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# Linear Algebra

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## Preface

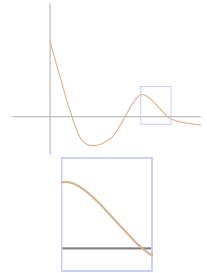
Up close, smooth things look flat—the picture behind differential calculus. In mathematical language, we can approximate smoothly varying functions by linear functions. In calculus of several variables, the resulting linear functions can be complicated: you need to study linear algebra.

Problems appear throughout the text, which you must learn to solve. They often provide vital results used in the course. Most of these problems have hints, particularly the more important ones. There are also review problems at the end of each section, and you should try to solve a few from each section. Try to solve each problem first before looking up the hint. Never use decimal approximations (for instance, from a calculator) on any problem, except to check your work; many problems are very sensitive to small errors and must be worked out precisely. Whenever ridiculously large numbers appear in the statement of a problem, this is a hint that they must play little or no role in the solution.

The prerequisites for this course are basic arithmetic and elementary algebra, typically learned in high school, and some comfort and facility with proofs, particularly using mathematical induction. You can't prove that all men are wearing hats just by pointing out one example of a man in a hat; most proofs require an argument, and not just examples. Polya [2] and Solow [3] explain induction and provide help with proofs. Bretscher [1] and Strang [5] are excellent introductory textbooks of linear algebra.

### For teachers

These notes are drawn from lectures given at University College Cork in the spring of 2006, for a first year introduction to linear algebra. The course aims for a complete proof of the spectral theorem, but with two gaps: (1) the proof for real symmetric matrices relies on the minimum principle, so requires the existence of a minimum of any quadratic function on the sphere; (2) the proof for complex self-adjoint matrices requires the fundamental theorem of algebra to prove that complex matrices have eigenvalues. We fill these gaps in appendices, but the students are not expected to work through the more difficult material provided in these appendices. There are a number of proofs that are not quite complete, giving only the idea behind the proof. In each case, giving a complete proof just requires adding in summation notation, which students often find confusing. I teach a small selection of the proofs.



I aim to make each chapter one lecture of material, but that hasn't always worked out. With that aim in mind, the chapters are unusually small, but students should find them easier to grasp. The book presents just the material required to reach the spectral theorem for self-adjoint matrices. This gives the course a natural focal point.

We first approach determinants by direct calculation, shying away from proofs via permutations, and from Cramer's rule and cofactor inversion, which are computationally infeasible. I hope that students learn how to compute with simple examples by hand, and then learn the theory. I ignored purely numerical topics and paid no attention to computational efficiency.

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## Matrix Calculations





## Chapter 1

### Solving Linear Equations

In this chapter, we learn how to solve systems of linear equations by a simple recipe, suitable for a computer.

#### Elimination

Consider equations

$$\begin{aligned} -6 - x_3 + x_2 &= 0 \\ 3x_1 + 7x_2 + 4x_3 &= 9 \\ 3x_1 + 5x_2 + 8x_3 &= 3. \end{aligned}$$

They are *linear* because they are sums of constants and constant multiples of variables. How can we solve them (or teach a computer to solve them)? To solve means to find values for each of the variables  $x_1, x_2$  and  $x_3$  satisfying all three of the equations.

#### Preliminaries

- a. Line up the variables:

$$\begin{aligned} x_2 - x_3 &= 6 \\ 3x_1 + 7x_2 + 4x_3 &= 9 \\ 3x_1 + 5x_2 + 8x_3 &= 3 \end{aligned}$$

All of the  $x_1$ 's are in the same column, etc. and all constants on the right hand side.

- b. Drop the variables and equals signs, just writing the numbers.

$$\begin{pmatrix} 0 & 1 & -1 & 6 \\ 3 & 7 & 4 & 9 \\ 3 & 5 & 8 & 3 \end{pmatrix}.$$

This saves rewriting the variables at each step. We put brackets around for decoration.

- c. Draw a box around the entry in the top left corner, and call that entry the *pivot*.

$$\begin{pmatrix} \boxed{0} & 1 & -1 & 6 \\ 3 & 7 & 4 & 9 \\ 3 & 5 & 8 & 3 \end{pmatrix}.$$

Forward elimination

- (1) If the pivot is zero, then swap rows with a lower row to get the pivot to be nonzero. This gives

$$\begin{pmatrix} \boxed{3} & 7 & 4 & 9 \\ 0 & 1 & -1 & 6 \\ 3 & 5 & 8 & 3 \end{pmatrix}.$$

(Going back to the linear equations we started with, we are swapping the order in which we write them down.) If you can't find any row to swap with (because every lower row also has a zero in the pivot column), then move pivot one step to the right  $\rightarrow$  and repeat step (1).

- (2) Add whatever multiples of the pivot row you need to each lower row, in order to kill off every entry under the pivot. ("Kill off" means "make into 0"). This requires us to add  $-(\text{row } 1)$  to row 3 to kill off the 3 under the pivot, giving

$$\begin{pmatrix} \boxed{3} & 7 & 4 & 9 \\ 0 & 1 & -1 & 6 \\ 0 & -2 & 4 & -6 \end{pmatrix}.$$

(Going back to the linear equations, we are adding equations together which doesn't change the answers—we could reverse this step by subtracting again.)

- (3) Make a new pivot one step down and to the right:  $\searrow$ .

$$\begin{pmatrix} \boxed{3} & 7 & 4 & 9 \\ 0 & \boxed{1} & -1 & 6 \\ 0 & -2 & 4 & -6 \end{pmatrix}.$$

and start again at step (1).

In our example, our next pivot, 1, must kill everything beneath it:  $-2$ . So we add  $2(\text{row } 2)$  to (row 3), giving

$$\begin{pmatrix} \boxed{3} & 7 & 4 & 9 \\ 0 & \boxed{1} & -1 & 6 \\ 0 & 0 & 2 & 6 \end{pmatrix}.$$

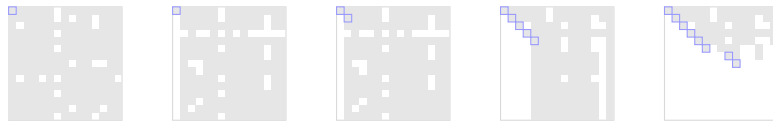


Figure 1.1: Forward elimination on a large matrix. The shaded boxes are nonzero entries, while zero entries are left blank. The pivots are outlined. You can see the first few steps, and then a step somewhere in the middle of the calculation, and then the final result.

We are done with that pivot. Move ↘.

$$\begin{pmatrix} \boxed{3} & 7 & 4 & 9 \\ 0 & \boxed{1} & -1 & 6 \\ 0 & 0 & \boxed{2} & 6 \end{pmatrix}.$$

Forward elimination is done. Let's turn the numbers back into equations, to see what we have:

$$\begin{array}{rclcl} \boxed{3x_1} & +7x_2 & +4x_3 & = & 9 \\ & \boxed{x_2} & -x_3 & = & 6 \\ & & \boxed{2x_3} & = & 6 \end{array}$$

Look from the bottom equation up: each pivot solves for one variable in terms of later variables.

**1.1** Apply forward elimination to

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

**1.2** Apply forward elimination to

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

**1.3** Apply forward elimination to

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



Figure 1.2: Back substitution on the large matrix from figure 1.1. You can see the first few steps, and then a step somewhere in the middle of the calculation, and then the final result. You can see the pivots turn into 1's.

### Back Substitution

Starting at the last pivot, and working up:

- a. Rescale the entire row to turn the pivot into a 1.
- b. Add whatever multiples of the pivot row you need to each higher row, in order to kill off every entry above the pivot.

Applied to our example:

$$\begin{pmatrix} 3 & 7 & 4 & 9 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 2 & 6 \end{pmatrix},$$

Scale row 3 by  $\frac{1}{2}$  :

$$\begin{pmatrix} 3 & 7 & 4 & 9 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Add row 3 to row 2,  $-4$ (row 3) to row 1.

$$\begin{pmatrix} 3 & 7 & 0 & -3 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Add  $-7$ (row 2) to row 1.

$$\begin{pmatrix} 3 & 0 & 0 & -66 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Scale row 1 by  $\frac{1}{3}$ .

$$\begin{pmatrix} 1 & 0 & 0 & -22 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Done. Turn back into equations:

$$\begin{aligned} x_1 &= -22 \\ x_2 &= 9 \\ x_3 &= 3. \end{aligned}$$

Forward elimination and back substitution together are called *Gauss–Jordan elimination* or just *elimination*. (Forward elimination is often called *Gaussian elimination*.) Forward elimination already shows us what is going to happen: which variables are solved for in terms of which other variables. So for answering most questions, we usually only need to carry out forward elimination, without back substitution.

## Examples

More than one solution:

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 &= 7 \\x_1 + 2x_3 &= 1 \\x_2 + x_3 &= 0\end{aligned}$$

Write down the numbers:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 7 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Kill everything under the pivot: add  $-(\text{row } 1)$  to row 2.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & -1 & 1 & -1 & -6 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Done with that pivot; move  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & -1 & 1 & -1 & -6 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Kill: add row 2 to row 3:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & -1 & 1 & -1 & -6 \\ 0 & 0 & 2 & -1 & -6 \end{pmatrix}.$$

Move  $\searrow$ . Forward elimination is done. Let's look at where the pivots lie:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & -1 & 1 & -1 & -6 \\ 0 & 0 & 2 & -1 & -6 \end{pmatrix}.$$

Let's turn back into equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 7 \\ -x_2 + x_3 - x_4 &= -6 \\ 2x_3 - x_4 &= -6 \end{aligned}$$

Look: each pivot solves for one variable, in terms of later variables. There was never any pivot in the  $x_4$  column, so  $x_4$  is a *free variable*:  $x_4$  can take on any value, and then we just use each pivot to solve for the other variables, bottom up.

1.4 Back substitute to find the values of  $x_1, x_2, x_3$  in terms of  $x_4$ .

No solutions: consider the equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 0 \\ 4x_1 + 3x_2 + 5x_3 &= 1. \end{aligned}$$

Forward eliminate:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 4 & 3 & 5 & 1 \end{pmatrix}$$

Add  $-2(\text{row } 1)$  to row 2,  $-4(\text{row } 1)$  to row 3.

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -3 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -3 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Turn back into equations:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ -x_2 - 3x_3 &= -2 \\ 0 &= -1. \end{aligned}$$

You can't solve these equations: 0 can't equal  $-1$ . So you can't solve the original equations either: there are no solutions. Two lessons that save you time and effort:

- a. If a pivot appears in the constants' column, then there are no solutions.
- b. You don't need to back substitute for this problem; forward elimination already tells you if there are any solutions.

## Summary

We can turn linear equations into a box of numbers. Start a pivot at the top left corner, swap rows if needed, move  $\rightarrow$  if swapping won't work, kill off everything under the pivot, and then make a new pivot  $\searrow$  from the last one. After forward elimination, we will say that the resulting equations are in *echelon form* (often called *row echelon form*).

The echelon form equations have the same solutions as the original equations. Each column except the last (the column of constants) represents a variable. Each pivot solves for one variable in terms of later variables (each pivot "binds" a variable, so that the variable is not free). The original equations have no solutions just when the echelon equations have a pivot in the column of constants. Otherwise there are solutions, and any pivotless column (besides the column of constants) gives a free variable (a variable whose value is not fixed by the equations). The value of any free variable can be picked as we like. So if there are solutions, there is either only one solution (no free variables), or there are infinitely many solutions (free variables). Setting free variables to different values gives different solutions. The number of pivots is called the

*rank*. Forward elimination makes the pattern of pivots clear; often we don't need to back substitute.

We often encounter systems of linear equations for which all of the constants are zero (the "right hand sides"). When this happens, to save time we won't write out a column of constants, since the constants would just remain zero all the way through forward elimination and back substitution.

**1.5** Use elimination to solve the linear equations

$$\begin{aligned}2x_2 + x_3 &= 1 \\4x_1 - x_2 + x_3 &= 2 \\4x_1 + 3x_2 + 3x_3 &= 4\end{aligned}$$

Review problems

**1.6** Apply forward elimination to

$$\begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

**1.7** Apply forward elimination to

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

**1.8** Apply forward elimination to

$$\begin{pmatrix} -1 & 2 & -2 & -1 \\ 1 & 2 & -2 & 2 \\ -2 & 0 & 0 & -1 \end{pmatrix}$$

**1.9** Apply forward elimination to

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

**1.10** Apply forward elimination to

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$



**1.11** Apply forward elimination to

$$\begin{pmatrix} 1 & 3 & 2 & 6 \\ 2 & 5 & 4 & 1 \\ 3 & 8 & 6 & 7 \end{pmatrix}$$

**1.12** Apply back substitution to the result of problem 1.3 on page 5.

**1.13** Apply back substitution to

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**1.14** Apply back substitution to

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

**1.15** Apply back substitution to

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

**1.16** Apply back substitution to

$$\begin{pmatrix} 3 & 0 & 2 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

**1.17** Use elimination to solve the linear equations

$$\begin{aligned} -x_1 + 2x_2 + x_3 + x_4 &= 1 \\ -x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_4 &= 2 \end{aligned}$$

**1.18** Use elimination to solve the linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ 2x_1 + 5x_2 + 7x_3 + 11x_4 &= 12 \\ x_2 + x_3 + 4x_4 &= 3 \end{aligned}$$

**1.19** Use elimination to solve the linear equations

$$\begin{aligned} -2x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - 2x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 - 2x_4 &= 0 \end{aligned}$$

**1.20** Write down the simplest example you can to show that adding one to each entry in a row can change the answers to the linear equations. So adding numbers to rows is *not* allowed.

**1.21** Write down the simplest systems of linear equations you can come up with that have

- One solution.
- No solutions.
- Infinitely many solutions.

**1.22** If all of the constants in some linear equations are zeros, must the equations have a solution?

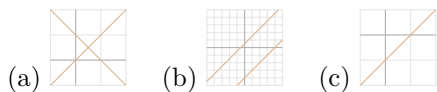
**1.23** Draw the two lines  $\frac{1}{2}x_1 - x_2 = -\frac{1}{2}$  and  $2x_1 + x_2 = 3$  in  $\mathbb{R}^2$ . In your drawing indicate the points which satisfy both equations.

**1.24** Which pair of equations cuts out which pair of lines? How many solutions does each pair of equations have?

$$\begin{aligned} x_1 - x_2 &= 0 & (1) \\ x_1 + x_2 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 &= 4 & (2) \\ -2x_1 + 2x_2 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 &= 1 & (3) \\ -3x_1 + 3x_2 &= -3 \end{aligned}$$



**1.25** Draw the two lines  $2x_1 + x_2 = 1$  and  $x_1 - 2x_2 = 1$  in the  $x_1x_2$ -plane. Explain geometrically where the solution of this pair of equations lies. Carry out forward elimination on the pair, to obtain a new pair of equations. Draw the lines corresponding to each new equation. Explain why one of these lines is parallel to one of the axes.

$$\begin{aligned}
 x_1 - x_2 + 2x_3 &= 2 & (1) \\
 -2x_1 + 2x_2 + x_3 &= -2 \\
 -3x_1 + 3x_2 - x_3 &= 0
 \end{aligned}$$

$$\begin{aligned}
 -x_1 - x_3 &= 0 & (2) \\
 x_1 - 2x_2 - x_3 &= 0 \\
 2x_1 - 2x_2 - 2x_3 &= -1
 \end{aligned}$$

$$\begin{aligned}
 x_1 + x_2 + x_3 &= 1 & (3) \\
 x_1 + x_2 + x_3 &= 0 \\
 x_1 + x_2 + x_3 &= -1
 \end{aligned}$$

$$\begin{aligned}
 -2x_1 + x_2 + x_3 &= -2 & (4) \\
 -2x_1 - x_2 + 2x_3 &= 0 \\
 -4x_2 + 2x_3 &= 4
 \end{aligned}$$

$$\begin{aligned}
 -2x_2 - x_3 &= 0 & (5) \\
 -x_1 - x_2 - x_3 &= -1 \\
 -3x_1 - 3x_2 - 3x_3 &= 0
 \end{aligned}$$

Table 1.1: Five systems of linear equations

**1.26** Find the quadratic function  $y = ax^2 + bx + c$  which passes through the points  $(x, y) = (0, 2), (1, 1), (2, 6)$ .

**1.27** Give a simple example of a system of linear equations which has a solution, but for which, if you alter one of the coefficients by a tiny amount (as tiny as you like), then there is no solution.

**1.28** If you write down just one linear equation in three variables, like  $2x_1 + x_2 - x_3 = -1$ , the solutions draw out a plane. So a system of three linear equations draws out three different planes. The solutions of two of the equations lie on the intersections of the two corresponding planes. The solutions of the whole system are the points where all three planes intersect. Which system of equations in table 1.1 draws out which picture of planes from figure 1.3 on the following page?

