be found in a paper by W. Hurewicz, "Grundriss der Mengerschen Dimensionstheorie", *Math. Annalen*, vol. XCVIII (1927), pp. 64-88.

APPENDIX

1. It is assumed that we know what is meant by a set of objects, *e.g.*, the set of books in a certain library, a set of chairs in a hall, a set of ideas, or even a set of sets. The objects constituting a set are said to be its elements, and the notation $p \in E$, $p \bar{\in} E$ is used to denote that p is or is not an element of E.

A set E is defined when of every element p it can be said whether $p \in E$ or $p \bar{\in} E$.

A set A is a subset of a set B, *i.e.* $A \subset B$ or $B \supset A$, if, whenever $p \in A$, then $p \in B$. If $A \subset B$ and $B \subset A$, A and B are identical, *i.e.* $A = B$. If $A \subset B$ and $A \neq B$, A is said to be a proper subset of B.

If two sets A and B are such that a (1, 1) correspondence can be established between their elements, then A and B are said to have the same *potency*. For example, the set of all odd integers less than 100 and the set of all even integers not greater than 100 have the same potency, for to every odd number may be correlated an even number greater by unity. The idea of potency may be extended to sets which are not finite; *e.g.*, the set of all natural numbers

$$1, 2, 3, \ldots, n, \ldots$$

and the set of all even integers

$$2, 4, 6, \ldots, 2n, \ldots$$

have the same potency.

Two finite sets have the same potency if and only if the number of elements in each set is the same. In the last example the given set and its subset have the same potency. A finite set cannot have the same potency as any of its subsets. *A set which has the same potency as one of its proper subsets is said to be infinite in the Dedekind sense.*

2. A set which has the same potency as the set of all natural numbers is said to be *countable.* The elements of a countable set can, therefore, be enumerated as a sequence

$$u_1, u_2, u_3, \ldots$$

with increasing indices. Conversely, the set of all terms of an infinite sequence is countable.

A subset of a countable set, if not finite, is obviously countable (since any subset of a sequence may be arranged as a sequence with increasing indices). Thus the sets of all odd numbers, all prime numbers, all squares are each countable.

The *sum* of two sets A and B, *i.e.* $A+B$, consists of elements p such that either $p \epsilon A$, or $p \epsilon B$; the *product* $A.B$ consists of elements p such that $p \epsilon A$ and $p \epsilon B$.

The sum of a finite set and a countable set is a countable set. For the sum of the set

$$u_1, u_2, \ldots, u_m,$$

and the set

$$v_1, v_2, \ldots$$

may be written as the infinite sequence

$$u_1, u_2, \ldots, u_m, v_1, v_2, v_3, \ldots$$

The sum of two countable sets is a countable set. In fact, the sum of the countable set

$$u_1, u_2, u_3, \ldots$$

and the countable set

$$v_1, v_2, v_3, \ldots$$

may be written down as the infinite sequence

$$u_1, v_1, u_2, v_2, u_3, v_3, \ldots$$

The definition of a sum of two sets may be easily extended to a sum of a finite or infinite sequence of sets. Given an infinite sequence of sets $E_1, E_2, E_3, \ldots$, the sum

$$S = E_1 + E_2 + E_3 + \ldots$$

is a set consisting of all elements p such that $p \epsilon E_i$ for at least one value of i. *The sum of a countable aggregate of countable sets is countable.* For, if $E_1, E_2, \ldots$ be an infinite sequence of countable sets, the elements of

$$S=E_1+E_2+E_3+\ldots$$

may be written down as a double sequence

$$\begin{array}{l} a_{11},\ a_{12},\ a_{13},\ \ldots \\ a_{21},\ a_{22},\ a_{23},\ \ldots \\ a_{31},\ a_{32},\ a_{33},\ \ldots \\ \ldots\ldots\ldots\ldots, \end{array}$$

where a_{k_1}, a_{k_2}, . . . are the elements of E_k. Arranging the elements of the double sequence into groups such that the nth group consists of all a_{kl}, $k+l=n+1$, we obtain the infinite sequence

$$a_{11},\ a_{21},\ a_{12},\ a_{31},\ a_{22},\ a_{13},\ a_{41},\ a_{32},\ \ldots$$

containing all terms of S.

The set of all rational numbers is countable. For let S be the set of all positive rational numbers. Denote by E_n the set of all positive rational numbers in lowest terms with n in the denominator; then

$$S=E_1+E_2+E_3+\ldots,$$

where E_n is countable. Hence S is countable. Similarly, the set T of all negative rational numbers is countable and, therefore, also the set of all rational numbers.

The set E of all finite sequences of natural numbers is countable. For a finite sequence $(n_1, n_2, \ldots, n_k)$ may be correlated in a unique way with the number

$$N=2^{n_1-1}+2^{n_1+n_2-1}+\ldots+2^{n_1+n_2+\ldots+n_k-1}.$$

The set of all polynomials with rational coefficients is countable, since a (1, 1) correspondence may be established between a polynomial and the finite sequence consisting of the coefficients. All such polynomials may, therefore, be represented as an infinite sequence

$$P_1,\ P_2,\ P_3,\ \ldots$$

A polynomial, as is well known, has at most a finite number of roots; writing down all the roots of P_1, then those of P_2 and so on, we obtain an infinite sequence

$$x_1,\ x_2,\ x_3,\ \ldots$$

consisting of the roots of all polynomials with rational coefficients, *i.e.* of all algebraic numbers. Hence, *the set of all algebraic numbers is countable.*

3. A set which is neither finite nor countable is said to be *non-countable.*

The set E of all infinite sequences of natural numbers is non-countable. For, if it were countable, it could be written as a double sequence

$$\begin{array}{c} n_{11},\ n_{12},\ n_{13},\ \ldots \\ n_{21},\ n_{22},\ n_{23},\ \ldots \\ n_{31},\ n_{32},\ n_{33},\ \ldots \\ \ldots\ldots\ldots\ldots \end{array} \tag{1}$$

But the infinite sequence

$$n_{11}+1,\ n_{22}+1,\ n_{33}+1,\ \ldots,\ n_{kk}+1,\ \ldots \tag{2}$$

differs from each of the sequences (1) and so $\bar{\epsilon}\ E$, which is contrary to the hypothesis that E consists of all infinite sequences of natural numbers. Similarly, it can be shown that for every infinite sequence of real numbers there exists a real number which is not a number of the sequence. Let

$$x_1,\ x_2,\ \ldots$$

be a sequence of real numbers. Every real number can be expressed in one, sometimes two, ways as an infinite decimal. Writing down these developments (one or both if there are two) of the successive terms of the sequence, we obtain an infinite sequence of infinite decimals

$$\begin{array}{l} c_1,\ c_{11}c_{12}c_{13}\ \ldots \\ c_2,\ c_{21}c_{22}c_{23}\ \ldots \\ c_3,\ c_{31}c_{32}c_{33}\ \ldots \\ \ldots\ldots\ldots\ldots \end{array}$$

We now construct the decimal

$$0.a_1a_2a_3\ \ldots,$$

where $a_1 \neq c_{11}$, $a_2 \neq c_{22}$, . . . generally $a_k \neq c_{kk}$ (we may choose, *e.g.*, $a_k = c_{kk}+1$ if $c_{kk} < 9$ and $a_k = 0$ if $c_{kk} = 9$).