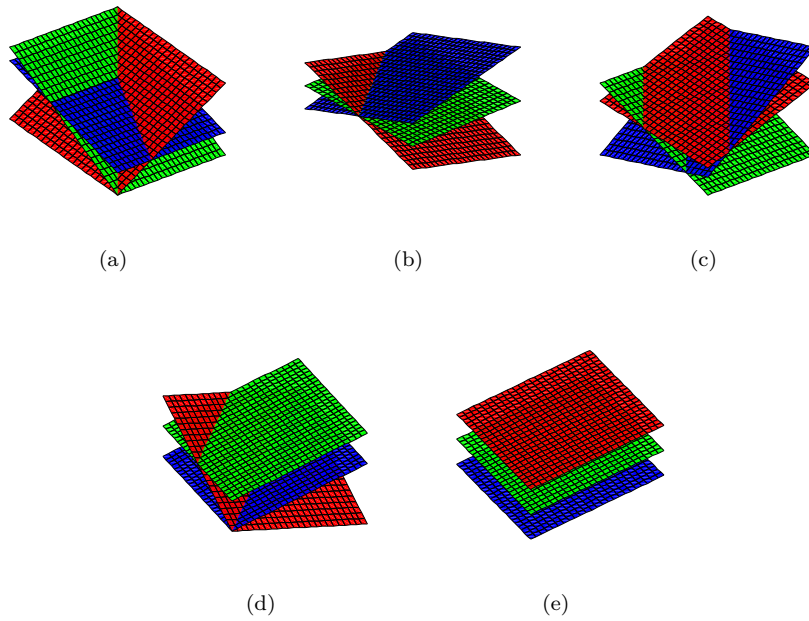


Figure 1.3: When you have three equations in three variables, each one draws a plane. Solutions of a pair of equations lie where their planes intersect. Solutions of all three equations lie where all three planes intersect.

## Chapter 2

### Matrices

The boxes of numbers we have been writing are called *matrices*. Let's learn the arithmetic of matrices.

#### Definitions

A *matrix* is a finite box  $A$  of numbers, arranged in rows and columns. We write it as

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1q} \\ A_{21} & A_{22} & \dots & A_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ A_{p1} & A_{p2} & \dots & A_{pq} \end{pmatrix}$$

and say that  $A$  is  $p \times q$  if it has  $p$  rows and  $q$  columns. If there are as many rows as columns, we will say that the matrix is *square*.

The entry  $A_{31}$  is in row 3, column 1. If we have 10 or more rows or columns (which won't happen in this book), we might write  $A_{1,1}$  instead of  $A_{11}$ . For example, we can distinguish  $A_{11,1}$  from  $A_{1,11}$ .

A matrix  $x$  with only one column is called a *vector* and written

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The collection of all vectors with  $n$  real number entries is called  $\mathbb{R}^n$ .

Think of  $\mathbb{R}^2$  as the  $xy$ -plane, writing each point as

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

instead of  $(x, y)$ . We draw a vector, for example the vector

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

as an arrow, pointing out of the origin, with the arrow head at the point  $x = 2, y = 3$ :



Figure 2.1: Echelon form: a staircase, each step down by only 1, but across to the right by maybe more than one. The pivots are the steps down, and below them, in the unshaded part, are zeros.



Similarly, think of  $\mathbb{R}^3$  as 3-dimensional space.

**2.1** Draw the vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

We draw vectors either as dots or more often as arrows. If there are too many vectors, pictures of arrows can get cluttered, so we prefer to draw dots. Sometimes we distinguish between points (drawn as dots) and vectors (drawn as arrows), but algebraically they are the same objects: columns of numbers.

### Review problems

**2.2** Find points in  $\mathbb{R}^3$  which form the vertices of a regular

- (a) cube
- (b) octahedron
- (c) tetrahedron.

### Echelon Form

A matrix is in *echelon form* if (as in figure 2.1) each row is either all zeros or starts with more zeros than any earlier row. The first nonzero entry of each row is called a *pivot*.

**2.3** Draw dots where the pivots are in figure 2.1.

**2.4** Give the simplest examples you can of two matrices which are not in echelon form, each for a different reason.

**2.5** The entries  $A_{11}, A_{22}, \dots$  of a square matrix  $A$  are called the *diagonal*. Prove that every square matrix in echelon form has all pivots lying on or above the diagonal.

**2.6** Prove that a square matrix in echelon form has a zero row just when it is either all zeroes or it has a pivot above the diagonal.

**2.7** Prove that a square matrix in echelon form has a column with no pivot just when it has a zero row. Thus all diagonal entries are pivots or else there is a zero row.

**Theorem 2.1.** *Forward elimination brings any matrix to echelon form, without altering the solutions of the associated linear equations.*

Obviously proof is by induction, but the result is clear enough, so we won't give a proof.

### Review problems

**2.8** In one colour, draw the locations of the pivots, and in another draw the “staircase” (as in figure 2.1 on the preceding page) for the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

### Matrices in Blocks

If

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix},$$

then write

$$\begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{pmatrix} \text{ and } \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}.$$

(We will often colour various rows and columns of matrices, just to make the discussion easier to follow. The colours have no mathematical meaning.)

Any matrix which has only zero entries will be written 0.

**2.9** What could  $\begin{pmatrix} 0 & 0 \end{pmatrix}$  mean?

**2.10** What could  $\begin{pmatrix} A & 0 \end{pmatrix}$  mean if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}?$$

## Matrix Arithmetic

Add matrices like:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}, A + B = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix}.$$

If two matrices have matching numbers of rows and columns, we add them by adding their components:  $(A + B)_{ij} = A_{ij} + B_{ij}$ . Similarly for subtracting.

**2.11** Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}.$$

Find  $A + B$ .

When we add matrices in blocks,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} C & D \\ A & B \end{pmatrix} = \begin{pmatrix} A+C & B+D \\ C+A & D+B \end{pmatrix}$$

(as long as  $A$  and  $C$  have the same numbers of rows and columns and  $B$  and  $D$  do as well).

**2.12** Draw the vectors

$$u = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, v = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

and the vectors  $0$  and  $u + v$ . In your picture, you should see that they form the vertices of a parallelogram (a quadrilateral whose opposite sides are parallel).

Multiply by numbers like:

$$7 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 \cdot 1 & 7 \cdot 2 \\ 7 \cdot 3 & 7 \cdot 4 \end{pmatrix}.$$

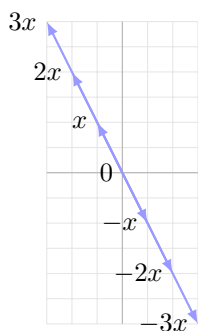
If  $A$  is a matrix and  $c$  is a number,  $cA$  is the matrix with  $(cA)_{ij} = cA_{ij}$ .

Suppose that

$$x = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

The multiples  $x, 2x, 3x, \dots$  and  $-x, -2x, -3x, \dots$  live on a straight line through  $0$ :





## Matrix Multiplication

Surprisingly, matrix multiplication is more difficult. To multiply a single row by a single column, just multiply entries in order, and add up:

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 4 = 3 + 8 = 11.$$

Put your left hand index finger on the row, and your right hand index finger on the column, and as you run your left hand along, run your right hand down:

$$\left( \begin{array}{cc} \rightarrow & \rightarrow \end{array} \right) \begin{pmatrix} \downarrow \\ \downarrow \end{pmatrix}.$$

As your fingers travel, you multiply the entries you hit, and add up all of the products.

### 2.13 Multiply

$$\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$$

To multiply the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix},$$

multiply any row of  $A$  by any column of  $B$ :

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 \end{pmatrix}.$$

As your left hand finger travels along a row, and your right hand down a column, you produce the entry in that row and column; the second row of  $A$  times the first column of  $B$  gives the entry of  $AB$  in second row, first column.

**2.14** Multiply

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix}$$

We write  $\sum_k$  in front of an expression to mean the sum for  $k$  taking on all possible values for which the expression makes sense. For example, if  $x$  is a vector with 3 entries,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

then  $\sum_k x_k = x_1 + x_2 + x_3$ .

If  $A$  is  $p \times q$  and  $B$  is  $q \times r$ , then  $AB$  is the  $p \times r$  matrix whose entries are  $(AB)_{ij} = \sum_k A_{ik} B_{kj}$ .

## Review problems

**2.15** If  $A$  is a matrix and  $x$  a vector, what constraints on dimensions need to be satisfied to multiply  $Ax$ ? What about  $xA$ ?

**2.16**

$$A = \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix}$$

Compute all of the following which are defined:

$$AB, AC, AD, BC, CA, CD.$$

**2.17** Find some  $2 \times 2$  matrices  $A$  and  $B$  with no zero entries for which  $AB = 0$ .

**2.18** Find a  $2 \times 2$  matrix  $A$  with no zero entries for which  $A^2 = 0$ .

**2.19** Suppose that we have a matrix  $A$ , so that whenever  $x$  is a vector with integer entries, then  $Ax$  is also a vector with integer entries. Prove that  $A$  has integer entries.

**2.20** A matrix is called *upper triangular* if all entries below the diagonal are zero. Prove that the product of upper triangular square matrices is upper triangular, and if, for example

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \dots & A_{1n} \\ & A_{22} & A_{23} & A_{24} & \dots & A_{2n} \\ & & A_{33} & A_{34} & \dots & A_{3n} \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \vdots \\ & & & & & A_{nn} \end{pmatrix},$$

(with zeroes under the diagonal) and

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & \dots & B_{1n} \\ & B_{22} & B_{23} & B_{24} & \dots & B_{2n} \\ & & B_{33} & B_{34} & \dots & B_{3n} \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \vdots \\ & & & & & B_{nn} \end{pmatrix},$$

then

$$AB = \begin{pmatrix} A_{11}B_{11} & * & * & * & \dots & * \\ & A_{22}B_{22} & * & * & \dots & * \\ & & A_{33}B_{33} & * & \dots & * \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & * \\ & & & & & A_{nn}B_{nn} \end{pmatrix}.$$

**2.21** Prove the analogous result for lower triangular matrices.

### Algebraic Properties of Matrix Multiplication

**2.22** If  $A$  and  $B$  matrices, and  $AB$  is defined, and  $c$  is any number, prove that  $c(AB) = (cA)B = A(cB)$ .

**2.23** Prove that matrix multiplication is associative:  $(AB)C = A(BC)$  (and that if either side is defined, then the other is, and they are equal).

**2.24** Prove that matrix multiplication is distributive:  $A(B + C) = AB + AC$  and  $(P + Q)R = PR + QR$  for any matrices  $A, B, C, P, Q, R$  (again if one side is defined, then both are and they are equal).

Running your finger along rows and columns, you see that blocks multiply like:

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = AC + BD$$

etc.

**2.25** To make sense of this last statement, what do we need to know about the numbers of rows and columns of  $A, B, C$  and  $D$ ?



## Chapter 3

### Inverses of Matrices

Just as a number has a reciprocal, some matrices have an inverse matrix.

#### The Identity Matrix

Define matrices

$$I_1 = (1), I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \dots$$

The  $n \times n$  matrix with 1's on the diagonal and zeros everywhere else is called the *identity matrix*, and written  $I_n$ . We often write it as  $I$  to be deliberately ambiguous about what size it is. An equivalent definition:

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

**3.1** What could  $I_{13}$  mean? (Careful: it has two meanings.) What does  $I_2$  mean?

**3.2** Prove that  $IA = AI = A$  for any matrix  $A$ .

**3.3** Suppose that  $B$  is an  $n \times n$  matrix, and that  $AB = A$  for any  $n \times n$  matrix  $A$ . Prove that  $B = I_n$ .

**3.4** Suppose that  $B$  is an  $n \times n$  matrix, and that  $BA = A$  for any  $n \times n$  matrix  $A$ . Prove that  $B = I_n$ .

**3.5** If  $A$  and  $B$  are two matrices and  $Ax = Bx$  for any vector  $x$ , prove that  $A = B$ .

The columns of  $I_n$  are vectors called  $e_1, e_2, \dots, e_n$ .

**3.6** Consider the identity matrix  $I_3$ . What are the vectors  $e_1, e_2, e_3$ ?

**3.7** The vector  $e_j$  has a one in which row? And zeroes in which rows?

**3.8** If  $A$  is a matrix, prove that  $Ae_1$  is the first column of  $A$ .

**3.9** If  $A$  is any matrix, prove that  $Ae_j$  is the  $j$ -th column of  $A$ .

If  $A$  is a  $p \times q$  matrix, by the previous exercise,

$$A = (Ae_1 \ Ae_2 \ \dots \ Ae_q).$$

In particular, when we multiply matrices

$$AB = (ABe_1 \ ABe_2 \ \dots \ ABe_q)$$

(and if either side of this equation is defined, then both sides are and they are equal). In other words, the columns of  $AB$  are  $A$  times the columns of  $B$ .

This next exercise is particularly vital:

**3.10** If  $A$  is a matrix, and  $x$  a vector, prove that  $Ax$  is a sum of the columns of  $A$ , each weighted by entries of  $x$ :

$$Ax = x_1 (Ae_1) + x_2 (Ae_2) + \dots + x_n (Ae_n).$$

### Review problems

**3.11** True or false (if false, give a counterexample):

- If the second column of  $B$  is 3 times the first column, then the same is true of  $AB$ .
- Same question for rows instead of columns.

**3.12** Can you find matrices  $A$  and  $B$  so that  $A$  is  $3 \times 5$  and  $B$  is  $5 \times 3$ , and  $AB = I$ ?

**3.13** Prove that the rows of  $AB$  are the rows of  $A$  multiplied by  $B$ .

**3.14** The *Fibonacci numbers* are the numbers  $x_0 = 1, x_1 = 1, x_{n+1} = x_n + x_{n-1}$ . Write down  $x_0, x_1, x_2, x_3$  and  $x_4$ . Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Prove that

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

**3.15** Let

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Draw these vectors in the plane. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$
$$D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

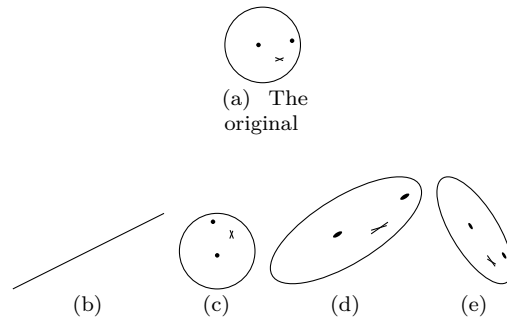


Figure 3.1: Faces formed by  $y = Ax$ . Each face is centered at the origin.

For each matrix  $M = A, B, C, D, E, F$  draw  $Mx$  and  $My$  (in a different colour for each matrix), and explain in words what each matrix is “doing” (for example, rotating, flattening onto a line, expanding, contracting, etc.).

**3.16** The first picture in figure 3.1 is the original in the  $x_1, x_2$  plane, and the center of the circular face is at the origin. If we pick a matrix  $A$  and set  $y = Ax$ , and draw the image in the  $y_1, y_2$  plane, which matrix below draws which picture?

$$\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

**3.17** Can you figure out which matrices give rise to the pictures in the last problem, just by looking at the pictures? Assume that you know that all the entries of each matrix are integers between -2 and 2.

**3.18** What are the simplest examples you can find of  $2 \times 2$  matrices  $A$  for which taking vectors  $x$  to  $Ax$

- contracts the plane,
- dilates the plane,
- dilates one direction, while contracting another,
- rotates the plane by a right angle,
- reflects the plane in a line,
- moves the vertical axis, but leaves every point of the horizontal axis where it is (a “shear”)?

### Inverses

A matrix is *square* if it has the same number of rows as columns. If  $A$  is a square matrix, an *inverse* is a square matrix  $B$  of the same size as  $A$  so that  $AB = BA = I$ .

**3.19** If  $A, B$  and  $C$  are square matrices, and  $AB = I$  and  $CA = I$ , prove that  $B = C$ . In particular, there is only one inverse (if there is one at all).

So we can unambiguously write the inverse of  $A$  (if there is one) as  $A^{-1}$ .

**3.20** If

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

check that

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

**3.21** Which  $1 \times 1$  matrices have inverses, and what are their inverses?

**3.22** By multiplying out the matrices, prove that any  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has inverse

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

as long as  $ad - bc \neq 0$ .

**3.23** If  $A$  and  $B$  are invertible matrices, prove that  $AB$  is invertible, and  $(AB)^{-1} = B^{-1}A^{-1}$ .

**3.24** Prove that  $(A^{-1})^{-1} = A$ , for any invertible square matrix  $A$ .

**3.25** If  $A$  is invertible, prove that  $Ax = 0$  only for  $x = 0$ .

**3.26** If  $A$  is invertible, and  $AB = I$ , prove that  $A = B^{-1}$  and  $B = A^{-1}$ .

### Review problems

**3.27** Write down a pair of nonzero  $2 \times 2$  matrices  $A$  and  $B$  for which  $AB = 0$ .

**3.28** If  $A$  is an invertible matrix, prove that  $Ax = Ay$  just when  $x = y$ .

**3.29** If a matrix  $M$  splits up into square blocks like

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

explain how to find  $M^{-1}$  in terms of  $A^{-1}$  and  $D^{-1}$ . (Warning: for a matrix which splits into blocks like

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

the inverse of  $M$  cannot be expressed in any elementary way in terms of the blocks and their inverses.)



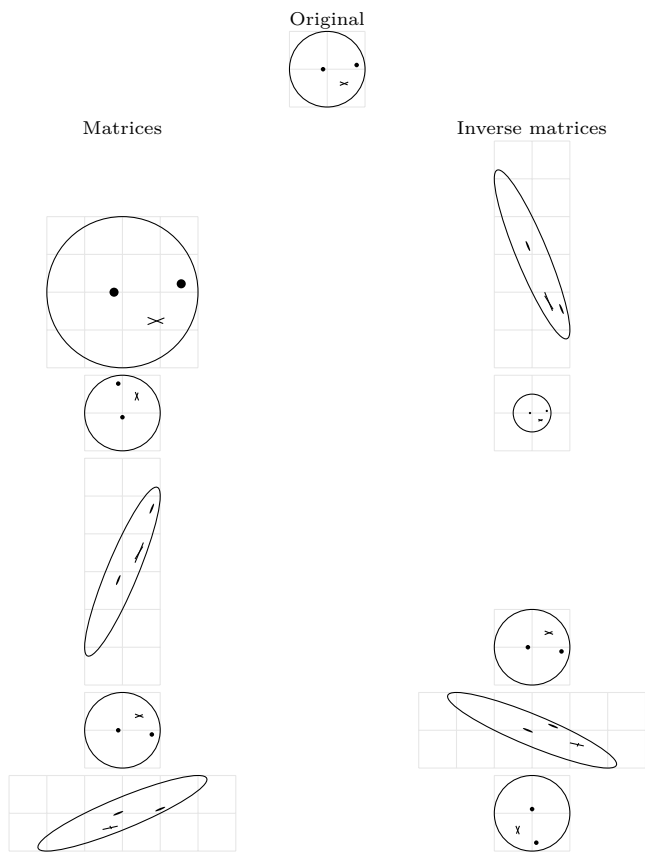


Figure 3.2: Images coming from some matrices, and from their inverses.

**3.30** Figure 3.2 shows how various matrices (on the left hand side) and their inverses (on the right hand side) affect vectors. But the two columns are scrambled up. Which right hand side picture is produced by the inverse matrix of each left hand side picture?

### Elimination by Matrix Multiplication

Take the  $3 \times 3$  identity matrix  $I$ , and swap the first two rows. Call the resulting matrix  $A$ :

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It turns out that, for any vector  $x$ , the vector  $Ax$  is just the vector  $x$  with the first two rows swapped. Why? First, let's see if this is true for the example of  $x = e_1$ . We know that  $Ae_1$  is the first column of  $A$ . So  $Ae_1$  is the first column

of  $I$ , but with the first two rows swapped. The first column of  $I$  is  $e_1$ . So  $Ae_1$  is  $e_1$  with the first two rows swapped. The same reasoning exactly works with  $e_1$  replaced by  $e_2$  or  $e_3$ . So if  $x = e_1$  or  $x = e_2$  or  $x = e_3$ , then  $Ax$  is  $x$  with the first two rows swapped. Since we can write any vector as  $x = x_1e_1 + x_2e_2 + x_3e_3$ , it is enough to check what happens if we take  $x = e_1$  and then check  $x = e_2$  and then check  $x = e_3$ , as we have done. So for any vector  $x$ , the vector  $Ax$  must be just  $x$  with the first two rows swapped. A *row operation* is the process of adding a multiple of a row to another row, swapping two rows, or rescaling a row. The same reasoning works exactly if we start with the  $n \times n$  identity matrix  $I$  and let  $A$  be the result of carrying out any of the row operations that we came across in elimination. Let's make that more precise and summarize what we have learned.

**Lemma 3.1.** *Carry out some row operation on  $I$ , or more generally you can carry out several row operations to  $I$ , as many as you like. Call the resulting matrix  $A$ . Then for any vector  $x$ , the vector  $Ax$  is the result of carrying out exactly those same row operations on  $x$ .*

For example, if we start with the  $3 \times 3$  identity matrix  $I$ , and add 7 row 1 to row 3 then

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}.$$

Our lemma claims that  $Ax$  is just  $x$  with 7 row 1 added to row 3. Let's check:

$$\begin{aligned} Ax &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 + 7x_1 \end{pmatrix}. \end{aligned}$$

**3.31** Which  $3 \times 3$  matrix  $S$  adds  $-5$  row 2 to row 3, and  $-7$  row 1 to row 2?

**3.32** Which  $4 \times 4$  matrix  $P$  takes row 1 to row 2, row 2 to row 3, row 3 to row 4, and row 4 to row 1?

**Corollary 3.2.** *Carry out several row operations on  $I$ , as many as you like. Call the resulting matrix  $A$ . Then for any matrix  $B$ , the matrix  $AB$  is just the result of applying exactly those same row operations to  $B$ .*

*Proof.* The columns of  $AB$  are  $A$  times the columns of  $B$ . □

**Corollary 3.3.** *If  $A$  is the matrix you get from  $I$  by carrying out some row operations, and  $B$  is the matrix that you get by carrying out some other row operations, then  $AB$  is the matrix that you get from  $I$  by carrying out first the row operations that gave you  $B$ , and then those that gave you  $A$ .*

*Proof.* The matrix  $A$  acts on a vector  $x$  by carrying out those row operations that gave us  $A$  from  $I$ , and the same is true for  $B$ . But  $(AB)x = A(Bx)$ , so  $AB$  is the matrix that carries out the row operations first of  $B$  and then of  $A$ . So then  $AB = ABI$  is the matrix you get by carrying out the row operations of  $B$  on  $I$ , and then the row operations of  $A$ .  $\square$

**Corollary 3.4.** *If  $A$  is the matrix that you get from  $I$  by carrying out some row operations, then the matrix  $A^{-1}$  is the matrix that you get from  $I$  by carrying out the inverse row operations, in the reverse order.*

*Proof.* Let  $B$  be the matrix that you get from  $I$  by carrying out the inverse row operations, in the reverse order. Then  $AB = I$  and  $BA = I$ , doing and then undoing various operations.  $\square$



## Chapter 4

### Matrices and Row Operations

We need some practice thinking about examples of matrices. In this chapter, we will encounter many different examples of simple types of matrices related to the row operations of elimination.

#### Permutation Matrices

A *permutation matrix* is a matrix obtained by scrambling up the rows of the identity matrix. As we just saw, if  $A$  is a permutation matrix, and  $x$  is a vector, then  $Ax$  is the result of permuting  $x$  by the same scrambling of rows that created  $A$  in the first place. And as we just saw, the product of any two permutation matrices  $A$  and  $B$  is a permutation matrix  $C = AB$ , and  $C$  scrambles up the rows of a vector  $x$  by  $Cx = A(Bx)$ , i.e. by permuting the rows via  $B$  and then via  $A$ .

**4.1** Prove that a matrix is the permutation matrix of some permutation just when

- its entries are all 0's or 1's and
- it has exactly one 1 in each column and
- it has exactly one 1 in each row.

**4.2** If  $A$  is a matrix, the *transpose* of  $A$  is the matrix  $B = A^t$  with the rows and columns swapped, so  $B_{ij} = A_{ji}$  for any  $i$  and  $j$ .

- Use the result of problem 4.1 to prove that if  $A$  is a permutation matrix, then  $A^t$  is also a permutation matrix.
- Prove that for any  $i$  and  $j$ ,  $A_{ij} = 1$  just when  $Ae_i = e_j$ .
- Prove that  $A^{-1} = A^t$ , a very fast method to find the inverse of any permutation matrix.

#### Strictly Lower Triangular Matrices

A square matrix is *strictly lower triangular* if it has the form

$$S = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

with 1's on the diagonal, 0's above the diagonal, and anything below.

**4.3** Let  $S$  be strictly lower triangular. Must it be true that  $S_{ij} = 0$  for  $i > j$ ? What about for  $j > i$ ?

**Lemma 4.1.** *If  $S$  is a strictly lower triangular matrix, and  $A$  any matrix, then  $SA$  is  $A$  with  $S_{ij}$  (row  $j$ ) added to row  $i$ . In particular,  $S$  adds multiples of rows to lower rows.*

*Proof.* For  $x$  a vector,

$$\begin{aligned}
 Sx &= \begin{pmatrix} 1 & & & \\ S_{21} & 1 & & \\ S_{31} & S_{32} & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \\
 &= \begin{pmatrix} x_1 \\ x_2 + S_{21}x_1 \\ x_3 + S_{31}x_1 + S_{32}x_2 \\ \vdots \end{pmatrix}
 \end{aligned}$$

adds  $S_{21}x_1$  to  $x_2$ , etc.

If  $A$  is any matrix then the columns of  $SA$  are  $S$  times columns of  $A$ .  $\square$

**4.4** Let  $S$  be a matrix so that for any matrix  $A$  (of appropriate size),  $SA$  is  $A$  with multiples of some rows added to later rows. Prove that  $S$  is strictly lower triangular.

**4.5** Prove that if  $R$  and  $S$  are strictly lower triangular then  $RS$  is too.

**4.6** Say that a strictly lower triangular matrix is *elementary* if it has only one nonzero entry below the diagonal. Prove that every strictly lower triangular matrix is a product of elementary strictly lower triangular matrices.

**Lemma 4.2.** *Every strictly lower triangular matrix is invertible, and its inverse is also strictly lower triangular.*

*Proof.* Clearly true for  $1 \times 1$  matrices. Let's consider an  $n \times n$  strictly lower triangular matrix  $S$ , and assume that we have already proven the result for all matrices of smaller size. Write

$$S = \begin{pmatrix} 1 & 0 \\ c & A \end{pmatrix}$$

where  $c$  is a column and  $A$  is a smaller strictly lower triangular matrix. Then

$$S^{-1} = \begin{pmatrix} 1 & 0 \\ -A^{-1}c & A^{-1} \end{pmatrix}$$

which is strictly lower triangular. □

A matrix  $M$  is *strictly upper triangular* if it has ones down the diagonal zeroes everywhere below the diagonal.

**4.7** For each fact proven above about strictly lower triangular matrices, prove an analogue for strictly upper triangular matrices.

**4.8** Draw a picture indicating where some vectors lie in the  $x_1x_2$  plane, and where they get mapped to in the  $y_1y_2$  plane by  $y = Ax$  with

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

### Diagonal Matrices

A diagonal matrix is one like

$$D = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix},$$

(with blanks representing 0 entries).

**4.9** Show by calculation that

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & \frac{8}{7} & \frac{9}{7} \end{pmatrix}.$$

**4.10** Prove that a diagonal matrix  $D$  is invertible just when none of its diagonal entries are zero. Find its inverse.

**Lemma 4.3.** *If*

$$D = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix},$$

*then  $DA$  is  $A$  with row 1 scaled by  $t_1$ , etc.*

*Proof.* For a vector  $x$ ,

$$Dx = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ = \begin{pmatrix} t_1 x_1 \\ t_2 x_2 \\ \vdots \\ t_n x_n \end{pmatrix}$$

(just running your fingers along rows and down columns). So  $D$  scales row  $i$  by  $t_i$ . For any matrix  $A$ , the columns of  $DA$  are  $D$  times columns of  $A$ .  $\square$

#### Review problems

**4.11** Which diagonal matrix  $D$  takes the matrix

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

to the matrix

$$DA = \begin{pmatrix} 1 & \frac{4}{3} \\ 5 & 6 \end{pmatrix}?$$

**4.12** Multiply

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}$$

**4.13** Draw a picture indicating where some vectors lie in the  $x_1x_2$  plane, and where they get mapped to in the  $y_1y_2$  plane by  $y = Ax$  with each of the following matrices playing the part of  $A$ :

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

#### Encoding Linear Equations in Matrices

Linear equations

$$\begin{aligned} x_2 - x_3 &= 6 \\ 3x_1 + 7x_2 + 4x_3 &= 9 \\ 3x_1 + 5x_2 + 8x_3 &= 3 \end{aligned}$$



can be written in matrix form as

$$\begin{pmatrix} 0 & 1 & -1 \\ 3 & 7 & 4 \\ 3 & 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}.$$

Any linear equations

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1q}x_q &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2q}x_q &= b_2 \\ &\vdots \\ A_{p1}x_1 + A_{p2}x_2 + \cdots + A_{pq}x_q &= b_p \end{aligned}$$

become

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix}$$

which we write as  $Ax = b$ .

**4.14** Write the linear equations

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ 3x_1 + 4x_2 &= 8 \end{aligned}$$

in matrices.

### Forward Elimination Encoded in Matrix Multiplication

Forward elimination on a matrix  $A$  is carried out by multiplying on the left of  $A$  by a sequence of permutation matrices and strictly lower triangular matrices. For example

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 3 & 7 & 4 \\ 3 & 5 & 8 \end{pmatrix}$$

Swap rows 1 and 2 (and let's write out the permutation matrix):

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 3 & 7 & 4 \\ 0 & 1 & -1 \\ 3 & 5 & 8 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to  $(\text{row } 3)$ :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 3 & 7 & 4 \\ 0 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix}$$

The string of matrices in front of  $A$  just gets longer at each step. Add  $2(\text{row } 2)$  to  $(\text{row } 3)$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 3 & 7 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Call this  $U$ . This is the echelon form:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

We won't write out these tedious matrices on the left side of  $A$  ever again, but it is important to see it done once. We will sum up this whole process by writing the last line as  $U = VA$ , where  $V$  is a product of permutation matrices and strictly lower triangular matrices. Back substitution is similarly carried out by multiplying by strictly upper triangular and invertible diagonal matrices.

### Review problems

**4.15** Let  $P$  be the  $3 \times 3$  permutation matrix which swaps rows 1 and 2. What does the matrix  $P^{99}$  do? Write it down.

**4.16** Let  $S$  be the  $3 \times 3$  strictly lower triangular matrix which adds  $2(\text{row } 1)$  to row 3. What does the  $3 \times 3$  matrix  $S^{101}$  do? Write it down.

**4.17** Which  $3 \times 3$  matrix adds twice the first row to the second row when you multiply by it?

**4.18** Which  $4 \times 4$  matrix swaps the second and fourth rows when you multiply by it?

**4.19** Which  $4 \times 4$  matrix doubles the second and quadruples the third rows when you multiply by it?

**4.20** If  $P$  is the permutation matrix of a permutation  $p$ , what is  $AP$ ?

**4.21** If we start with

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 6 \end{pmatrix}$$

and end up with

$$PA = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

what permutation matrix is  $P$ ?

**4.22** If  $A$  is a  $2 \times 2$  matrix, and  $AP = PA$  for every  $2 \times 2$  permutation matrix  $P$  or strictly lower triangular matrix, then prove that  $A = cI$  for some number  $c$ .

**4.23** If the third and fourth columns of a matrix  $A$  are equal, are they still equal after we carry out forward elimination? After back substitution?

**4.24** How many pivots can there be in a  $3 \times 5$  matrix in echelon form?

**4.25** Write down the simplest  $3 \times 5$  matrices you can come up with in echelon form and for which

- The second and third variables are the only free variables.
- There are no free variables.
- There are pivots in precisely the columns 3 and 4.

**4.26** Write down the simplest matrices  $A$  you can for which the number of solutions to  $Ax = b$  is

- 1 for any  $b$ ;
- 0 for some  $b$ , and  $\infty$  for other  $b$ ;
- 0 for some  $b$ , and 1 for other  $b$ ;
- $\infty$  for any  $b$ .

**4.27** Suppose that  $A$  is a square matrix. Prove that all entries of  $A$  are positive just when, for any nonzero vector  $x$  which has no negative entries, the vector  $Ax$  has only positive entries.

**4.28** Prove that short matrices kill. A matrix is called *short* if it is wider than it is tall. We say that a matrix  $A$  *kills* a vector  $x$  if  $x \neq 0$  but  $Ax = 0$ .