The real number thus constructed is obviously different from every term of the given sequence. It follows, therefore, that *the set of all real numbers is non-countable.*

There exists, as is well known, a (1, 1) correspondence between the set of real numbers and the points on a straight line; hence, the set of points on a straight line is non-countable.

If a finite or countable set be removed from a non-countable set, the remaining set is non-countable. For, let P be a non-countable set, Q finite or countable, and R the remainder. Hence, $P=Q+R$; if R were finite or countable then P would be finite or countable, contrary to the assumption that it is non-countable. The set R is, therefore, neither finite nor countable and so must be non-countable. After removing from the non-countable set of real numbers the countable set of algebraic numbers, there remains a non-countable set of real numbers which are known as the transcendental numbers.

CARDINAL NUMBERS

4. Let all sets be divided into classes, two sets belonging to the same class if and only if they have the same potency; then all sets of a given class have a common characteristic. The symbols used to designate classes of sets of equal potency are called *cardinal numbers.* The cardinal number corresponding to the class of all countable sets is denoted by $\aleph_0$ and the one corresponding to the class of all sets of the same potency as the set of all real numbers by $\mathfrak{c}$. It follows from the definition that to every set corresponds a cardinal number (namely that number which serves to designate the class containing the given set). The cardinal number corresponding to a set E is frequently denoted by $\bar{\bar{E}}$ and is called *the potency of the set E.* Sets with cardinal $\mathfrak{c}$ are said to have *the potency of the continuum.*

Cardinal numbers different from the natural numbers are called *transfinite numbers.* There exist different transfinite numbers, *e.g.* $\aleph_0$ and $\mathfrak{c}$.

The sum $\mathfrak{m}+\mathfrak{n}$ of two cardinal numbers is the cardinal number of the set $M+N$, where M and N are mutually exclusive[1] sets and

[1]Two sets A and B are said to be *mutually exclusive* if $A.B=0$.

$\overline{\overline{M}} = \mathfrak{m}$ and $\overline{\overline{N}} = \mathfrak{n}$. It is easily seen that the addition of cardinal numbers is commutative and associative.

It follows from § 2 that

$$(1) \qquad \mathfrak{n} + \aleph_0 = \aleph_0,$$

$$(2) \qquad \aleph_0 + \aleph_0 = \aleph_0,$$

$$\aleph_0 + \aleph_0 + \dots = \aleph_0 \ (\aleph_0 \text{ terms}).$$

Consider the set E of all real numbers, and let N be the set of all rational numbers and M the remainder. Then

$$\overline{\overline{E}} = \overline{\overline{M}} + \overline{\overline{N}};$$

but $\overline{\overline{E}} = \mathfrak{c}$, $\overline{\overline{N}} = \aleph_0$, and let $\overline{\overline{M}} = \mathfrak{m}$; then

$$(3) \qquad \mathfrak{c} = \mathfrak{m} + \aleph_0,$$

$$\mathfrak{c} + \aleph_0 = (\mathfrak{m} + \aleph_0) + \aleph_0 = \mathfrak{m} + (\aleph_0 + \aleph_0) = \mathfrak{m} + \aleph_0 = \mathfrak{c};$$

hence

$$(4) \qquad \mathfrak{c} + \aleph_0 = \mathfrak{c}.$$

For n a natural number

$$(5) \qquad \mathfrak{c} + \mathfrak{n} = (\mathfrak{c} + \aleph_0) + \mathfrak{n} = \mathfrak{c} + (\aleph_0 + \mathfrak{n}) = \mathfrak{c} + \aleph_0 = \mathfrak{c}.$$

The relation $y = \dfrac{x}{1 + |x|}$ establishes a (1, 1) correspondence between the set of all real numbers x and the set of all real numbers y in the interval $(-1, 1)$. The cardinal of the latter set is, therefore, $\mathfrak{c}$. Let a and $b > a$ be two given real numbers. The relation $z = \dfrac{b-a}{2} y + \dfrac{a+b}{2}$ establishes clearly a (1, 1) correspondence between the set of all real numbers y which satisfy the inequality $-1 < y < 1$ and the set of all real numbers z satisfying the inequality $a < z < b$; hence, the two sets have the same potency, *i.e.* the potency $\mathfrak{c}$. This potency will not change if we add one element to the set. Hence, for every a and $b > a$, the set of all real numbers in the interval (a, b) has potency $\mathfrak{c}$ (the end-points being included or excluded).

In particular, the set M of all real numbers x satisfying the inequality $0 \leq x < 1$, the set N of all real numbers x such that

$1 \leq x < 2$, and the set S of all real numbers x such that $0 \leq x < 2$ have all potency $\mathfrak{c}$. But M and N are mutually exclusive sets and $M+N = S$; therefore, $\overline{\overline{M}}+\overline{\overline{N}}=\overline{\overline{S}}$, and so

$$\mathfrak{c}+\mathfrak{c}=\mathfrak{c}. \tag{6}$$

The definition of a sum of cardinal numbers may be extended to an infinite sequence of cardinal numbers. If

$$\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots$$

is an infinite sequence of cardinal numbers, and $M_k (k=1, 2, \ldots)$ are mutually exclusive sets such that $\overline{\overline{M_k}} = \mathfrak{m}_k$, then the sum of the infinite series

$$\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3+ \ldots$$

is the cardinal number of the set

$$M_1+M_2+M_3+ \ldots$$

Thus, it may be easily seen that

$$\aleph_0 = 1+1+1+1+ \ldots$$

(it is sufficient here to take M_k to consist of one number k). Also

$$\aleph_0 = 1+2+3+4+ \ldots$$

(here, M_k may be taken to consist of k natural numbers n satisfying the inequality $\frac{1}{2}(k-1)k < n \leq \frac{1}{2}k(k+1)$, for $k=1, 2, \ldots$
Similarly,

$$\aleph_0 = \aleph_0+\aleph_0+\aleph_0+ \ldots, \tag{7}$$

and

$$\mathfrak{c} = \mathfrak{c}+\mathfrak{c}+\mathfrak{c}+ \ldots, \tag{8}$$

for it would be sufficient to consider in the first case the (countable) set M_k consisting of the natural numbers

$$2k-1,\ 2(2k-1),\ 2^2(2k-1),\ 2^3(2k-1), \ldots$$

and in the second case the set M_k of all real numbers x satisfying the inequality $k-1 < x < k$ $(k=1, 2, \ldots)$, noting that the set of all real positive numbers has cardinal $\mathfrak{c}$.