## Summary

The many steps of elimination can each be encoded into a matrix multiplication. The resulting matrices can all be multiplied together to give the single equation  $U = VA$ , where  $A$  is the matrix we started with,  $U$  is the echelon matrix we end up with and  $V$  is the product of the various matrices that carry out all of our elimination steps. There is a big idea at work here: encode a possibly huge number of steps into a single algebraic equation (in this case the tiny little equation  $U = VA$ ), turning a large computation into a simple piece of algebra. We will use this tiny equation many times.



## Chapter 5

### Finding the Inverse of a Matrix

Let's use elimination to calculate the inverse of a matrix.

#### Finding the Inverse of a Matrix By Elimination

If  $Ax = y$  then multiplying both sides by  $A^{-1}$  gives  $x = A^{-1}y$ , solving for  $x$ . We can write out  $Ax = y$  as linear equations, and solve these equations for  $x$ . For example, if

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix},$$

then writing out  $Ax = y$ :

$$\begin{aligned} x_1 - 2x_2 &= y_1 \\ 2x_1 - 3x_2 &= y_2. \end{aligned}$$

Let apply Gauss–Jordan elimination, but watch the equations instead of the matrices. Add  $-2(\text{equation } 1)$  to equation 2.

$$\begin{aligned} x_1 - 2x_2 &= y_1 \\ x_2 &= -2y_1 + y_2. \end{aligned}$$

Add  $2(\text{equation } 2)$  to equation 1.

$$\begin{aligned} x_1 &= -3y_1 + 2y_2 \\ x_2 &= -2y_1 + y_2. \end{aligned}$$

So

$$A^{-1} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}.$$

**Theorem 5.1.** *Let  $A$  be a square matrix. Suppose that Gauss–Jordan elimination applied to the matrix  $(A \ I)$  ends up with  $(U \ V)$  with  $U$  and  $V$  square matrices.  $A$  is invertible just when  $U = I$ , in which case  $V = A^{-1}$ .*

Before the proof, let's have an example. Let's invert

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}.$$

$$\left( \begin{array}{c|cc} A & & I \end{array} \right) = \left( \begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array} \right)$$

Add  $-2(\text{row } 1)$  to row 2.

$$\left( \begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array} \right)$$

Make a new pivot  $\searrow$ .

$$\left( \begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array} \right)$$

Add  $2(\text{row } 2)$  to row 1.

$$\begin{aligned} & \left( \begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array} \right) \\ & = \left( \begin{array}{c|cc} U & & V \end{array} \right). \end{aligned}$$

Obviously these are the same steps we used in the example above; the shaded part represents coefficients in front of the  $y$  vector above. Since  $U = I$ ,  $A$  is invertible and

$$\begin{aligned} A^{-1} &= V \\ &= \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}. \end{aligned}$$

*Proof.* Gauss–Jordan elimination on  $(A \ I)$  is carried out by multiplying by various invertible matrices (strictly lower triangular, permutation, invertible diagonal and strictly upper triangular), say like

$$\left( \begin{array}{c|cc} U & & V \end{array} \right) = M_N M_{N-1} \dots M_2 M_1 \left( \begin{array}{c|cc} A & & I \end{array} \right).$$

So

$$\begin{aligned} U &= M_N M_{N-1} \dots M_2 M_1 A \\ V &= M_N M_{N-1} \dots M_2 M_1, \end{aligned}$$

which we summarize as  $U = VA$ . Clearly  $V$  is a product of invertible matrices, so invertible. Thus  $U$  is invertible just when  $A$  is.

First suppose that  $U$  has pivots all down the diagonal. Every pivot is a 1. Entries above and below each pivot are 0, so  $U = I$ . Since  $U = VA$ , we find  $I = VA$ . Multiply both sides on the left by  $V^{-1}$ , to see that  $V^{-1} = A$ . But

then multiply on the right by  $V$  to see that  $I = AV$ . So  $A$  and  $V$  are inverses of one another.

Next suppose that  $U$  doesn't have pivots all down the diagonal. We always start Gauss–Jordan elimination on the diagonal, so we fail to place a pivot somewhere along the diagonal just because we move  $\rightarrow$  during forward elimination. That move makes a pivotless column, hence a free variable for the equation  $Ax = 0$ . Setting the free variable to a nonzero value produces a nonzero  $x$  with  $Ax = 0$ . By problem 3.25 on page 26,  $A$  is not invertible.  $\square$

## Review problems

**5.1** Find the inverse of

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

**5.2** Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

**5.3** Find the inverse of

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

**5.4** Find the inverse of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix}.$$

**5.5** Is there a faster method than Gauss–Jordan elimination to find the inverse of a permutation matrix?

## Invertibility and Forward Elimination

**Proposition 5.2.** *A square matrix  $U$  in echelon form is invertible just when  $U$  has pivots all the way down the diagonal, which occurs just when  $U$  has no zero rows.*

*Proof.* Applying back substitution to a matrix  $U$  which is already in echelon form preserves the locations of the pivots, and just rescales them to be 1, killing everything above them. So back substitution takes  $U$  to  $I$  just when  $U$  has pivots all the way down the diagonal.  $\square$

For example,

$$\begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix}$$

is invertible, while

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

is *not* invertible.

**Theorem 5.3.** *A square matrix  $A$  is invertible just when its echelon form  $U$  is invertible.*

So we can quickly decide if a matrix is invertible by forward elimination. We only need back substitution if we actually need to compute out the inverse.

*Proof.*  $U = VA$ , and  $V$  is invertible, so  $U$  is invertible just when  $A$  is.  $\square$

For example,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

has echelon form

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

so  $A$  is invertible.

**5.6** Is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

invertible?

**5.7** Is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

invertible?

**5.8** Prove that a square matrix  $A$  is invertible just when the only solution  $x$  to the equation  $Ax = 0$  is  $x = 0$ .



## Inversion and Solvability of Linear Equations

**Theorem 5.4.** Take a square matrix  $A$ .

- If the matrix  $A$  is invertible then, for any vector  $b$ , the equation  $Ax = b$  has a unique solution  $x$ .
- If the matrix  $A$  is not invertible then the equation  $Ax = b$  has either no solution or infinitely many, and both of these possibilities occur for different choices of  $b$ .

*Proof.* If  $A$  is invertible, then multiplying both sides of  $Ax = b$  by  $A^{-1}$ , we see that we have to have  $x = A^{-1}b$ .

On the other hand, suppose that  $A$  is not invertible. There is a free variable for  $Ax = b$ , so no solutions or infinitely many. Lets see that for different choices of  $b$  both possibilities occur. Carry out forward elimination, say  $U = VA$ . Then  $U$  has a zero row, say row  $n$ . We can't solve  $Ux = e_n$  (look at row  $n$ ). So set  $b = V^{-1}e_n$  and we can't solve  $Ax = b$ . But now instead set  $b = 0$  and we can solve  $Ax = 0$  (for example with  $x = 0$ ) and therefore solve  $Ax = 0$  with infinitely many solutions  $x$ , since there is a free variable.  $\square$

The equations

$$\begin{aligned}x_1 + 2x_2 &= 9845039843453455938453 \\x_1 - 2x_2 &= 90853809458394034464578\end{aligned}$$

have a unique solution, because they are  $Ax = b$  with

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$$

which has echelon form

$$U = \begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{-4} \end{pmatrix}.$$

**5.9** Suppose that  $A$  and  $B$  are  $n \times n$  matrices and  $AB = I$ . Prove that  $A$  and  $B$  are both invertible, and that  $B = A^{-1}$  and that  $A = B^{-1}$ .

**5.10** Prove that for square matrices  $A$  and  $B$  of the same size

$$(AB)^{-1} = B^{-1}A^{-1}$$

(and if either side is defined, then the other is and they are equal).

## Review problems

**5.11** Is

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

invertible?

**5.12** How many solutions are there to the following equations?

$$x_1 + 2x_2 + 3x_3 = 284905309485083$$

$$x_1 + 2x_2 + x_3 = 92850234853408$$

$$x_2 + 15x_3 = 4250348503489085.$$

**5.13** Let  $A$  be the  $n \times n$  matrix which has 1 in every entry on or under the diagonal, and 0 in every entry above the diagonal. Find  $A^{-1}$ .

**5.14** Let  $A$  be the  $n \times n$  matrix which has 1 in every entry on or above the diagonal, and 0 in every entry below the diagonal. Find  $A^{-1}$ .

**5.15** Give an example of a  $3 \times 3$  invertible matrix  $A$  for which  $A$  and  $A^t$  have different values for their pivots.

**5.16** Imagine that you start with a  $4 \times 4$  matrix  $A$  which might not be invertible, and carry out forward elimination on  $(A \ I)$ . Suppose you arrive at

$$(U \ V) = \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 2 & 8 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 2 \end{pmatrix},$$

with some pivots somewhere on the first two rows of  $U$ . Fact: you can solve  $Ax = b$  just for those vectors  $b$  which solve the equations

$$\begin{aligned} 2b_1 + 8b_2 + 3b_3 + 9b_4 &= 0 \\ b_2 + 5b_3 + 2b_4 &= 0 \end{aligned} .$$

Explain why.

## Chapter 6

### The Determinant

We can see whether a matrix is invertible by computing a single number, the *determinant*. We will learn how to calculate the determinant, and tricks to make it easy to find determinants of some types of matrices.

**6.1** Use forward elimination to prove that a  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible just when  $ad - bc \neq 0$ .

For any  $2 \times 2$  matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , the determinant is  $ad - bc$ . For larger matrices, the determinant is complicated.

#### Definition

Determinants are computed as in figure 6.1 on the following page. To compute a determinant, run your finger down the first column, writing down plus and minus signs in the pattern  $+, -, +, -, \dots$  in front the entry your finger points at, and then writing down the determinant of the matrix you get by deleting the row and column where your finger lies (always the first column), and add up.

**6.2** Prove that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

#### Review problems

**6.3** Find the determinant of

$$\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\begin{aligned}
\det \begin{pmatrix} 3 & 2 & 1 \\ 1 & 4 & 5 \\ 6 & 7 & 2 \end{pmatrix} &= + (3) \det \begin{pmatrix} 3 & 2 & 1 \\ 1 & 4 & 5 \\ 6 & 7 & 2 \end{pmatrix} \\
&\quad - (1) \det \begin{pmatrix} 3 & 2 & 1 \\ 1 & 4 & 5 \\ 6 & 7 & 2 \end{pmatrix} \\
&\quad + (6) \det \begin{pmatrix} 3 & 2 & 1 \\ 1 & 4 & 5 \\ 6 & 7 & 2 \end{pmatrix} \\
&= 3 \det \begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix} - \det \begin{pmatrix} 2 & 1 \\ 7 & 2 \end{pmatrix} + 6 \det \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \\
&= 3(4 \cdot 2 - 5 \cdot 7) - (2 \cdot 2 - 1 \cdot 7) + 6(2 \cdot 5 - 1 \cdot 4).
\end{aligned}$$

Figure 6.1: Computing a  $3 \times 3$  determinant.

**6.4** Find the determinant of

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

**6.5** Does  $A_{11}^2$  appear in the expression for  $\det A$ , when you expand out all of the determinants in the expression completely?

**6.6** Prove that the determinant of

$$A = \begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix}$$

is zero, no matter what number we put in place of the \*'s, even if the numbers are all different.

**6.7** Give an example of a matrix all of whose entries are positive, even though its determinant is zero.

**6.8** What is  $\det I$ ? Justify your answer.

**6.9** Prove that

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \det C,$$

for  $A$  and  $C$  any square matrices, and  $B$  any matrix of appropriate size to fit in here.

### Easy Determinants

Lets find the determinant of

$$A = \begin{pmatrix} 7 & & \\ & 4 & \\ & & 2 \end{pmatrix}.$$

(There are zeros wherever entries are not written.) Running down the first column, we only hit the 7. So

$$\det A = 7 \det \begin{pmatrix} 4 & \\ & 2 \end{pmatrix}.$$

By the same trick:

$$\begin{aligned} \det A &= (7)(4) \det(2) \\ &= (7)(4)(2). \end{aligned}$$

Summing up:

**Lemma 6.1.** *The determinant of a diagonal matrix*

$$A = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{pmatrix}$$

is  $\det A = p_1 p_2 \dots p_n$ .

We can easily do better: recall that a matrix  $A$  is *upper triangular* if all entries below the diagonal are 0. By the same trick again:

**Lemma 6.2.** *The determinant of an upper triangular square matrix*

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} & \dots & U_{1n} \\ & U_{22} & U_{23} & U_{24} & \dots & U_{2n} \\ & & U_{33} & U_{34} & \dots & U_{3n} \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \vdots \\ & & & & & U_{nn} \end{pmatrix}$$

is the product of the diagonal terms:  $\det U = U_{11}U_{22} \dots U_{nn}$ .

**Corollary 6.3.** *A square matrix  $A$  is invertible just when  $\det U \neq 0$ , with  $U$  obtained from  $A$  by forward elimination.*

*Proof.* The matrix  $U$  is upper triangular. The fact that  $\det U \neq 0$  says just precisely that all diagonal entries of  $U$  are not zero, so are pivots—a pivot in every column. Apply theorem 5.3 on page 42.  $\square$

### Review problems

**6.10** Find

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

**6.11** Suppose that  $U$  is an invertible upper triangular matrix.

- Prove that  $U^{-1}$  is upper triangular.
- Prove that the diagonal entries of  $U^{-1}$  are the reciprocals of the diagonal entries of  $U$ .
- How can you calculate by induction the entries of  $U^{-1}$  in terms of the entries of  $U$ ?

**6.12** Let  $U$  be any upper triangular matrix with integer entries. Prove that  $U^{-1}$  has integer entries just when  $\det U = \pm 1$ .

### Tricks to Find Determinants

**Lemma 6.4.** *Swapping any two neighbouring rows of a square matrix changes the sign of the determinant. For example,*

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = - \det \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}.$$

*Proof.* It is obvious for  $1 \times 1$  (you can't swap anything). It is easy to check for a  $2 \times 2$ . Picture a  $3 \times 3$  matrix  $A$ : look at example 6.1 on page 46. For simplicity, let's swap rows 1 and 2. Then the plus sign of row 1 and the minus sign of row 2 are clearly switched in the 1st and 2nd terms in the determinant. In the 3rd term, the leading plus sign is not switched. Look at the determinant in the 3rd term: rows 1 and 2 don't get crossed out, and have been switched, so the determinant factor changes sign. So all terms in the determinant formula have changed sign, and therefore the determinant has changed sign. The argument goes through identically with any size of matrix (by induction) and any two neighboring rows, instead of just rows 1 and 2.  $\square$

**Lemma 6.5.** *Swapping any two rows of a square matrix changes the sign of the determinant, so  $\det PA = -\det A$  for  $P$  the permutation matrix of a transposition.*

*Proof.* Suppose that we want to swap two rows, not neighboring. For concreteness, imagine rows 1 and 4. Swapping the first with the second, then second with third, etc., a total of 3 swaps will drive row 1 into place in row 4, and drives the old row 4 into row 3. Two more swaps (of row 3 with row 2, row 2 with row 1) puts everything where we want it.

More generally, to swap two rows, start by swapping the higher of the two with the row immediately under it, repeatedly until it fits into place. Some number  $s$  of swaps will do the trick. Now the row which was the lower of the two has become the higher of the two, and we have to swap it  $s - 1$  swaps into place. So  $2s - 1$  swaps in all, an odd number.  $\square$

**6.13** If a square matrix has two rows the same, prove that it has determinant 0.

**6.14** Find  $2 \times 2$  matrices  $A$  and  $B$  for which

$$\det(A + B) \neq \det A + \det B.$$

So  $\det$  doesn't behave well under adding matrices. But it *does* behave well under adding rows of matrices.

Watch each row:

$$\det \begin{pmatrix} 1+5 & 2+6 \\ 3 & 4 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \det \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}.$$

**Theorem 6.6.** *The determinant of any square matrix scales when you scale across any row like*

$$\det \begin{pmatrix} 7 \cdot 1 & 7 \cdot 2 \\ 3 & 4 \end{pmatrix} = 7 \cdot \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

*or when you scale down any column like*

$$\det \begin{pmatrix} 7 \cdot 1 & 2 \\ 7 \cdot 3 & 4 \end{pmatrix} = 7 \cdot \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

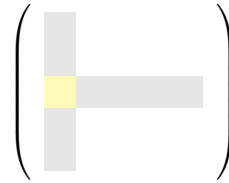
*It adds when you add across any row like*

$$\det \begin{pmatrix} 1+5 & 2+6 \\ 3 & 4 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \det \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$$

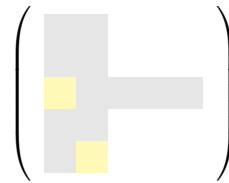
*or when you add down any column like*

$$\det \begin{pmatrix} 1 & 2+5 \\ 3 & 4+6 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \det \begin{pmatrix} 1 & 5 \\ 3 & 6 \end{pmatrix}.$$

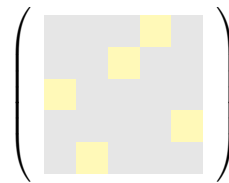
*Proof.* To compute a determinant, you pick an entry from the first column, and then delete its row and column. You then multiply it by the determinant of what is left over, which is computed by picking out an entry from the second column, not from the same row, etc. If we ignore for a moment the plus and minus signs, we can see the pattern emerging: you just pick something from the first column, and cross out its row and column,



and then something from the second column, and cross out its row and column,



and so on:



Finally, you have picked one entry from each column, all from different rows. In this example, we picked  $A_{31}, A_{52}, A_{23}, A_{14}, A_{45}$ . Multiply these together, and you get just one term from the determinant:  $A_{31}A_{52}A_{23}A_{14}A_{45}$ . Your term has exactly one entry from the first column, and then you crossed out the first column and moved on. Suppose that you double all of the entries in the first column. Your term contains exactly one entry from that column,  $A_{31}$  in our example, so your term doubles. Adding up the terms, the determinant doubles.