The product $\mathfrak{m}.\mathfrak{n}$ of the cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ is the cardinal number of the set P consisting of all pairs (x, y) where $x \epsilon M$, $y \epsilon N$, and where $\overline{\overline{M}}=\mathfrak{m}$, $\overline{\overline{N}}=\mathfrak{n}$. It is easily seen that the multiplication of cardinal numbers is commutative and associative. It follows readily that

$$\aleph_0.\aleph_0=\aleph_0.$$

For let M and N be each the set of natural members. The product P will then be the set of all pairs (m, n) of natural numbers, *i.e.* the set of all elements of the double sequence

$$\begin{array}{l}(1, 1), (1, 2), (1, 3), \ldots \\ (2, 1), (2, 2), (2, 3), \ldots \\ (3, 1), (3, 2), (3, 3), \ldots \\ \ldots\ldots\ldots\ldots\ldots\end{array}$$

which may be ordered as a single infinite sequence by the diagonal method. Hence $\overline{\overline{P}}=\aleph_0$, and since $\overline{\overline{M}}.\overline{\overline{N}}=\overline{\overline{P}}$, we have $\aleph_0.\aleph_0=\aleph_0$.

If $\mathfrak{m}$ be a finite cardinal and $\mathfrak{n}$ any cardinal number, we have

(9) $$\mathfrak{m}.\mathfrak{n}=\mathfrak{n}+\mathfrak{n}+\ \ldots\ +\mathfrak{n}\ (m \text{ terms});$$

for, let M be the set of natural numbers $1, 2, \ldots, m$, N a set such that $\overline{\overline{N}}=\mathfrak{n}$, and P the set of all pairs (m, n), where $m \epsilon M$ and $n \epsilon N$. Hence $\overline{\overline{P}}=\mathfrak{m}.\mathfrak{n}$. Denoting for a given k the set of all pairs (k, n) by P_k we get obviously $\overline{\overline{P_k}}=\mathfrak{n}$, for $k=1, 2, \ldots, m$, and $P=P_1+P_2+\ \ldots\ +P_m$, where the sets P_k are mutually exclusive. Since $\overline{\overline{P}}=\mathfrak{m}.\mathfrak{n}$, (9) follows at once.

In particular, for $\mathfrak{n}=\aleph_0$, (9) and (2) give

$$\mathfrak{m}.\aleph_0=\aleph_0\ (\mathfrak{m}=1, 2, \ldots).$$

Similarly, for $\mathfrak{n}=\mathfrak{c}$, (9) and (6) give

$$\mathfrak{m}.\mathfrak{c}=\mathfrak{c}\ (\mathfrak{m}=1, 2, \ldots).$$

Changing slightly the proof of (9), it can be easily shown that, for every cardinal number $\mathfrak{n}$,

(10) $$\aleph_0.\mathfrak{n}=\mathfrak{n}+\mathfrak{n}+\mathfrak{n}+\ldots$$

and so for $\mathfrak{n}=\mathfrak{c}$

$$\aleph_0.\mathfrak{c}=\mathfrak{c}+\mathfrak{c}+\mathfrak{c}+\ldots=\mathfrak{c}, \text{ by (8)}.$$

To obtain the product $\mathfrak{c}.\mathfrak{c}$, let M and N be each the set of all positive real numbers not exceeding 1. The set P will, therefore, be the set of all pairs (x, y), where x and y are real numbers satisfying the inequalities $0<x\leqq 1$ and $0<y\leqq 1$. It can be shown that the set P has the same potency as the set E of all real numbers z satisfying the inequality $0<z\leqq 1$.

In fact, let (x, y) be an element of P. Develop x and y as infinite decimal fractions, *e.g.*

$$x=0\cdot 4\,3\,0\,0\,7\,0\,9\,5\,0\,0\,0\,8\,3\ldots,$$
$$y=0\cdot 0\,5\,6\,0\,0\,3\,0\,0\,0\,1\,4\,0\,2\ldots.$$

Divide the digits to the right of the decimal point into groups by means of a stroke after each significant figure; we thus get an infinite sequence of groups:

$$4\,|3\,|0\,0\,7\,|0\,9\,|5\,|0\,0\,0\,8\,|3\,|\ldots$$
$$0\,5\,|6\,|0\,0\,3\,|0\,0\,0\,0\,1\,|4\,|0\,2\,|\ldots$$

Place the groups of the second sequence between the successive groups of the first, we thus get a new sequence of groups:

$$4\,|0\,5\,|3\,|6\,|0\,0\,7\,|0\,0\,3\,|0\,9\,|0\,0\,0\,1\,|5\,|4\,|0\,0\,0\,8\,|0\,2\,|3\,|\ldots$$

and omitting the strokes we get an infinite sequence of digits, which is the decimal part of a certain number

$$z=0\cdot 4\,0\,5\,3\,6\,0\,0\,7\,0\,0\,3\,0\,9\,0\,0\,0\,1\,5\,4\,0\,0\,0\,8\,0\,2\,3\ldots,$$

which we correlate with the pair (x, y).

It is easily seen that such a correlation establishes a (1, 1) correspondence between the elements of the set P and those of E. But $\overline{\overline{E}}=\mathfrak{c}$; hence, $\overline{\overline{P}}=\overline{\overline{E}}=\mathfrak{c}$, and since $\overline{\overline{P}}=\overline{\overline{M}}.\overline{\overline{N}}=\mathfrak{c}.\mathfrak{c}$ we have

(11) $$\mathfrak{c}.\mathfrak{c}=\mathfrak{c}.$$

It follows from the above that the set of all pairs (x, y) of real numbers x, y has the same potency as the set of all real numbers.

Geometrically, this means that the set of all points in the plane has the same potency as the set of all points in a straight line and, therefore, also as the set of all points in a finite segment.

The definition of a product of two cardinal numbers may be extended to an infinite sequence. It follows readily that if $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ are given cardinal numbers and $M_1, M_2 \ldots, M_n$ sets such that $\overline{\overline{M_k}} = \mathfrak{m}_k$, for $k=1, 2, \ldots, n$, then the cardinal number $\mathfrak{m}_1 . \mathfrak{m}_2 \ldots \mathfrak{m}_n$ is the potency of the set of all combinations $(m_1, m_2, \ldots, m_n)$, where $m_k \in M_k$, for $k=1, 2, \ldots, n$. Similarly, the infinite product

$$\mathfrak{m}_1 . \mathfrak{m}_2 . \mathfrak{m}_3 \ldots$$

of cardinal numbers is the potency of the set P of all the infinite sequences

$$m_1, m_2, m_3, \ldots,$$

where $m_k \in M_k$, for $k=1, 2, \ldots$ and $\overline{\overline{M_k}} = \mathfrak{m}_k$ $(k=1, 2, \ldots)$.

In particular, let M_k be the set consisting of the numbers 0 and 1. The set P will, therefore, be the set of all infinite sequences

$$a_1, a_2, a_3, \ldots \tag{12}$$

consisting of the numbers 0 and 1. Denote by Q the set of the sequences belonging to P in which there is an infinite number of ones, and by R the remainder of P. R consists, therefore, of all those sequences in which, from a certain stage onwards, there are only zeros and so has the same potency as the set of all finite sequences consisting of 0 and 1, which is a countable set. The set Q, however, has the same potency as the set X of all positive real numbers ≤ 1 and so potency $\mathfrak{c}$. The (1, 1) correspondence between the elements of Q and those of X may be easily established if we correlate the sequence (12) with the number

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots$$

(which obviously belongs to X).

Hence $\overline{\overline{P}} = \mathfrak{c}$, and so

$$\mathfrak{c} = 2.2.2 \ldots . \tag{13}$$

Similarly, it can be shown that

$$\mathfrak{c} = \aleph_0 . \aleph_0 . \aleph_0 \ldots \tag{14}$$

for, it is sufficient to take as M_k ($k=1, 2, \ldots$) the set of all natural numbers, and so P will be the set of all infinite sequences

$$n_1, n_2, n_3, \ldots \tag{15}$$

of natural numbers, which has the same potency as the set X of all positive real numbers ≤ 1. To establish a (1, 1) correspondence between the elements of P and X it is sufficient to correlate the number

$$\frac{1}{2^{n_1}} + \frac{1}{2^{n_1+n_2}} + \frac{1}{2^{n_1+n_2+n_3}} + \ldots$$

of the set X with the sequence (15).
Furthermore,

$$\mathfrak{c} = \mathfrak{c}.\mathfrak{c}.\mathfrak{c}. \ldots \tag{16}$$

To prove this let $M_k = X$, for $k=1, 2, \ldots$; we show that the set P of all infinite sequences consisting of elements of X has the same potency as the set X. To establish a (1, 1) correspondence between the elements of P and X, correlate the sequence

$$x_1, x_2, x_3, \ldots \tag{17}$$

which belongs to P with a number x in the form

$$g_1{}'g_2{}'g_1{}''g_3{}'g_2{}''g_1{}'''g_4{}'g_3{}'' \ldots, \tag{18}$$

where $g_n{}'g_n{}''g_n{}''' \ldots$ designates x_n, and the sequence (17) is designated by the double sequence

$$\begin{array}{l} g_1{}'g_1{}''g_1{}''' \ldots \\ g_2{}'g_2{}''g_2{}''' \ldots \\ g_3{}'g_3{}''g_3{}''' \ldots \\ \ldots\ldots\ldots\ldots, \end{array}$$

from which (18) is obtained by the diagonal method.

5. Let P and Q be two given sets. If with every element of P there is correlated an element of Q, where the same element of Q may be correlated with several elements of P, we obtain a mapping

of the set P on the set Q. Let now $\mathfrak{m}$ and $\mathfrak{n}$ be two cardinal numbers and M, N two sets such that $\overline{\overline{M}}=\mathfrak{m}$, $\overline{\overline{N}}=\mathfrak{n}$; then the power $\mathfrak{m}^{\mathfrak{n}}$ is defined to be the potency of the set of all mappings of the set N on the set M. It can be easily shown that for any three cardinal numbers $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ we have

$$\mathfrak{m}^{\mathfrak{n}+\mathfrak{p}} = \mathfrak{m}^{\mathfrak{n}}.\mathfrak{m}^{\mathfrak{p}}$$

$$(\mathfrak{m}\mathfrak{n})^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{p}}.\mathfrak{n}^{\mathfrak{p}}$$

$$(\mathfrak{m}^{\mathfrak{n}})^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{n}\mathfrak{p}}.$$

If n be a natural number, we have obviously

$$\mathfrak{m}^{n} = \mathfrak{m}.\mathfrak{m}\ldots\mathfrak{m} \ (n \text{ factors}).$$

It follows also readily from the definitions of a power and of an infinite product of cardinal numbers (§ 4), that

(1) $$\mathfrak{m}^{\aleph_0} = \mathfrak{m}.\mathfrak{m}.\mathfrak{m}.\ldots$$

In particular, for $\mathfrak{m}=2$ we obtain from (13)

(2) $$2^{\aleph_0} = \mathfrak{c}.$$

From (1), (14), and (16) we obtain

$$\aleph_0^{\aleph_0} = \mathfrak{c}^{\aleph_0} = \mathfrak{c}.$$

Let N be a set of potency $\mathfrak{n}$; then $2^{\mathfrak{n}}$ will be the cardinal number of the set E of all subsets of N, the null set and the set N being included. Thus $2^{\aleph_0}$ or $\mathfrak{c}$ is the cardinal number of all subsets of the set of natural numbers, and $2^{\mathfrak{c}}$ is the cardinal of the set of subsets of the set of all real numbers and so the cardinal of the set of all functions of a real variable.

6. Given two cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$, we say that $\mathfrak{m} < \mathfrak{n}$ if the set M of potency $\mathfrak{m}$ has equal potency with a subset of the set N whose potency is $\mathfrak{n}$, and if there is no subset of M of equal potency with N. We cannot, however, as yet state that every two cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ are related to each other by one of the three signs $>$, $=$, $<$.

It follows at once from the definition of inequality of cardinal numbers, that

$$\mathfrak{n} < \aleph_0, \text{ for } \mathfrak{n} = 1, 2, 3, \ldots,$$

and

$$\aleph_0 < \mathfrak{c}.$$

For, if M be the set of all natural numbers and N the set of all real numbers, then $\overline{\overline{M}} = \aleph_0$, $\overline{\overline{N}} = \mathfrak{c}$, where M has the same potency as a certain subset of N but not conversely (since N is non-countable).

It is, however, still unknown whether there are cardinal numbers $\mathfrak{m}$ satisfying the inequality $\aleph_0 < \mathfrak{m} < \mathfrak{c}$. The assumption that there are no such cardinal numbers is known as the *Hypothesis of the continuum*. The assumption that there is no cardinal number between $\mathfrak{m}$ and $2^{\mathfrak{m}}$, whatever be the transfinite number $\mathfrak{m}$, is known as the *Cantor aleph-hypothesis*. It can be shown that every cardinal number $\mathfrak{m}$ satisfies the inequality

$$(1) \qquad 2^{\mathfrak{m}} > \mathfrak{m};$$

in other words, *the set of all subsets of a given aggregate has potency greater than that of the aggregate.* From (1) we get at once the infinite sequence of inequalities

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < 2^{2^{2^{\aleph_0}}} < \ldots,$$

which shows that there is an infinite number of transfinite cardinal numbers.