In the same way, scaling any column, you scale your entry from that column, so you scale your term. You scale all of the terms, so you scale the determinant.

When you cobbled together your term, you picked out an entry from some row, and then crossed out that row. So you didn't use the same row twice. There are as many rows as columns, and you picked an entry in each column, so you have picked as many entries as there are rows, never using the same row twice. So you must have picked out exactly one entry from each row. In our example term above, we see this clearly: the rows used were 3, 5, 2, 1, 4. By the same argument as for columns, if you scale row 2, you must scale the entry  $A_{23}$ , any only that entry, so you scale the term. Adding up all possible terms, you scale the determinant.



Lets see why we can add across rows. If I try to add entries across the first row, a single term looks like

$$\begin{pmatrix} 1+6 & 2+7 & 3+8 & 4+9 & 5+10 \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \end{pmatrix} = (4+9)(\dots)$$

where the (...) indicates all of the other factors from the lower rows, which we will leave unspecified,

$$= 4(\dots) + 9(\dots) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \end{pmatrix} + \begin{pmatrix} 6 & 7 & 8 & 9 & 10 \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \end{pmatrix}$$

since we keep all of the entries in the lower rows exactly the same in each matrix. This shows that each term adds when you add across a single row, so the sum of the terms, the determinant, must add. This reasoning works for any size of matrix in the same way. Moreover, it works for columns just in the same way as for rows.  $\square$

**6.15** What happens to the determinant if I double the first row and then triple the second row?

The determinant is the sum over all choices you could make of rows to pick at each step; and of course, there are some plus and minus signs which we are still ignoring.

**6.16** Draw pictures like those in the proof above of patterns of crossing out rows and columns, and explain which term each one computes, for determinants of

- a.  $2 \times 2$ ,
- b.  $3 \times 3$ , and
- c.  $4 \times 4$

matrices.

**Proposition 6.7.** *Suppose that  $S$  is the strictly upper or strictly lower triangular matrix which adds a multiple of one row to another row. Then*

$$\det SA = \det A.$$

*i.e. we can add a multiple of any row to any other row without affecting the determinant.*

*Proof.* We can always swap rows as needed, to get the rows involved to be the first and second rows. Then swap back again. This just changes signs somehow, and then changes them back again. So we need only work with the first and second rows. For simplicity, picture a  $3 \times 3$  matrix as 3 rows:

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Adding  $s$  (row 1) to (row 2) gives

$$\begin{pmatrix} a_1 \\ a_2 + s a_1 \\ a_3 \end{pmatrix},$$

which has determinant

$$\det \begin{pmatrix} a_1 \\ a_2 + s a_1 \\ a_3 \end{pmatrix} = \det \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + s \det \begin{pmatrix} a_1 \\ a_1 \\ a_3 \end{pmatrix}.$$

by the last lemma. The second determinant vanishes because it has two identical rows. The general case is just the same with more notation: we stuff more rows around the three rows we had above.  $\square$

**6.17** Which property of the determinant is illustrated in each of these examples?

(a)

$$\det \begin{pmatrix} 10 & -5 & -5 \\ -1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} = 5 \det \begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix}$$

(b)

$$\det \begin{pmatrix} 1 & -2 & -3 \\ -1 & 0 & 2 \\ -3 & 0 & 2 \end{pmatrix} = -\det \begin{pmatrix} 1 & -2 & -3 \\ -3 & 0 & 2 \\ -1 & 0 & 2 \end{pmatrix}$$

(c)

$$\det \begin{pmatrix} 1 & -1 & -2 \\ 4 & 0 & 2 \\ 2 & -2 & -1 \end{pmatrix} = \det \begin{pmatrix} 1 & -1 & 2 \\ 4 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

## Chapter 7

### The Determinant via Elimination

The fast way to compute the determinant of a large matrix is via elimination.

The fast formula for the determinant

**Theorem 7.1.** *Via forward elimination,*

$$\det A = \begin{cases} \pm (\text{product of the pivots}) & \text{if there is a pivot in each column,} \\ 0 & \text{otherwise.} \end{cases}$$

where

$$\pm = \begin{cases} + & \text{if we make an even number of row swaps during forward elimination,} \\ - & \text{otherwise.} \end{cases}$$

In particular,  $A$  is invertible just when  $\det A \neq 0$ .

Forward elimination takes

$$A = \begin{pmatrix} 0 & 7 \\ 2 & 3 \end{pmatrix} \text{ to } U = \begin{pmatrix} \boxed{2} & 3 \\ 0 & \boxed{7} \end{pmatrix}$$

with one row swap so  $\det A = -(2)(7) = -14$ .

The fast formula isn't actually any faster for small matrices, so for a  $2 \times 2$  or  $3 \times 3$  you wouldn't use it. But we need the fast formula anyway; each of the two formulas gives different insight.

*Proof.* We can see how the determinant changes during elimination: adding multiples of rows to other rows does nothing, swapping rows changes sign.  $\square$

**7.1** Use the fast formula to find the determinant of

$$A = \begin{pmatrix} 2 & 5 & 5 \\ 2 & 5 & 7 \\ 2 & 6 & 11 \end{pmatrix}$$

**7.2** Just by looking, find

$$\det \begin{pmatrix} 1001 & 1002 & 1003 & 1004 \\ 2002 & 2004 & 2006 & 2008 \\ 2343 & 6787 & 1938 & 4509 \\ 9873 & 7435 & 2938 & 9038 \end{pmatrix}.$$

**7.3** Prove that a square matrix is invertible just when its determinant is not zero.

Review problems

**7.4** Find the determinant of

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

**7.5** Find the determinant of

$$\begin{pmatrix} 0 & 2 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

**7.6** Find the determinant of

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 0 \\ 2 & -1 & -1 \end{pmatrix}$$

**7.7** Find the determinant of

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 2 & 2 & -1 \end{pmatrix}$$

**7.8** Find the determinant of

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

**7.9** Find the determinant of

$$\begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

**7.10** Prove that a square matrix with a zero row has determinant 0.

**7.11** Prove that  $\det PA = (-1)^N \det A$  if  $P$  is the permutation matrix of a product of  $N$  transpositions.

**7.12** Use the fast formula to find the determinant of

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \\ 3 & 5 & 2 \end{pmatrix}$$

**7.13** Prove that the determinant of any lower triangular square matrix

$$L = \begin{pmatrix} L_{11} & & & & & \\ L_{21} & L_{22} & & & & \\ L_{31} & L_{32} & L_{33} & & & \\ L_{41} & L_{42} & L_{43} & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ L_{n1} & L_{n2} & L_{n3} & \dots & L_{n(n-1)} & L_{nn} \end{pmatrix}$$

(with zeroes above the diagonal) is the product of the diagonal terms:  $\det L = L_{11}L_{22} \dots L_{nn}$ .

### Determinants Multiply

**Theorem 7.2.**  $\det(AB) = \det(A) \det(B)$ , for any square matrices  $A$  and  $B$  of the same size.

*Proof.* Suppose that  $\det A = 0$ . By the fast formula,  $A$  is not invertible. Problem 5.10 on page 43 tells us that therefore  $AB$  is not invertible, and both  $\det(AB)$  and  $\det(A) \det(B)$  are 0. So we can safely suppose that  $\det A \neq 0$ . Via Gauss-Jordan elimination, any invertible matrix is a product of matrices each of which adds a multiple of one row to another, or scales a row, or swaps two rows. Write  $A$  as a product of such matrices, and peel off one factor at a time, applying lemma 6.4 on page 48 and proposition 6.7 on page 51.  $\square$

If

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 7 & 5 & 4 \end{pmatrix},$$

then it is hard to compute out  $AB$ , and then compute out  $\det AB$ . But  $\det AB = \det A \det B = (1)(2)(3)(1)(2)(4) = 48$ .

## Transpose

The *transpose* of a matrix  $A$  is the matrix  $A^t$  whose entries are  $A_{ij}^t = A_{ji}$  (switching rows with columns). Flip over the diagonal:

$$A = \begin{pmatrix} 10 & 2 \\ 3 & 40 \\ 5 & 6 \end{pmatrix}, \quad A^t = \begin{pmatrix} 10 & 3 & 5 \\ 2 & 40 & 6 \end{pmatrix}.$$

**7.14** Find the transpose of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}.$$

**7.15** Prove that

$$(AB)^t = B^t A^t.$$

(The transpose of the product is the product of the transposes, in the *reverse* order.)

**7.16** Prove that the transpose of any permutation matrix is a permutation matrix. How is the permutation of the transpose related to the original permutation?

**Corollary 7.3.**

$$\det A = \det A^t$$

*Proof.* Forward elimination gives  $U = VA$ ,  $U$  upper triangular and  $V$  a product of permutation and strictly lower triangular matrices. Transpose:  $U^t = A^t V^t$ . But  $V^t$  is a product of permutation and strictly upper triangular matrices, with the same number of row swaps as  $V$ , so  $\det V^t = \det V = \pm 1$ . The matrix  $U^t$  is lower triangular, so  $\det U^t$  is the product of the diagonal entries of  $U^t$  (by problem 7.13 on the preceding page), which are the diagonal entries of  $U$ , so  $\det U^t = \det U$ .  $\square$

## Expanding Down Any Column or Across Any Row

Consider the “checkerboard pattern”

$$\begin{array}{cccc} + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & & & \end{array}$$

**Theorem 7.4.** We can compute the determinant of any square matrix  $A$  by picking any column (or any row) of  $A$ , writing down plus and minus signs from the same column (or row) of the checkerboard pattern matrix, writing down the

entries of  $A$  from that column (or row), multiplying each of these entries by the determinant obtained from deleting the row and column of that entry, and adding all of these up.

For

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 4 & 5 \\ 6 & 7 & 2 \end{pmatrix},$$

if we expand along the second row, we get

$$\begin{aligned} \det A &= - (1) \det \begin{pmatrix} 3 & 2 & 1 \\ 6 & 7 & 2 \end{pmatrix} \\ &+ (4) \det \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \\ &- (5) \det \begin{pmatrix} 3 & 2 \\ 6 & 7 \end{pmatrix} \end{aligned}$$

*Proof.* By swapping columns (or rows), we change signs of the determinant. Swap columns (or rows) to get the required column (or row) to slide over to become the first column (or row). Take the sign changes into account with the checkboard pattern: changing all plus and minus signs for each swap.  $\square$

**7.17** Use this to calculate the determinant of

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 839 & -1702 & 1 & 493 \end{pmatrix}.$$

## Summary

### Determinants

- (a) scale when you scale across a row (or down a column),
- (b) add when you add across a row (or down a column),
- (c) switch sign when you swap two rows, (or when you swap two columns),
- (d) don't change when you add a multiple of one row to another row (or a multiple of one column to another column),

- (e) don't change when you transpose,
- (f) multiply when you multiply matrices.

The determinant of

- (a) an upper (or lower) triangular matrix is the product of the diagonal entries.
- (b) a permutation matrix is  $(-1)^{\# \text{ of transpositions}}$ .
- (c) a matrix is not zero just when the matrix is invertible.
- (d) any matrix is  $\det A = (-1)^N \det U$ , if  $A$  is taken by forward elimination with  $N$  row swaps to a matrix  $U$ .

**7.18** If  $A$  is a square matrix, prove that

$$\det(A^k) = (\det A)^k$$

for  $k = 1, 2, 3, \dots$

**7.19** Use this last exercise to find

$$\det(A^{2222444466668888})$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1234567890 \end{pmatrix}.$$

**7.20** If  $A$  is invertible, prove that

$$\det(A^{-1}) = \frac{1}{\det A}.$$

**Review problems**

**7.21** What are all of the different ways you know to calculate determinants?

**7.22** How many solutions are there to the following equations?

$$\begin{aligned} x_1 + 1010x_2 + 130923x_3 &= 2839040283 \\ 2x_2 + 23932x_3 &= 2390843248 \\ 3x_3 &= 98234092384 \end{aligned}$$

**7.23** Prove that no matter which entry of an  $n \times n$  matrix you pick ( $n > 1$ ), you can find some invertible  $n \times n$  matrix for which that entry is zero.



## Bases and Subspaces



## Chapter 8

### Span

We want to think not only about vectors, but also about lines and planes. We will find a convenient language in which to describe lines and planes and similar objects.

#### The Problem

Look at a very simple linear equation:

$$x_1 + 2x_2 + x_3 = 0. \quad (8.1)$$

There are many solutions. Each is a point in  $\mathbb{R}^3$ , and together they draw out a plane. But how do we write down this plane? The picture is useless—we can't see for sure which vectors live on it. We need a clear method to write down planes, lines, and similar things, so that we can communicate about them (e.g. over the telephone or to a computer).

One method to describe a plane is to write down an equation, like  $x_1 + 2x_2 + x_3 = 0$ , cutting out the plane. But there is another method, which we will often prefer, building up the plane out of vectors.



Figure 8.1: The solutions of an equation forming a plane.

#### Span

Consider the equations

$$\begin{aligned} x_1 + 2x_2 - 7x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

Solutions have

$$\begin{aligned} x_1 &= -2x_2 + 7x_4 \\ x_3 &= -x_4, \end{aligned}$$

giving

$$\begin{aligned}x &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} -2x_2 + 7x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \\ &= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ 0 \\ -1 \\ 1 \end{pmatrix}\end{aligned}$$

But  $x_2$  and  $x_4$  are free—they can be anything. The solutions are just arbitrary “combinations” of

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 7 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

We can just remember these two vectors, to describe all of the solutions.

A *multiple* of a vector  $v$  is a vector  $cv$  where  $c$  is a number. A *linear combination* of some vectors  $v_1, v_2, \dots, v_p$  in  $\mathbb{R}^n$  is a vector

$$v = c_1v_1 + c_2v_2 + \cdots + c_pv_p,$$

for some numbers  $c_1, c_2, \dots, c_p$  (a sum of multiples). The *span* of some vectors is the collection of all of their linear combinations.

### The Solution

We can describe the plane of solutions of equation 8.1 on the previous page: it is the span of the vectors

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

That isn't obvious. (You can apply forward elimination to check that it is correct.) But immediately we see our next problem: you might describe it as this span, and I might describe it as the span of

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

How do we see that these are the same thing?

### How to Tell if a Vector Lies in a Span

If we have some vectors, lets say

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

how do we tell if another vector lies in their span? Lets ask if

$$y = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

lies in the span of  $x_1$  and  $x_2$ . So we are asking if  $y$  is a linear combination  $c_1x_1 + c_2x_2$ .

Solving the linear equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 &= 4 \\ 3c_1 - c_2 &= 7 \end{aligned}$$

just means finding numbers  $c_1$  and  $c_2$  for which

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix},$$

writing  $y$  as a linear combination of  $x_1$  and  $x_2$ . Solving linear equations is *exactly the same problem* as asking whether one vector is a linear combination of some other vectors.

**8.1** Write down some linear equations, so that solving them is the same problem as asking whether

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

is a linear combination of

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

A *pivot column* of a matrix  $A$  is a column in which a pivot appears when we forward eliminate  $A$ .

The matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

has echelon form

$$U = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

so columns 1 and 3 of  $A$ :

$$A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

are pivot columns.

**Lemma 8.1.** Write some vectors into the columns of a matrix, say

$$A = \begin{pmatrix} x_1 & x_2 & \dots & x_p & y \end{pmatrix}$$

and apply forward elimination. Then  $y$  lies in the span of  $x_1, x_2, \dots, x_p$  just when  $y$  is not a pivot column.