7. The following is the so-called multiplicative axiom stated by Zermelo in 1904 and tacitly implied in several of the preceding results:

For every aggregate M consisting of sets E, non-null and mutually exclusive, there exists (at least one) aggregate N containing one, and one only, element of each set E.

The meaning of this axiom may be explained by the following examples:

Divide all real numbers into sets assigning two numbers to the same set if and only if their difference is rational. We thus get an aggregate M of mutually exclusive, non-null sets. By the multiplicative axiom, there exists a set N containing one and only one number of each set E. No one, however, has been able so far to construct the set N, for it is impossible in this case to put down a law of selection which would pick out a certain element of the set E. This has lead some mathematicians to doubt even the probability of the truth of the axiom. Consider another example. Divide all countable sets of points on a straight line which are not symmetrical with respect to the point 0 into classes, assigning to the same class those sets which are symmetrical images of each other with respect to the point 0. There will obviously be two sets in each class. By the multiplicative axiom there exists a set N containing one set only of each pair, but we cannot devise any rule which would enable us to select this set. The existence of the set N is, therefore, deduced only on the basis of the multiplicative axiom.

If, however, all points of a straight line be divided into classes, assigning to the same class two sets if and only if they are mutually exclusive and their sum gives the whole line, then the set N may be actually constructed; for it is sufficient to assign to N that set of each class which contains the point 0.

We shall next consider some of the applications of the axiom. Let M be an aggregate of potency $\mathfrak{m}$ consisting of mutually exclusive non-null sets. By the multiplicative axiom there exists a set N containing one and only one element of each set belonging to M. We shall evidently have $\overline{\overline{N}}=\overline{\overline{M}}$; hence $\overline{\overline{N}}=\mathfrak{m}$. On the other hand, N being obviously a subset of the sum S of all sets constituting M, we have

$$\overline{\overline{S}}\geqq\overline{\overline{N}}, \text{ and so } \overline{\overline{S}}\geqq\mathfrak{m}.$$

We thus arrive at the following result: *If any aggregate be divided into mutually exclusive sets, the set of these sets has potency $\leqq$ the potency of the original aggregate.*

It follows readily from the above that the potency of any set of points in the plane is not less than the potency of the set of its projections. For the given set may be divided into subsets consisting of all points which project into the same point.

Furthermore, if a set of potency $\mathfrak{c}$ be divided into two parts, one of them at least has potency $\mathfrak{c}$. For let P be the set of all points in the plane and so of potency $\mathfrak{c}$. It will be sufficient to show that there is no division $P=A+B$, where both A and B have potency $<\mathfrak{c}$. Suppose, on the contrary, that such a division exists. The projection of the set A on the x-axis has by the above potency $\leq$ the potency of A, hence $<\mathfrak{c}$. There exists, therefore, an abscissa x_0 such that the straight line $x=x_0$ does not contain points of A. We conclude, similarly, that there exists an ordinate y_0 such that the straight line $y=y_0$ does not contain points of B. Hence the point (x_0, y_0) belongs neither to A nor to B, contrary to the fact that $P=A+B$. The above statement is, therefore, proved.

There are other more general forms of the multiplicative axiom, *e.g.*, the following (Hilbert):

There exists a correspondence which correlates to each property W possessed by at least one object a certain element $\tau(W)$ possessing the property W.

This axiom leads to the so-called *general principle of selection* (Zermelo). E being any set, denote by W_E the property of belonging to the set E. If E is not a null set there exists obviously at least one object which has the property W_E, whereas $\tau(W_E)$ will be an element of E. Hence, *there exists a correlation which assigns to every non-null set an element of that set.*

There exists, therefore, for every given set a correlation which assigns to every non-null subset of the given set a certain element belonging to that subset.

It can be shown, proceeding from the above, that *every non-null set which is not finite contains a countable subset.* In fact, let E be a given non-null set, which is not finite. Then to each non-null subset C of E corresponds a certain element $\alpha(C)$ of C. Put $p_1=\alpha(E)$, and let E_1 be the set obtained on removing p_1 from E. If E_1 were a null set or finite, then the set E would be finite. Hence E_1 is neither null nor finite. Let further $p_2=\alpha(E_1)$, and let E_2 be the set obtained from E_1 after removing p_2. As above, E_2 is neither null nor finite. Let now $p_3=\alpha(E_2)$, and so on.

We thus obtain an infinite sequence of different elements of the set E

(1) $$p_1, p_2, p_3, \ldots$$

which forms a countable subset of the set E.

A set which contains a countable subset has the same potency as certain of its subsets (*i.e.* it is infinite in the Dedekind sense). For, retaining the above notation, we can establish a (1, 1) correspondence between the sets E and E_1 as follows: correlate every element of E which is not in (1) with itself and every element which belongs to (1) with its successor in (1).

Let now $\mathfrak{u}$ be a cardinal number which is not finite and U a set of potency $\mathfrak{u}$. The set U is, therefore, neither null nor finite and so must contain a subset of potency $\aleph_0$. Hence,

$$\mathfrak{u} \geq \aleph_0$$

for every cardinal number $\mathfrak{u}$ which is not finite. For a finite cardinal number $\mathfrak{n}$ we have obviously the inequality $\mathfrak{n} < \aleph_0$; hence, every cardinal number is $\gtreqless \aleph_0$.

If a set E is such that $\overline{\overline{E}} < \aleph_0$, then E is a finite set. If $\overline{\overline{E}} = \aleph_0$, then E is countable; and if finally $\overline{\overline{E}} > \aleph_0$, then E is non-countable.

Let E be a set which is neither null nor finite. Hence E contains a countable subset P. Remove from E the elements belonging to P, and denote the remainder by R; hence, $E = P + R$, and (since P and R are mutually exclusive)

(2) $$\overline{\overline{E}} = \overline{\overline{P}} + \overline{\overline{R}}.$$

Add to the set E any countable set Q distinct from E; therefore,

(3) $$\overline{\overline{P}} + \overline{\overline{Q}} = \aleph_0 = \overline{\overline{P}},$$

and putting $E + Q = S$, we shall get from (2) and (3)

$$\overline{\overline{S}} = \overline{\overline{E}} + \overline{\overline{Q}} = \overline{\overline{P}} + \overline{\overline{R}} + \overline{\overline{Q}} = \overline{\overline{P}} + \overline{\overline{R}} = \overline{\overline{E}}.$$

Hence, *the potency of a set which is neither null nor finite does not change if we add to it a countable set of elements.*

Let now E denote a non-countable set, P its countable subset, R a set defined as above. Hence R is neither a null set nor finite (since then $E = P + R$ would be countable), and so it will not change its potency if we add to it the countable set P, which gives $\overline{\overline{E}}=\overline{\overline{R}}$. We have thus proved that *the potency of a non-countable set does not change if we remove from it a countable set of elements.*

ORDERED SETS

8. A set E is said to be *ordered* if there exists a convention according to which it can be said of any two distinct elements of the set that one element precedes the other in the set. This is expressed in writing by $a \prec b$, *i.e.* a comes before b or $a \succ b$, *i.e.* a comes after b. Whatever this convention may be the following two conditions must be satisfied:

1. Relation $a \prec b$ excludes the relation $b \prec a$ (asymmetry).
2. If $a \prec b$ and $b \prec c$ then $a \prec c$ (transitivity).