*Proof.* As in our example on page on the preceding page, the problem is precisely whether we can solve the linear equations whose matrix is  $A$ , with  $y$  the column of constants. We already know that linear equations have solutions just when the column of constants is not a pivot column.  $\square$

Applied to our example, this gives

$$\begin{aligned} A &= \begin{pmatrix} x_1 & x_2 & y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \\ 3 & -1 & 7 \end{pmatrix} \end{aligned}$$

to which we apply forward elimination:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \\ 3 & -1 & 7 \end{pmatrix}$$

Add  $-2(\text{row } 1)$  to row 2, and  $-3(\text{row } 1)$  to row 3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & -4 & 4 \end{pmatrix}$$

Add  $-2(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

There is no pivot in the last column, so  $y$  is a linear combination of  $x_1$  and  $x_2$ , i.e. lies in their span. (In fact, in the echelon form, we see that the last column is twice the first column minus the second column. So this must hold in the original matrix too:  $y = 2x_1 - x_2$ .)

**8.2** What if we have a lot of vectors  $y$  to test? Prove that vectors  $y_1, y_2, \dots, y_q$  all lie in the span of vectors  $x_1, x_2, \dots, x_p$  just when the matrix

$$\begin{pmatrix} x_1 & x_2 & \dots & x_p & y_1 & y_2 & \dots & y_q \end{pmatrix}$$

has no pivots in the last  $q$  columns.

Review problems

**8.3** Is the span of the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

the same as the span of the vectors

$$\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}?$$

**8.4** Describe the span of the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

**8.5** Does the vector

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

lie in the span of the vectors

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}?$$

**8.6** Does the vector

$$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

lie in the span of the vectors

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}?$$

**8.7** Does the vector

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

lie in the span of the vectors

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}?$$

**8.8** Does the vector

$$\begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$$

lie in the span of the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}?$$

**8.9** Find a linear equation satisfied on the span of the vectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

## Subspaces

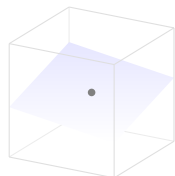
Picture a straight line through the origin, or a plane through the origin. We generalize this picture: a *subspace*  $P$  of  $\mathbb{R}^n$  is a collection of vectors in  $\mathbb{R}^n$  so that

- a.  $P$  is not empty (i.e. some vector belongs to the collection  $P$ )
- b. If  $x$  belongs to  $P$ , then  $ax$  does too, for any number  $a$ .

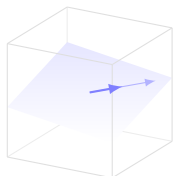


c. If  $x$  and  $y$  belong to  $P$ , then  $x + y$  does too.

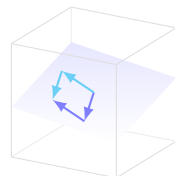
We can see in pictures that a plane through the origin is a subspace:



My plane is not empty: the origin lies in my plane



Scale a vector from my plane: it stays in that plane



Add vectors from my plane: the sum also lies in my plane

**8.10** Prove that  $0$  belongs to every subspace.

**8.11** Prove that if a subspace contains some vectors, then it contains their span.

Intuitively, a subspace is a flat object, like a line or a plane, passing through the origin  $0$  of  $\mathbb{R}^n$ . The set  $P$  of vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

for which  $x_1 + 2x_2 = 0$  is a subspace of  $\mathbb{R}^2$ , because

- $x = 0$  satisfies  $x_1 + 2x_2 = 0$  (so  $P$  is not empty).
- If  $x$  satisfies  $x_1 + 2x_2 = 0$ , then  $ax$  satisfies

$$\begin{aligned} (ax)_1 + 2(ax)_2 &= ax_1 + 2ax_2 \\ &= a(x_1 + 2x_2) \\ &= 0. \end{aligned}$$

c. If  $x$  and  $y$  are points of  $P$ , satisfying

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ y_1 + 2y_2 &= 0 \end{aligned}$$

then  $x + y$  satisfies

$$\begin{aligned} (x_1 + y_1) + 2(x_2 + y_2) &= (x_1 + 2x_2) + (y_1 + 2y_2) \\ &= 0. \end{aligned}$$

**8.12** Is the set  $S$  of all points

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

of the plane with  $x_2 = 1$  a subspace?

**8.13** Is the set  $P$  of all points

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with  $x_1 + x_2 + x_3 = 0$  a subspace?

The word “subspace” really means just the same as “the span of some vectors,” as we will eventually see.

**Proposition 8.2.** *The span of a set of vectors is a subspace; in fact, it is the smallest subspace containing those vectors. Conversely, every subspace is a span: the span of all of the vectors inside it.*

In order to make this proposition true, we have to change our definitions just a little: if we have an empty collection of vectors (i.e. we don’t have any vectors at all), then we will declare that the span of that empty collection is the origin.

If we have an infinite collection of vectors, then their span just means the collection of all linear combinations we can build up from all possible choices we can make of any finite number of vectors from our collection. We don’t allow infinite sums. We would really like to avoid using spans of infinite sets of vectors; we will address this problem in chapter 9.

*Proof.* Given any set of vectors  $X$  in  $\mathbb{R}^n$ , let  $U$  be their span. So any vector in  $U$  is a linear combination of vectors from  $X$ . Scaling any linear combination yields another linear combination, and adding two linear combinations yields a further linear combination, so  $U$  is a subspace. If  $W$  is any other subspace containing  $X$ , then we can add and scale vectors from  $W$ , yielding more vectors from  $W$ , so we can make linear combinations of any vectors from  $W$  making more vectors from  $W$ . Therefore  $W$  contains the span of  $X$ , i.e. contains  $U$ .

Finally, if  $V$  is any subspace, then we can add and scale vectors from  $V$  to make more vectors from  $V$ , so  $V$  is the span of all vectors in  $V$ .  $\square$

**8.14** Prove that every subspace is the span of the vectors that it contains. (Warning: this fact isn’t very helpful, because any subspace will either contain only the origin, or contain infinitely many vectors. We would really rather only think about spans of finitely many vectors. So we will have to reconsider this problem later.)

**8.15** What are the subspaces of  $\mathbb{R}$ ?

**8.16** If  $U$  and  $V$  are subspaces of  $\mathbb{R}^n$ :

- Let  $W$  be the set of vectors which either belong to  $U$  or belong to  $V$ . Is  $W$  a subspace?
- Let  $Z$  be the set of vectors which belong to  $U$  and to  $V$ . Is  $Z$  a subspace?

## Review problems

**8.17** Is the set  $X$  of all points

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

of the plane with  $x_2 = x_1^2$  a subspace?

**8.18** Which of the following are subspaces of  $\mathbb{R}^4$ ?

- The set of points  $x$  for which  $x_1x_4 = x_2x_3$ .
- The set of points  $x$  for which  $2x_1 = 3x_2$ .
- The set of points  $x$  for which  $x_1 + x_2 + x_3 + x_4 = 0$ .
- The set of points  $x$  for which  $x_1, x_2, x_3$  and  $x_4$  are all  $\geq 0$ .

**8.19** Is a circle in the plane a subspace? Prove your answer. Draw pictures to explain your answer.

**8.20** Which lines in the plane are subspaces? Draw pictures to explain your answer.

## Summary

We have solved the problem of this chapter: to describe a subspace. You write down a set of vectors spanning it. If I write down a different set of vectors, you can check to see if mine are linear combinations of yours, and if yours are linear combinations of mine, so you know when yours and mine span the same subspace.



## Chapter 9

### Bases

Our goal in this book is to greatly simplify equations in many variables by changing to new variables. In linear algebra, the concept of changing variables is replaced with the more concrete concept of a *basis*.

#### Definition

A basis is a list of “just enough” vectors to span a subspace. For example, we should be able to span a line by writing down just one vector lying in it, a plane with just two vectors, etc.

A *linear relation* among some vectors  $x_1, x_2, \dots, x_p$  in  $\mathbb{R}^n$  is an equation

$$c_1x_1 + c_2x_2 + \dots + c_px_p = 0,$$

where  $c_1, c_2, \dots, c_p$  are not all zero.

A set of vectors is *linearly independent* if the vectors admit no linear relation. A set of vectors is a *basis* of  $\mathbb{R}^n$  if (1) the vectors are linearly independent and (2) adding any other vector into the set would render them no longer linearly independent.

The vectors

$$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

satisfy the linear relation  $2x_1 - x_2 = 0$ .

#### Properties

**Lemma 9.1.** *The columns of a matrix are linearly independent just when each one is a pivot column.*

*Proof.* Obvious from lemma 8.1 on page 64. □

**9.1** Is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

a basis of  $\mathbb{R}^2$ ?

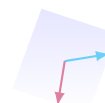


Figure 9.1: Only this plane contains 0 and these two vectors. Three-legged tables don't wobble, unless all of the feet of the table legs lie on the same straight line.

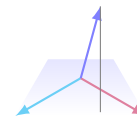


Figure 9.2: The vector sticking up is linearly independent of the other two vectors.

**9.2** The *standard basis* of  $\mathbb{R}^n$  is the basis  $e_1, e_2, \dots, e_n$  (where  $e_1$  is the first column of  $I_n$ , etc.). Prove that the standard basis of  $\mathbb{R}^n$  is a basis.

**9.3** Prove that there is a linear relation between some vectors  $w_1, w_2, \dots, w_q$  just when one of those vectors, say  $w_k$ , is a linear combination of earlier vectors  $w_1, w_2, \dots, w_{k-1}$ .

**Theorem 9.2.** *Every linearly independent set of vectors in  $\mathbb{R}^n$  consists in at most  $n$  vectors, and consists in exactly  $n$  vectors just when it is a basis.*

*Proof.* Suppose that  $x_1, x_2, \dots, x_p$  are a linearly independent. Let

$$A = \begin{pmatrix} x_1 & x_2 & \dots & x_p \end{pmatrix}.$$

There is either one pivot or no pivot in each row. So the number of rows is at least as large as the number of pivots. There are  $n$  rows. There is one pivot in each column, so  $p$  pivots. So  $p \leq n$ .

If  $p = n$ , we have one pivot in each row, so adding another vector (another column) can't add another pivot. Therefore adding any other vector to the vectors  $x_1, x_2, \dots, x_p$  would break linear independence.

If  $p < n$ , then we have zero rows after forward elimination. Suppose that forward elimination yields  $U = VA$ . Then  $(U \ e_{p+1})$  has more pivot columns than  $U$  has, so  $(A \ V^{-1}e_{p+1})$  has more pivot columns than  $A$  has. Thus adding a new vector  $x_{p+1} = V^{-1}e_{p+1}$  to the collection of vectors  $x_1, x_2, \dots, x_p$ , we have a larger linearly independent collection.  $\square$

**9.4** Prove that every linearly independent set of vectors in  $\mathbb{R}^n$  belongs to a basis.

**Lemma 9.3.** *A set of vectors  $u_1, u_2, \dots, u_n$  is a basis of  $\mathbb{R}^n$  just when every vector  $b$  in  $\mathbb{R}^n$  can be written as a linear combination*

$$b = a_1u_1 + a_2u_2 + \dots + a_nu_n,$$

for a unique choice of numbers  $a_1, a_2, \dots, a_n$ .

*Proof.* Let

$$A = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix},$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

and apply theorem 5.4 on page 43 to the equation  $Aa = b$ .  $\square$

## Review problems

**9.5** Are the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

linearly independent?

**9.6** Are the vectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

a basis?

**9.7** Can you find matrices  $A$  and  $B$  so that  $A$  is  $3 \times 5$  and  $B$  is  $5 \times 3$ , and  $AB = 1$ ?

**9.8** Suppose that  $A$  is  $3 \times 5$  and  $B$  is  $5 \times 3$ , and that  $AB$  is invertible. Must the columns of  $B$  be linearly independent? the rows of  $B$ ? the columns of  $A$ ? the rows of  $A$ ?

**9.9** Give an example of a  $3 \times 3$  matrix for which any two columns are linearly independent, *but* the three columns together are not linearly independent. Can such a matrix be invertible?

## The Change of Basis Matrix

The *change of basis matrix*  $F$  associated to a basis  $u_1, u_2, \dots, u_n$  of  $\mathbb{R}^n$  is the matrix

$$F = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}.$$

Note that  $Fe_1 = u_1, Fe_2 = u_2, \dots, Fe_n = u_n$ . So taking  $x$  to  $Fx$  is a “change of basis”, taking the standard basis to the new basis.

**9.10** Prove that an  $n \times n$  matrix  $C$  is the change of basis matrix of a basis just when the equation  $Cx = 0$  has  $x = 0$  as its only solution, which occurs just when  $C$  is invertible.

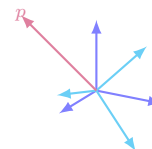
Suppose that you and I look at the sky and watch a falling star. You measure its position against the fixed choice of basis  $e_1, e_2, e_3$ , while I measure against some funny choice of basis  $u_1, u_2, u_3$ .

The actual position is some vector  $p$  in  $\mathbb{R}^3$ . Lets say

$$\begin{aligned} p &= x_1e_1 + x_2e_2 + x_3e_3 \text{ as you measure it,} \\ &= y_1u_1 + y_2u_2 + y_3u_3 \text{ as I measure it.} \end{aligned}$$

Let

$$F = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$$



be the change of basis matrix, so that  $Fe_1 = u_1$ , etc. So  $F$  takes your basis to mine. If we let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

then in your basis:

$$\begin{aligned} p &= x_1e_1 + x_2e_2 + x_3e_3 \\ &= x \end{aligned}$$

but in mine:

$$\begin{aligned} p &= y_1u_1 + y_2u_2 + y_3u_3 \\ &= y_1Fe_1 + y_2Fe_2 + y_3Fe_3 \\ &= F(y_1e_1 + y_2e_2 + y_3e_3) \\ &= Fy. \end{aligned}$$

So  $x = Fy$  converts my measurements to yours.

Suppose that we change variables by  $x = Fy$  and so  $y = F^{-1}x$ , with  $F$  some invertible matrix. Then any matrix  $A$  acting on the  $x$  variables by taking  $x$  to  $Ax$  is represented in  $y$  variables as

$$\begin{array}{ccccc} \xleftarrow{F^{-1}} & \xleftarrow{A} & \xleftarrow{F} & & \\ \text{turn } x\text{'s to } y\text{'s} & \text{act on } x\text{'s} & \text{turn } y\text{'s to } x\text{'s} & & \end{array}$$

the matrix  $F^{-1}AF$ .

**9.11** Take

$$F = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Compute  $F^{-1}AF$ .

**9.12** A shower of falling stars fall to Earth. Each star falls from a position

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

to a position on the ground

$$Ax = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$



What is the matrix  $A$ ? Suppose that I measure the positions of the stars against the basis

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Find the change of basis matrix  $F$ , and find  $F^{-1}AF$ , the matrix that describes how each star falls from the sky as measured against my basis.

Review problems

**9.13** Is

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a basis of  $\mathbb{R}^3$ ?

**9.14** If  $A$  is a matrix, show how each vector which  $A$  kills determines a linear relation between the columns of  $A$ , and vice versa.

**9.15** Are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

linearly independent?

**9.16** Write down a basis of  $\mathbb{R}^2$  other than the standard basis, and prove that your basis really is a basis.

**9.17** Is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

a basis of  $\mathbb{R}^2$ ?

**9.18** Is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

a change of basis matrix? If so, for what basis?

**9.19** If  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  are two bases of  $\mathbb{R}^n$ , prove that there is a unique invertible matrix  $A$  so that  $Ax_1 = y_1, Ax_2 = y_2$ , etc.

## Bases of Subspaces

We can write down a subspace, by writing down a spanning set of vectors. But you might write down more vectors than you need to. We want to squeeze the description down to the bare simplest minimum, throwing out redundant information.

If  $V$  is a subspace of  $\mathbb{R}^n$ , a *basis* of  $V$  is a set of linearly independent vectors from  $V$ , so that adding any other vector from  $V$  into the set would render them no longer linearly independent.

The vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are a basis for the subspace  $V$  in  $\mathbb{R}^3$  of vectors of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$

Obviously:

**Lemma 9.4.** *If some vectors span a subspace, then putting them into the columns of a matrix, the pivot columns form a basis of the subspace.*

Lets find a basis for the span of the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Put them into a matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Forward eliminate:

$$\begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so the first and second columns are pivot columns. Therefore

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

are a basis for the span.

**Proposition 9.5.** *Every subspace of  $\mathbb{R}^n$  has a basis. Moreover, any basis  $v_1, v_2, \dots, v_p$  of a subspace  $V$  of  $\mathbb{R}^n$  lives in a basis  $v_1, v_2, \dots, v_p, w_1, w_2, \dots, w_q$  of  $\mathbb{R}^n$ .*

*Proof.* If  $V$  only contains the 0 vector, then we can take no vectors as a basis for  $V$ , and let  $w_1, w_2, \dots, w_n$  be any basis for  $\mathbb{R}^n$ . On the other hand, if  $V$  contains a nonzero vector, then pick as many linearly independent vectors from  $V$  as possible. By theorem 9.2 on page 72, we could only pick at most  $n$  vectors. They must span  $V$ , because otherwise we could pick another one. If  $V = \mathbb{R}^n$ , then we are finished. Otherwise, pick as many vectors from  $\mathbb{R}^n$  as possible which are linearly independent of  $v_1, v_2, \dots, v_p$ . Clearly we stop just when we hit a total of  $n$  vectors.  $\square$

### Dimension

Do all bases look pretty much the same?

**Theorem 9.6.** *Any two bases of a subspace have the same number of vectors.*

*Proof.* Imagine two bases, say  $x_1, x_2, \dots, x_p$  and  $y_1, y_2, \dots, y_q$ , for the same subspace. Forward eliminate

$$\left( \begin{array}{cccc|cccc} x_1 & x_2 & \dots & x_p & y_1 & y_2 & \dots & y_q \end{array} \right)$$

yielding

$$\left( \begin{array}{cccc|cccc} \text{shaded triangle} & & & & & & & \\ & \text{shaded square} & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array} \right).$$

Each  $x$  vector generates a pivot,  $p$  pivots in all, straight down the diagonal. Forward eliminate the right hand portion of the matrix, yielding

$$\left( \begin{array}{cccc|cccc} \text{shaded triangle} & & & & & & & \\ & \text{shaded triangle} & & & & & & \\ & & \text{shaded triangle} & & & & & \\ & & & \text{shaded triangle} & & & & \\ & & & & \text{shaded triangle} & & & \\ & & & & & \text{shaded triangle} & & \\ & & & & & & \text{shaded triangle} & \\ & & & & & & & \text{shaded triangle} \end{array} \right),$$

giving at most  $p$  pivots because of the zero rows. Each  $y$  vector generates a pivot. So there aren't more than  $p$  of these  $y$  vectors. Thus no more  $y$  vectors than  $x$  vectors. Reversing the roles of  $x$  and  $y$  vectors, we find that there can't be more  $x$  vectors than  $y$  vectors.  $\square$

**9.20** Prove that every subspace of  $\mathbb{R}^n$  has a basis with at most  $n$  vectors.

The *dimension* of a subspace is the number of vectors in any basis. Write the dimension of a subspace  $U$  as  $\dim U$ .

### Review problems

**9.21** Consider the vectors in  $\mathbb{R}^n$  of the form  $e_i - e_j$  (for all possible values of  $i$  and  $j$  from 1 to  $n$ ). Find a basis for the subspace they span.

### Summary

- A *subspace* is a flat thing passing through 0.
- A *basis* for a subspace is a collection of just enough vectors to span the subspace.
- A *change of basis matrix* is a basis organized into the columns of a matrix.

### Uniqueness of Reduced Echelon Form

When we carry out elimination, we choose rows to swap.

**9.22** Find the simplest matrix  $A$  you can with two different ways of carrying out forward elimination, with different results.

Recall that Gauss–Jordan elimination means forward elimination followed by back substitution. The matrix resulting from Gauss–Jordan elimination is said to be in *reduced echelon form*.

**Theorem 9.7.** *The result of Gauss–Jordan elimination does not depend on the choices made of which rows to swap.*

*Proof.* Suppose that  $U$  and  $W$  are two different eliminations of the same matrix  $A$ , obtained using different choices of rows to swap. The first pivot column is just the first nonzero column of  $A$ . The second pivot column is the earliest column which is linearly independent of the first pivot column, etc. This is true for  $A$ , and doesn't change under forward elimination or back substitution. Therefore  $A$  and  $U$  and  $W$  have the same pivot columns. After elimination, the first pivot column becomes  $e_1$ , the second becomes  $e_2$ , etc. So all of the pivot columns of  $U$  and  $W$  must be identical.

Every pivotless column is a linear combination of earlier pivot columns, and the coefficients in this linear combination are not affected by Gauss–Jordan

elimination. Therefore the pivotless columns of  $U$  and  $W$  are the same linear combinations of the pivot columns. The pivot columns are the same, so all columns are the same.  $\square$

**9.23** The rank is the number of pivots in the forward elimination. Prove that the rank of a matrix does not depend on which rows you choose when forward eliminating.



## Chapter 10

### Kernel and Image

Each matrix  $A$  has two important subspaces associated to it: its *kernel* (the vectors it kills), and its *image* (the vectors  $b$  for which you can solve  $Ax = b$ ).

#### Kernel

If  $A$  is any matrix, say  $p \times q$ , then the vectors  $x$  in  $\mathbb{R}^q$  for which  $Ax = 0$  (vectors “killed” by  $A$ ) form a subspace of  $\mathbb{R}^q$  called the *kernel* of  $A$ , and written  $\ker A$ .

The kernel is a subspace, because

- 0 belongs to the kernel of any matrix  $A$ , since  $A0 = 0$  (everything kills 0).
- If  $Ax = 0$  and  $Ay = 0$ , then  $A(x + y) = Ax + Ay = 0$  (when you kill two vectors, you kill their sum).
- If  $Ax = 0$ , then  $A(ax) = aAx = 0$  (when you kill a vector, you kill its multiples).

**10.1** If a matrix is wider than it is tall (a “short” matrix), then its kernel contains nonzero vectors.

**10.2** Prove that the kernel of  $AB$  contains the kernel of  $B$ . Does it have to contain the kernel of  $A$ ?

We will often need to find kernels of matrices. To rapidly calculate the kernel of a matrix, for example

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{pmatrix}$$

- Carry out forward elimination and back substitution.

$$\begin{pmatrix} \boxed{1} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \boxed{1} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Cut out all zero rows.

$$\begin{pmatrix} \boxed{1} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \boxed{1} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

c. Change the signs of all entries after each pivot.

$$\begin{pmatrix} \boxed{1} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \boxed{1} & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

(This corresponds to changing equations like  $x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = 0$  to  $x_1 = -\frac{1}{2}x_3 - \frac{1}{2}x_4$ . Think of it as moving everything after the pivot over to the right hand side, although we won't actually move anything.)

d. Stuff in whatever rows from the identity matrix you need into your matrix, so that it ends up with nonzero entries all down the diagonal.

$$\begin{pmatrix} \boxed{1} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \boxed{1} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We won't mark the new rows with pivots. Each new row corresponds to setting one of the free variables to 1 and the others to 0.

e. Cut out all of the pivot columns. The remaining columns are a basis for the kernel.

$$\begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}.$$

**10.3** Apply this algorithm to the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{pmatrix}$$

and check that  $Ax = 0$  for each vector  $x$  in your resulting basis for the kernel.

**Lemma 10.1.** *The algorithm works, giving a basis for the kernel of any matrix.*

*Proof.* Each vector in the kernel is obtained by setting arbitrary values for the free variables, and letting the pivots solve for the other variables. Let  $v_1$  be the vector in the kernel which has value 1 for the 1<sup>st</sup> free variable, and 0 for all other free variables. Similarly, make a vector  $v_2, v_3, \dots, v_s$  for each free variable—suppose that there are  $s$  free variables. The kernel is a subspace, so each linear combination

$$c_1v_1 + c_2v_2 + \dots + c_s v_s$$

lies in the kernel. This linear combination has value  $c_1$  for the first free variable,  $c_2$  for the second, etc. (just looking at the rows of the free variables). Each



vector in the kernel has some values  $c_1, c_2, \dots, c_s$  for the free variables. So each vector in the kernel is a unique linear combination of  $v_1, v_2, \dots, v_s$ . Suppose that we find a linear relation among  $v_1, v_2, \dots, v_s$ , say  $c_1 v_1 + c_2 v_2 + \dots + c_s v_s = 0$ . Look at the row in which  $v_1$  has a 1 and all of the other vectors have 0's: the linear relation gives  $c_1 = 0$  in that row. Similarly all of  $c_1, c_2, \dots, c_s$  must vanish, so there is no linear relation among these vectors. Therefore the vectors  $v_1, v_2, \dots, v_s$  form a basis for the kernel.

Finally, we need to see why these vectors  $v_1, v_2, \dots, v_s$  are precisely the vectors which come out of our process above. First, look at our example. The reduced echelon form turns back into equations as

$$\begin{array}{rclcl} x_1 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & = & 0 \\ & x_2 & +\frac{3}{2}x_3 & +\frac{1}{2}x_4 & = & 0 \end{array}$$

Solving for pivots means subtracting off:

$$\begin{array}{rcl} x_1 & = & -\frac{1}{2}x_3 - \frac{1}{2}x_4 \\ & x_2 & = & -\frac{3}{2}x_3 - \frac{1}{2}x_4. \end{array}$$

All free variables line up on the right hand side, and we have changed the signs of their coefficients. Setting  $x_3 = 1$  and  $x_4 = 0$ , go down the right hand side, killing the  $x_4$  entries, and putting  $x_3 = 1$  in each  $x_3$  entry, i.e. writing down just the entries from the  $x_3$  column:

$$v_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix}.$$

The general algorithm works in the same way: if we put all free variables on to the right hand side, and then set one free variable to 1 (“turn it on”) and the others to 0’s (“turn them off”), we can picture this as “turning on” the column associated to that free variable. Each pivot solves for a pivot variable—the value of that pivot variable is the entry in the corresponding row of the “turned on” column.  $\square$

**10.4** Give an example of a square matrix whose kernel is *not* the kernel of its transpose.

**10.5** Draw a picture of the kernel for each of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, C = (1), D = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

**Corollary 10.2.** *The dimension of the kernel of a matrix is the number of pivotless columns after forward elimination.*

Another way to say it: the dimension of the kernel of a matrix  $A$  is the number of free variables in the equation  $Ax = 0$ .

Review problems

**10.6** Find a basis for the kernel of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**10.7** Find a basis for the kernel of

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

**10.8** Find a basis for the kernel of

$$\begin{pmatrix} 1 & -1 & -1 & -1 & 1 \\ 2 & 2 & 0 & 0 & -1 \\ 2 & 2 & 0 & 1 & 0 \\ -1 & 0 & 2 & -1 & 2 \end{pmatrix}$$

**10.9** Find a basis for the kernel of

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

**10.10** Find a basis for the kernel of

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

**10.11** Find the dimension of the kernel of

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**10.12** Prove that the kernel of  $A$  is the kernel of

$$\begin{pmatrix} A \\ A \end{pmatrix}.$$

**10.13** If you know the kernel of a  $p \times q$  matrix  $A$ , how do you find the dimension of the kernel of

$$B = \begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix}?$$

### Image

The *image* of a matrix is the set of vectors  $y$  of the form  $y = Ax$  for some vector  $x$ , written  $\text{im } A$ .

**10.14** Prove that the image of a matrix is the span of its columns.

The image of

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

is the span of

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}.$$

**10.15** Prove that the equation  $Ax = b$  has a solution  $x$  just when  $b$  lies in the image of  $A$ .

### Review problems

**10.16** Describe the image of the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}.$$

**10.17** (a) Suppose that  $A$  is a  $2 \times 2$  matrix which, when taking a vector  $x$  to the vector  $Ax$ , takes multiples of  $e_1$  to multiples of  $e_1$ . Show that  $A$  is upper triangular.

(b) Similarly, if  $A$  is a  $3 \times 3$  matrix which takes multiples of  $e_1$  to multiples of  $e_1$ , and takes linear combinations of  $e_1$  and  $e_2$  to other such linear combinations, then  $A$  is upper triangular.

(c) Generalize this to  $\mathbb{R}^n$ , and use it to show that the inverse of an invertible upper triangular matrix is upper triangular.

**10.18** Prove that if a matrix is taller than it is wide (a “tall” matrix), then some vector does not belong to its image.

**10.19** Find a basis for the image of

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 2 & 2 & -1 \end{pmatrix}.$$

Explain your answer.

**10.20** Find a basis for the image of

$$\begin{pmatrix} 2 & -6 & -6 \\ -6 & 18 & 18 \\ 0 & 0 & 0 \end{pmatrix}.$$

Explain your answer.

**10.21** Find a basis for the image of

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & -1 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

Explain your answer.

**10.22** Find a basis for the image of

$$\begin{pmatrix} 2 & -1 & 3 \\ -6 & 3 & -9 \\ 0 & 0 & 0 \end{pmatrix}.$$

Explain your answer.

**10.23** Find a basis for the image of

$$\begin{pmatrix} -3 & 1 & 0 \\ 3 & -1 & 0 \\ 3 & -1 & 0 \end{pmatrix}.$$

Explain your answer.

**10.24** Find a basis for the image of

$$\begin{pmatrix} 2 & 4 & 0 \\ -6 & -12 & 0 \\ 4 & 8 & 0 \end{pmatrix}.$$

Explain your answer.

**10.25** Find a basis for the image of

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

Explain your answer.

**10.26** Find a basis for the image of

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & -1 \end{pmatrix}.$$

Explain your answer.

**10.27** Find a basis for the image of

$$\begin{pmatrix} -1 & 4 & 4 \\ 3 & -12 & -12 \\ 3 & -12 & -12 \end{pmatrix}.$$

Explain your answer.

**10.28** Find a basis for the image of

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Explain your answer.

**10.29** Find a basis for the image of

$$\begin{pmatrix} 2 & -6 & -4 \\ -6 & 18 & 12 \\ 0 & 0 & 0 \end{pmatrix}.$$

Explain your answer.

**10.30** Find a basis for the image of

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Explain your answer.

## Kernel and Image

**10.31** If  $A$  and  $B$  are matrices, and there is an invertible matrix  $C$  for which  $B = CA$ , prove that  $A$  and  $B$  have the same kernel.

**10.32** If  $A$  and  $B$  are two matrices, and  $B = CA$  for an invertible matrix  $C$ , prove that  $A$  and  $B$  have images of the same dimension.

**Theorem 10.3.** For any matrix  $A$ ,

$$\dim \ker A + \dim \operatorname{im} A = \text{number of columns.}$$

*Proof.* The image of  $A$  is the span of the columns. By lemma 8.1 on page 64, each pivotless column is a linear combination of earlier pivot columns. So the pivot columns span the image. Pivot columns are linearly independent: a basis. Each pivotless column contributes (in our algorithm) to our basis for the kernel.  $\square$

The matrix

$$A = \begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -2 & 2 & -1 & 1 \end{pmatrix}$$

has echelon form

$$U = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

So columns 1, 3 and 4 of  $A$  (not of  $U$ ) are a basis for the image of  $A$ :

$$\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The image has dimension 3 because there are 3 pivot columns. The kernel has dimension 1, because there is one pivotless column. *The pivotless column is not a basis for the kernel.* It just shows you the *dimension* of the kernel. (In this example, the pivotless column isn't even in the kernel.)

**10.33** Find the rank of

$$A = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

and explain what this tells you about image and kernel.

## Review problems

**10.34** Find two matrices  $A$  and  $B$ , which have different images, but for which  $B = CA$  for an invertible matrix  $C$ . Prove that the images really are different, but of the same dimension.

**10.35** Prove that  $\text{rank } A = \text{rank } A^t$  for any matrix  $A$ .

## Summary

- The *kernel* of a matrix is the set of vectors it kills. It is large just when linear equations  $Ax = b$  with one solution have lots of solutions (measures plurality of solutions when they exist).
- Our algorithm makes a basis for the kernel out of the pivotless columns.
- The *image* of a matrix is the stuff that comes out of it—the vectors  $b$  for which you can solve  $Ax = b$  (measures existence of solutions).
- The pivot columns are a basis of the image.

## Review problems

**10.36** Suppose that  $Ax = b$ . Prove that  $A(x + y) = b$  too, just when  $y$  lies in the kernel. So the kernel measures the plurality of solutions of equations, while the image measures existence of solutions.

**10.37** What is the maximum possible rank of a  $4 \times 3$  matrix? A  $3 \times 5$  matrix?

**10.38** If a  $3 \times 5$  matrix  $A$  has rank 3 must the equation  $Ax = b$  have a solution  $x$ ? Can it have more than one solution? If it has one solution, must it have infinitely many?

**10.39** As for the previous question, but with a  $5 \times 3$  matrix  $A$ .

**10.40** If  $A = BC$  and  $B$  is  $5 \times 4$  and  $C$  is  $4 \times 5$ , prove that  $\det A = 0$ .

**10.41** Write down a  $2 \times 2$  matrix  $A$  so that if I choose any vector  $x$  with positive entries, then the vector  $Ax$  also has positive entries, and lies between (but not on) the horizontal axis and a diagonal line.

**10.42** The Fredholm Alternative: for any matrix  $A$  and vector  $b$ , prove that just one of the following two problems has a solution: (1)  $Ax = b$  or (2)  $A^t y = 0$  with  $b^t y \neq 0$ .