An element of E, which is not preceded by any other, is said to be the *first* element; and one which is not followed by any other is called the *last* element of the set E.

The set of natural numbers apart from its usual order may be also ordered according to the following convention. Of two numbers the one with the least number of different prime factors will come first, and in case of an equal number of different prime factors the one of smaller value. It is easily seen that this agreement orders the set of natural numbers (*i.e.* conditions 1 and 2 are satisfied). Hence we get

$$1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 7 \prec 6 \prec 34 \prec 35 \prec 30 \prec \ldots$$

Two ordered sets G and Γ are said to be *similar*, *i.e.* $G \simeq \Gamma$, if there exists a (1, 1) correspondence between their elements which leaves the order relations between corresponding pairs of elements invariant. Thus if a, b are any two elements of G and α, β their corresponding elements in Γ, then the relation

$$a \prec b$$

implies the relation

$$\alpha \prec \beta$$

and conversely.

It is easily seen that an ordered set is similar to itself and two sets similar to a third are similar to each other. (The relation of similarity is, therefore, symmetrical and transitive.)

Divide all ordered sets into classes assigning two sets to the same class if and only if they are similar. Then sets belonging to the same class are said to be of the same *ordinal type*. Ordinal types thus serve as symbols to designate the various classes.

Two ordered sets of the same type have obviously the same potency, but the converse is not necessarily true. The set of all natural numbers and the set of all rational numbers have the same potency (both countable), but when ordered according to their magnitude, are evidently of different types.

The ordinal type of a set E is denoted according to Cantor by $\overline{\overline{E}}$. If n be a natural number, then all ordered sets consisting of n elements are easily seen to be similar to the set of the first n natural numbers. We are, therefore, led to assume n for the symbol of the corresponding ordinal type.

The ordinal type of the class which contains the set of all natural numbers in their successive order is denoted by ω, again following Cantor. The set of all negative integers $\ldots -4 \prec -3 \prec -2 \prec -1$ ordered according to their algebraic magnitude belongs to a different type ordered in the opposite direction to that of ω and is denoted by ω^*.

Generally, if α be a given type, then the type reversed in order to that one is denoted by α^*. It may happen that $\alpha^* = \alpha$; this is the case for every finite type, also for the type η of the set of all rational numbers ordered according to their magnitude, as well as for the type λ of the set of all real numbers ordered according to their magnitude.

9. A set E is said to be *dense* if between every two of its elements there is at least one element of E and, therefore, an infinite number of them. Thus the set of all rational numbers, and the set of all real numbers, each ordered according to magnitude, are both dense.

It can be proved that *two countable, dense, and ordered sets, which have neither a first nor a last element, are similar, and are, therefore, of type* η. Similarly, it can be easily proved that *every countable ordered set is similar to a certain set of rational numbers which are ordered according to their magnitude.*

A *cut* of an ordered set E is a division of all the elements of the set into two non-null classes A and B such that every element of the class A precedes every element of the class B. Such a division is denoted by $[A, B]$.

If in a given cut $[A, B]$ the class A has a last element and the class B a first element, then this cut is said to give rise to a *jump*. Thus in the set of natural numbers each cut supplies a jump. Obviously, in order that an ordered set be dense it is necessary and sufficient that none of its cuts gives rise to a jump.

If in a cut $[A, B]$ the class A has no last term and the class B no first term, the cut is said to produce a *gap*. Thus in the set of all rational numbers different from zero the cut into the class of negative rational numbers and the class of positive rational numbers produces a gap.

A set which has neither jumps nor gaps is said to be *continuous*.

If a given ordered set E has gaps, these may be removed by the addition of new elements in the following way. To each cut $[A, B]$ which produces a gap, we assign a new element not contained in E which is considered as following all the elements of A and preceding all those of B. Of two elements assigned to different cuts $[A, B]$, $[A_1, B_1]$ we consider the first as preceding the second when A is a proper subset of A_1 and as following the second when B is a proper subset of B_1.

It can be easily shown that adding such new elements to E we obtain a new ordered set F which has no gaps.

10. Let ϕ_1 and ϕ_2 be two ordinal types, O_1 and O_2 two mutually exclusive ordered sets such that $\overline{O_1} = \phi_1$ and $\overline{O_2} = \phi_2$. Put $O = O_1 + O_2$ and order O as follows: two elements of O which belong both to O_1 or both to O_2 are to retain the ordinal relation which they had in their respective sets. Of two elements of O, one belonging to O_1 and one to O_2, the one belonging to O_1 will precede the other.

The set O is thus easily seen to be ordered and its type $\phi = \overline{O}$ will depend solely on the types ϕ_1, ϕ_2 and not on the sets O_1, O_2 which correspond to these types. We call ϕ the sum of ϕ_1 and ϕ_2 and write

$$\phi = \phi_1 + \phi_2.$$

It follows from the definition of the types ω and ω^* that

$$\omega^*+\omega$$

is the ordinal type of the set of all integers ordered according to their algebraic magnitude, *i.e.*

$$\ldots \prec -2 \prec -1 \prec 0 \prec 1 \prec 2 \prec \ldots,$$

while the sum

$$\omega+\omega^*$$

is the ordinal type of the class containing the set of the reciprocals of all the integers (zero excluded) ordered according to their algebraic magnitudes, *i.e.*

$$-\frac{1}{1} \prec -\frac{1}{2} \prec -\frac{1}{3} \prec \ldots \prec \frac{1}{3} \prec \frac{1}{2} \prec \frac{1}{1}.$$

The ordinal types $\omega^*+\omega$ and $\omega+\omega^*$ are different, for the first one does not contain a first nor last element whereas the second has both. The first type has no gaps, the second has a gap. Hence

$$\omega^*+\omega \neq \omega+\omega^*,$$

and so addition of ordinal types is not necessarily commutative.

Similarly, it may be shown that

$$1+\omega \neq \omega+1,$$

but if we put $\xi=\omega+\omega^*$ we find that

$$1+\xi=\xi+1$$

(since each sum is equal to ξ).

Furthermore, it is easily seen that

$$\eta+\eta=\eta,\ \lambda+\lambda\neq\lambda,$$

and the relation

$$(\alpha+\beta)^*=\beta^*+\alpha^*$$

is true for every type α and β.

The definition of a sum of ordinal types may be extended immediately to any finite number of types, and such a sum is easily seen to satisfy the associative law. Thus

$$(\omega+1)+\omega=\omega+(1+\omega)=\omega+\omega.$$

Similarly,

$$\eta+1+\eta=\eta,\ \lambda+1+\lambda=\lambda.$$

The sum of ordinal numbers may be further extended to infinite sequences. Let

(1) $$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots$$

be an infinite sequence of ordinal types and

(2) $$O_1,\ O_2,\ O_3,\ \ldots$$

be mutually exclusive ordered sets such that $\overline{O_n}=\alpha_n$, for $n=1, 2, 3, \ldots$ Put

(3) $$O=O_1+O_2+O_3+\ldots$$

and let O be ordered as follow: if two elements of O belong to the same set O_n, then they retain in O the ordinal relation which they had in O_n, but if two elements of O belong to different sets, that element will come first which belongs to the earlier set in the sequence. It is easily seen that the set O will be ordered by the above procedure and its type will depend solely on the sequence (1) of types and not on the sets of sequence (2) corresponding to those types.

We may, therefore, say that every infinite series of ordinal types has a definite (well-defined) sum. *E.g.*,

$$\omega=1+1+1+\ldots$$

but also

$$\omega=2+2+2+\ldots=1+2+3+4+\ldots=2+2^2+2^3+\ldots$$

We also note that

$$\eta=\eta+\eta+\eta+\ldots,\ \lambda=\lambda+1+\lambda+1+\lambda+1+\lambda+\ldots$$

Let now ϕ and ψ be two ordinal types, U and V two ordered sets such that $\overline{U}=\phi$, $\overline{V}=\psi$. Denote by P the set of all pairs (u, v), where $u \in U$ and $v \in V$, and order P, assuming that

$$(u, v) \prec (u_1, v_1)$$

if $v \prec v_1$ (in V) or if $v=v_1$ and $u \prec u_1$ (in U).

It is easily seen that such an agreement will order P (*i.e.* conditions 1 and 2 of § 8 will be satisfied) and that the type of P will depend only on the types ϕ and ψ. The ordinal type of P is defined to be the product of the types ϕ and ψ and is written $\phi.\psi$.

In order to obtain the product $2.\omega$, consider the set U of type 2 consisting of the numbers 1 and 2 and the set of natural numbers ordered according to increasing magnitude. The set P will consist of all pairs (u, v) where u is 1 or 2 and v a natural number; ordering P as above we obtain the sequence

$$(1, 1) \prec (2, 1) \prec (1, 2) \prec (2,2) \prec (1, 3) \prec (2, 3) \prec (1, 4) \prec (2, 4) \prec \dots,$$

which is of type ω. Hence

$$2.\omega = \omega.$$

Similarly,

$$n.\omega = \omega, \text{ for every natural } n.$$

The product $\omega.2$ is the type of the set

$$(1, 1) \prec (2, 1) \prec (3, 1) \prec (4, 1) \prec \dots \prec (1, 2) \prec (2, 2) \prec (3, 2) \prec \dots$$

and is, therefore, of type $\omega + \omega$. Hence

$$\omega.2 = \omega + \omega,$$

and, therefore,

$$\omega.2 \neq 2.\omega.$$

Multiplication of ordinal types is thus seen to be non-commutative. We also note that $\eta.2 = \eta + \eta = \eta$, but $2.\eta \neq \eta$ (since the type 2η contains jumps). Similarly $2.\lambda \neq \lambda.2$, $\eta.\lambda \neq \lambda.\eta$.

The multiplication of ordinal types is, however, associative and distributive if the second factor is a sum. Thus

$$(\phi.\psi).\theta = \phi.(\psi.\theta),$$

$$\phi.(\psi + \theta) = \phi.\psi + \phi.\theta,$$

but

$$(1+1).\omega \neq 1.\omega + 1.\omega,$$

since the left hand side is equal to ω, whereas on the right we have $\omega + \omega$.

We have obviously for every ordinal type ϕ and every natural number n the product $\phi.n$ equal to the sum of n terms each equal to ϕ. Similarly,

$$\phi.\omega = \phi + \phi + \phi + \ldots$$

11. A set is said to be *well-ordered* if each of its non-null subsets has a first element.

Every finite ordered set is well-ordered. Sets whose types are ω, $\omega+1$, $\omega+\omega$, $\omega.\omega$ are evidently well-ordered; but the sets whose types are ω^*, η, λ are not well-ordered.

A well-ordered set cannot contain an infinite subset

$$a_1 \succ a_2 \succ a_3 \succ \ldots ,$$

i.e. one of type ω^*; for in such case it would contain a subset without a first element, contrary to definition.

A non-null subset of a well-ordered set is obviously well-ordered.

Well-ordered sets have the following important property which is known as the principle of transfinite induction:

If a certain theorem T

1, *is true for the first element of a well-ordered set W,*

2, *is true for an element a of W, if it is true for every element preceding a, then T is true for every element of W.*

Indeed, suppose that a certain theorem T satisfies conditions 1 and 2 but that there exist elements of W for which it is not true; let N be the set of such elements. N will, therefore, be a non-null subset of a well-ordered set and so will have a first element a, say. It follows from the definition of N that T must be true for every element x of W which is such that $x \prec a$; but by condition 2, T must be true for a, which is contrary to the fact that $a \epsilon N$. The principle of transfinite induction for well-ordered sets is, therefore, proved.

A well-ordered set may be similar to a proper subset of itself, *e.g.* the set of all natural numbers ordered according to their increasing magnitudes is similar to its subset consisting of the even numbers. We shall now prove that *if a well-ordered set W is similar to a proper subset S of itself, then an element of W cannot be correlated with an element of S which precedes it.*

For suppose that in the correlation of W and its subset S to the element a_1 of W corresponds a_2 of S such that $a_2 \prec a_1$; let a_3 be the element of S which corresponds to a_2 of W, hence $a_3 \prec a_2$, since $a_2 \prec a_1$ in W. Let now a_4 be the element of S corresponding to a_3 in W, and, since $a_3 \prec a_2$, we have $a_4 \prec a_3$. Arguing thus repeatedly we are led to an infinite sequence

$$a_1 \succ a_2 \succ a_3 \succ \ldots$$

of W, which is impossible, and so the above statement is true.

Let W be a well-ordered set and a one of its elements. The set of all elements of W preceding a is called a *section* of W determined by the element a and denoted by $A(a)$. It follows from the above theorem that *a well-ordered set cannot be similar to a section of itself nor to any subsets of such a section;* for, in the similar correlation between the set W and a set $S \subset A(a)$ to an element a of W, there would have to correspond an element a' of the section $A(a)$, and so an element preceding a, contrary to the above theorem.

Given two well-ordered sets A and B it can be easily shown that either $A \simeq B$ or $A(a_0) \simeq B$ or else $A \simeq B(b_0)$, *i.e.* two well-ordered sets are either similar, or one of them is similar to a section of the other.

12. Ordinal types of well-ordered sets are called *ordinal numbers.*

If ϕ and ψ be two different ordinal numbers, then, as seen above, of two sets of these types just one is similar to a section of the other. In one case we write $\phi < \psi$, in the other $\psi < \phi$ (or $\phi > \psi$). It is convenient to include the number zero in the set of ordinal numbers, it being defined as the smallest of all ordinal numbers.

Let W be a well-ordered set of type ϕ. Let, further, a be an element of W, and $\psi(a)$ the ordinal type of the section $A(a)$, where $\psi(a) = 0$ if a be the first element of W; we shall have obviously $\psi(a) < \phi$ and $\psi(a_1) < \psi(a_2)$, for $a_1 \prec a_2$. Hence, to every element of W there corresponds an ordinal number $\psi < \phi$, and to a later element corresponds a larger number. Conversely, every ordinal number $\psi < \phi$ corresponds to some element of W; in fact, if $\psi < \phi$, then the set W_1 of type ψ is similar to a certain section $A(a)$ of W, and so $\psi = \psi(a)$. Hence, *a well-ordered set of type ϕ is similar to the set of all ordinal numbers $< \phi$ (0 included), which are ordered according to increasing magnitudes.*

The elements of a well-ordered set may, therefore, be denoted by the symbol a_ψ, where the subscripts $\psi = \psi(a)$ are ordinal numbers (including 0 which is the subscript of the first element a_0). Thus, the n elements of a finite set may be denoted by

$$a_0, a_1, \ldots, a_{n-1},$$

the elements of a set of type ω by

$$a_0, a_1, a_2, \ldots,$$

the elements of a set of type $\omega+n$ (n a natural number), by

$$a_0, a_1, \ldots, a_\omega, a_{\omega+1}, \ldots, a_{\omega+n-1},$$

and so on. Generally, the elements of a well-ordered set of type ϕ may be written down as a transfinite sequence of type ϕ, *i.e.*

$$a_0, a_1, \ldots, a_\omega, \ldots, a_\xi, \ldots (\xi < \phi).$$

Every set E of ordinal numbers is well-ordered. For, let ϕ be any number of E; the set E_1 of all ordinal numbers $<\phi$ is, as previously shown, well-ordered of type ϕ. If ϕ is not the smallest number of E then the set of all numbers of E which belong to E_1 is not null, and so, as a subset of a well-ordered set, will have a first element, α say. It is easily seen that α is the smallest element of E. Hence, every set E of ordinal numbers has a smallest number, and this proves the above statement.

It can be easily shown that *the sum of two positive ordinal numbers is always greater than the first number and* $\geq$ *either of the numbers.* From this it follows at once that for every ordinal number α

$$\alpha+1 > \alpha.$$

The number $\alpha+1$ is said to be the *successor* of α. It can be easily shown that there is no ordinal number ξ satisfying the inequalities $\alpha < \xi < \alpha+1$. Hence, every ordinal number has a successor. But not every ordinal number has a *predecessor*, *i.e.* a number for which the given one is a successor. Thus, the numbers ω, $\omega+\omega$ have no predecessors. Ordinal numbers which possess predecessors, *i.e.* those of the form $\alpha+1$ are said to be of the *first kind*, and those without a predecessor are of the *second kind*.

Let E denote a well-ordered set of type α whose elements are ordinal numbers; we may, therefore, represent these elements by the symbols ϕ_ξ, where ξ is any ordinal number $< \alpha$ (0 included). In other words, the elements of the set E may be represented by the transfinite sequence

$$(1) \qquad \phi_0, \phi_1, \phi_2, \ldots, \phi_\omega, \phi_{\omega+1}, \ldots, \phi_\xi, \ldots, (\xi < \alpha),$$

of type α.

If for $\xi < \eta < \alpha$ we have $\phi_\xi < \phi_\eta$, the sequence is said to be *increasing*. In such a case the smallest ordinal number λ which exceeds every term of (1) is called the limit of the sequence, and we write

$$\lambda = \lim_{\xi < \alpha} \phi_\xi.$$

Thus,

$$\omega = \lim_{n<\omega} n = \lim_{n<\omega} n^2 = \lim_{n<\omega} 2^n;$$

$$\omega + \omega = \lim_{n<\omega} (\omega + n);$$

every number α of the second kind may, therefore, be written as

$$\alpha = \lim_{\xi<\alpha} \xi,$$

i.e. every number of the second kind is the limit of all ordinal numbers less than it.

All finite ordinal numbers (0 included) are said to be numbers of the *first class*. All ordinal numbers which are ordinal types of countable sets constitute the *second class* of numbers.

The set E of all numbers of the first and second classes is non-countable. Indeed, suppose it is countable; E being a set of ordinal numbers is, as shown previously, a set well-ordered according to the magnitude of the numbers. Let Ω be its type, and so, as the type of a well-ordered countable set, it would be a number of the second class, *i.e.* an element of E, ϕ say.

But every ordinal number is the type of the set of all ordinal numbers less than it; hence $\Omega = \phi$ would be the type of a section of the set E determined by the element ϕ of this set. Thus, the set E (which is of type Ω) would be similar to a section of itself, which is impossible.

Hence, the set of all numbers of the first and second classes is non-countable. The potency of this set is denoted by $\aleph_1$. Obviously, $\aleph_0 < \aleph_1$, and it may be easily seen that there is no cardinal number between $\aleph_0$ and $\aleph_1$. For, suppose $\mathfrak{m}$ is a cardinal number such that

$$\aleph_0 < \mathfrak{m} < \aleph_1;$$

then there exists a subset E_1 of the set E such that $\overline{\overline{E_1}} = \mathfrak{m}$. But since $\mathfrak{m} < \aleph_1$, E_1 must be similar to a section of E determined by some element, ϕ say; since $\phi \in E$ and so is a number of the first or second class, the section of E determined by ϕ is at most countable. We have, therefore, $\mathfrak{m} \leq \aleph_0$, contrary to hypothesis.

The cardinal number $\aleph_1$ follows, therefore, immediately after $\aleph_0$. It is, however, still unknown whether $\aleph_1 = \mathfrak{c}$, or $\aleph_1 \neq \mathfrak{c}$. The assumption that $\aleph_1 = \mathfrak{c}$ is equivalent to the *hypothesis of the continuum.*

All ordinal numbers which are types of well-ordered sets of potency $\aleph_1$ constitute the numbers of the *third class.* The smallest of them is easily seen to be Ω.

It can be shown that the set of all the numbers of the third class has potency $> \aleph_1$; its potency is denoted by $\aleph_2$. The potency of a well-ordered set is generally called *aleph* (denoted by $\aleph$), and it can be shown that if a cardinal number is an aleph, then

$$\aleph + \aleph = \aleph \, . \, \aleph = \aleph.$$

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