**10.43** Prove that the image of  $AB$  is contained in the image of  $A$ .

**10.44** Prove that the rank of  $AB$  is never more than the rank of  $A$  or of  $B$ .

**10.45** Prove that the rank of a sum of matrices is never more than the sum of the ranks.

**10.46** Which of the following can change when you carry out forward elimination?

- a. image,
- b. kernel,
- c. dimension of image,
- d. dimension of kernel?

**10.47** Prove that the rank of  $AB$  is no larger than the ranks of  $A$  and  $B$ .



Eigenvectors



## Chapter 11

### Eigenvalues and Eigenvectors

In this chapter, we study certain special vectors, called *eigenvectors*, associated to a square matrix.

#### What are eigenvectors

When a vector  $x$  is struck by a matrix  $A$ , it becomes a new vector  $Ax$ . Usually the new vector is unrelated to the old one. Rarely, the new vector might just be the old one stretched or squished; we will then call  $x$  an *eigenvector* of  $A$ . If we have a basis worth of eigenvectors, then the matrix  $A$  just squishes or stretches each one, and we can completely recover the matrix if we know the basis of eigenvectors and their eigenvalues.



Figure 11.1: An eigenvector just gets stretched

#### Eigenvalues and the Characteristic Polynomial

An *eigenvector*  $x$  of a square matrix  $A$  is a nonzero vector for which  $Ax = \lambda x$  for some number  $\lambda$ , called the *eigenvalue* of  $x$ . The *eigenvalue* is the factor that the eigenvector gets stretched by.

**11.1** For each of the pictures in problem 3.16 on page 25, calculate the eigenvalues of the associated matrix and draw on each face the directions that the eigenvectors point in.

**Lemma 11.1.** *A number  $\lambda$  is an eigenvalue of a square matrix  $A$  (which is to say that there is an eigenvector  $x$  with that number as eigenvalue) just when*

$$\det(A - \lambda I) = 0.$$

*Proof.* Rewrite the equation  $Ax = \lambda x$  as  $(A - \lambda I)x = 0$ . Recall that the equation  $Bx = 0$  (with  $B$  a square matrix) has a nonzero solution  $x$  just when  $B$  is not invertible, so just when  $\det B = 0$ . Lets pick  $B$  to be  $A - \lambda I$ ; there is an eigenvector  $x$  with eigenvalue  $\lambda$  just when  $\det(A - \lambda I) = 0$ .  $\square$

The expression  $\det(A - \lambda I)$  is called the *characteristic polynomial* of the matrix  $A$ .

We can restate the lemma:

**Lemma 11.2.** *The eigenvalues of a square matrix  $A$  are precisely the roots of its characteristic polynomial.*

The matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

has characteristic polynomial

$$\begin{aligned} \det \left( \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) &= \det \begin{pmatrix} 2 - \lambda & 0 \\ 0 & 3 - \lambda \end{pmatrix} \\ &= (2 - \lambda)(3 - \lambda). \end{aligned}$$

So the eigenvalues are  $\lambda = 2$  and  $\lambda = 3$ .

**11.2** Prove that the eigenvalues of an upper (or lower) triangular matrix are the diagonal entries. For example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

has eigenvalues  $\lambda = 1, \lambda = 4, \lambda = 6$ .

**11.3** Find a  $2 \times 2$  matrix  $A$  whose eigenvalues are *not* the same as its diagonal entries.

**11.4** Find  $2 \times 2$  matrices  $A$  and  $B$  for which  $A + B$  has an eigenvalue which is not a sum of some eigenvalue of  $A$  with some eigenvalue of  $B$ .

The set of eigenvalues of a matrix is called its *spectrum*.

The matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has characteristic polynomial  $\det(A - \lambda I) = \lambda^2 + 1$ , which has no roots, so there are no eigenvalues (among real numbers  $\lambda$ ).

**11.5** Why is the characteristic polynomial a polynomial in  $\lambda$ ?

**11.6** What is the highest order term of the variable  $\lambda$  in the characteristic polynomial of a matrix  $A$ ?

Appendix: Why the Fast Formula is So Slow

We use the slow formula to calculate determinants when we compute out characteristic polynomials. Why not the fast formula? Lets try it on an example. Take

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & 4 \end{pmatrix}.$$

Lets hunt down the eigenvalues of  $A$ , by computing the characteristic polynomial as before. But this time, lets try the fast formula for the determinant, to find  $\det(A - \lambda I)$ . We apply forward elimination to

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 1 & 2 \\ 2 & 3 - \lambda & 0 \\ 1 & 1 & 4 - \lambda \end{pmatrix}.$$

to find

$$\begin{pmatrix} 1 - \lambda & 1 & 2 \\ 2 & 3 - \lambda & 0 \\ 1 & 1 & 4 - \lambda \end{pmatrix}$$

Add  $-\frac{2}{1-\lambda}$ (row 1) to row 2,  $-\frac{1}{1-\lambda}$ (row 1) to row 3.

$$\begin{pmatrix} 1 - \lambda & 1 & 2 \\ 0 & -\frac{1-4\lambda+\lambda^2}{-1+\lambda} & 4(-1+\lambda)^{-1} \\ 0 & \frac{\lambda}{-1+\lambda} & -\frac{2-5\lambda+\lambda^2}{-1+\lambda} \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 - \lambda & 1 & 2 \\ 0 & -\frac{1-4\lambda+\lambda^2}{-1+\lambda} & 4(-1+\lambda)^{-1} \\ 0 & \frac{\lambda}{-1+\lambda} & -\frac{2-5\lambda+\lambda^2}{-1+\lambda} \end{pmatrix}$$

Add  $\frac{\lambda}{1-4\lambda+\lambda^2}$ (row 2) to row 3.

$$\begin{pmatrix} 1 - \lambda & 1 & 2 \\ 0 & -\frac{1-4\lambda+\lambda^2}{-1+\lambda} & 4(-1+\lambda)^{-1} \\ 0 & 0 & -\frac{\lambda^3-8\lambda^2+15\lambda-2}{1-4\lambda+\lambda^2} \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 - \lambda & 1 & 2 \\ 0 & -\frac{1-4\lambda+\lambda^2}{-1+\lambda} & 4(-1+\lambda)^{-1} \\ 0 & 0 & -\frac{\lambda^3-8\lambda^2+15\lambda-2}{1-4\lambda+\lambda^2} \end{pmatrix}$$

The point: at each step, the expressions are rational functions of  $\lambda$ , accumulating to become more complicated at each step. This is not any faster than the slow process, which gives:

$$\begin{aligned} \det(A - \lambda I) &= + (1 - \lambda) \det \begin{pmatrix} 3 - \lambda & 0 \\ 1 & 4 - \lambda \end{pmatrix} \\ &\quad - 2 \det \begin{pmatrix} 1 & 2 \\ 1 & 4 - \lambda \end{pmatrix} \\ &\quad + 1 \det \begin{pmatrix} 1 & 2 \\ 3 - \lambda & 0 \end{pmatrix} \\ &= 2 - 15\lambda + 8\lambda^2 - \lambda^3. \end{aligned}$$

Always use the slow process when searching for eigenvalues. There is actually a faster method to find eigenvalues of large matrices, but it is slower on small matrices, and we won't ever want to work with large matrices.

### Review problems

**11.7** Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

**11.8** Prove that a square matrix  $A$  and its transpose  $A^t$  have the same eigenvalues.

**11.9** Prove that

$$\det(F^{-1}AF - \lambda I) = \det(A - \lambda I)$$

for any square matrix  $A$ , and any invertible matrix  $F$ . So the characteristic polynomial is unchanged by change of basis.

**11.10** If all of the entries of a square matrix are positive, are its eigenvalues positive?

**11.11** Are the eigenvalues of  $AB$  equal to those of  $BA$ ?

**11.12** Give an example of  $2 \times 2$  matrices  $A$  and  $B$  for which the eigenvalues of  $AB$  are *not* products of eigenvalues of  $A$  with those of  $B$ .

**11.13** What are the eigenvalues of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}?$$

The *multiplicity* of an eigenvalue  $\lambda_j$  is the number of factors of  $\lambda - \lambda_j$  appearing in the characteristic polynomial.

**11.14** Suppose that the characteristic polynomial of some  $n \times n$  matrix  $A$  splits into a product of linear factors. Prove that the determinant of  $A$  is the product of its eigenvalues (each taken with multiplicity), by setting  $\lambda = 0$  in the characteristic polynomial.

**11.15** From the previous exercise, if a  $2 \times 2$  matrix  $A$  has eigenvalues 0 and 1, what is its rank?

**11.16** Write out the characteristic polynomial of an  $n \times n$  matrix  $A$  as

$$\det(A - \lambda I) = s_0(A) - s_1(A)\lambda + s_2(A)\lambda^2 + \cdots + (-1)^n s_n(A)\lambda^n.$$

- Find  $s_n(A)$ .
- Prove that  $s_0(A) = \det A$ .
- Prove that  $s_j(A)$  is a sum of products of precisely  $n - j$  entries of  $A$ . In particular,  $s_{n-1}(A)$  is a polynomial of degree 1 as a function of each entry of  $A$ .
- Use this to prove that  $s_{n-1}(A) = A_{11} + A_{22} + \cdots + A_{nn}$ . (This quantity  $A_{11} + A_{22} + \cdots + A_{nn}$  is called the *trace* of  $A$ ).
- Prove that  $s_j(F^{-1}AF) = s_j(A)$  for any invertible matrix  $F$ , so the coefficients of the characteristic polynomial are unchanged by change of basis.
- Take a basis  $u_1, u_2, \dots, u_n$  for which the vectors  $u_{r+1}, u_{r+2}, \dots, u_n$  form a basis of the kernel, let  $F$  be the associated change of basis matrix, and look at  $F^{-1}AF$ . Prove that

$$F^{-1}AF = \begin{pmatrix} P & 0 \\ Q & 0 \end{pmatrix}$$

for some invertible  $r \times r$  matrix  $P$ , and some matrix  $Q$ .

- If  $A$  has rank  $r$ , prove that  $s_k(A) = 0$  for  $k \leq n - r$ .
- Write down two  $2 \times 2$  matrices of different ranks with the same characteristic polynomial.

### How to find eigenvectors

To find the eigenvectors of a matrix  $A$ : once you have the eigenvalues, pick each eigenvalue  $\lambda$ , and find the kernel of  $A - \lambda I$ .

The matrix

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

has eigenvalues  $\lambda = 2$  and  $\lambda = 3$ .

Lets start with  $\lambda = 2$ :

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Our algorithm (from section 10) for finding the kernel yields a basis

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

for the  $\lambda = 2$ -eigenvectors.

**11.17** Do the same for  $\lambda = 3$ .

**11.18** Find the eigenvectors and eigenvalues of

$$A = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Put it all together. How do we calculate the eigenvectors of

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}?$$

a. Find the eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 3 - \lambda & 2 \\ 0 & 1 - \lambda \end{pmatrix} \\ &= (3 - \lambda)(1 - \lambda) \end{aligned}$$

So the eigenvalues are  $\lambda = 3$  and  $\lambda = 1$ .

b. Find the eigenvectors: for each eigenvalue  $\lambda$ , compute a basis for the kernel of  $A - \lambda I$ . For  $\lambda = 3$ :

$$\begin{aligned} A - 3I &= \begin{pmatrix} 3 - 3 & 2 \\ 0 & 1 - 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} \end{aligned}$$



The kernel of  $A - 3I$  has basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The nonzero linear combinations of this basis are the eigenvectors with eigenvalue  $\lambda = 3$ .

For  $\lambda = 1$ ,

$$\begin{aligned} A - I &= \begin{pmatrix} 3 - 1 & 2 \\ 0 & 1 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

The kernel of  $A - I$  has basis

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The nonzero linear combinations of this basis are the  $\lambda = 1$ -eigenvectors.

Review problems

**11.19** Without any calculation, what are the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}?$$

**11.20** Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

**11.21** Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

**11.22** Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 3 \\ 2 & 0 & 3 \end{pmatrix}$$

**11.23** Prove that every eigenvector of any square matrix  $A$  is an eigenvector of  $A^{-1}$ , of  $A^2$ , of  $3A$  and of  $A - 7I$ . How are the eigenvalues related?

**11.24** Forward elimination messes up eigenvalues and eigenvectors. Back substitution messes them up further. Give the simplest examples you can.

**11.25** What are the eigenvalues and eigenvectors of the permutation matrix of a transposition?

**11.26** What are the eigenvalues and eigenvectors of a  $2 \times 2$  strictly lower triangular matrix?

**11.27** What are all of the real numbers that can occur as values of the

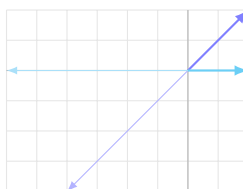
- (a) determinant
- (b) trace
- (c) pivots
- (d) eigenvalues

of  $n \times n$  permutation matrices? Justify your answers.

**11.28** Find a matrix with the given eigenvectors.



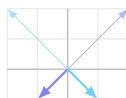
**11.29** Find a matrix with the given eigenvectors.



**11.30** Find a matrix with the given eigenvectors.



**11.31** Find a matrix with the given eigenvectors.



11.32 Find a matrix with the given eigenvectors.



11.33 Find a matrix with the given eigenvectors.



11.34 Find a matrix with the given eigenvectors.



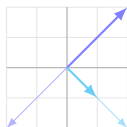
11.35 Find a matrix with the given eigenvectors.



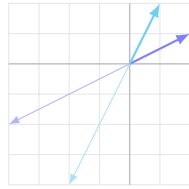
11.36 Find a matrix with the given eigenvectors.



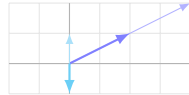
11.37 Find a matrix with the given eigenvectors.



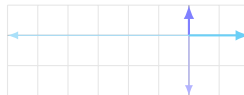
11.38 Find a matrix with the given eigenvectors.



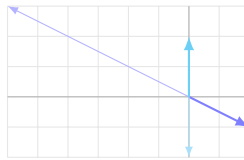
**11.39** Find a matrix with the given eigenvectors.



**11.40** Find a matrix with the given eigenvectors.



**11.41** Find a matrix with the given eigenvectors.



**11.42** Find a matrix with the given eigenvectors.



## Chapter 12

### Bases of Eigenvectors

In this chapter, we try (and don't always succeed) to organize eigenvectors into bases.

#### Eigenspaces

The  $\lambda$ -*eigenspace* of a square matrix  $A$  is the set of vectors  $x$  for which  $(A - \lambda I)x = 0$  (i.e. the kernel of  $A - \lambda I$ ). The eigenvectors are precisely the *nonzero* vectors in the eigenspace. In particular, if  $\lambda$  is not an eigenvalue, then the  $\lambda$ -eigenspace is just the 0 vector.

**12.1** Prove that for any value  $\lambda$ , the  $\lambda$ -eigenspace of any square matrix is a subspace.

#### Review problems

**12.2** Suppose that  $A$  and  $B$  are  $n \times n$  matrices, and  $AB = BA$ . Prove that if  $x$  is in the  $\lambda$ -eigenspace of  $A$ , then so is  $Bx$ .

#### Bases of Eigenvectors

Diagonal matrices are very easy to work with:

$$\begin{pmatrix} 2 & \\ & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix},$$

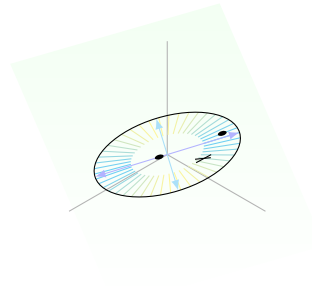


Figure 12.1: An eigenspace with eigenvalue  $\lambda = 2$ : anything you draw in that subspace get doubled.

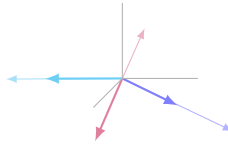


Figure 12.2: A basis of eigenvectors of a matrix. Each vector starts off as a thickly drawn vector, and gets stretched into the thinly drawn vector. A negative stretching factor reverses the direction of the vector. We can recover the entire matrix  $A$  from this picture: if we know the directions of the basis vectors that are stretched and how much the matrix stretches vectors in each of those directions, we can find the matrix.

each variable simply getting scaled by a factor. The next easiest are matrices that become diagonal when we change variables.

**Theorem 12.1** (Decoupling Theorem). *If  $u_1, u_2, \dots, u_n$  is a basis of  $\mathbb{R}^n$ , and each of  $u_1, u_2, \dots, u_n$  is an eigenvector of a square matrix  $A$ , say  $Au_1 = \lambda_1 u_1, Au_2 = \lambda_2 u_2, \dots, Au_n = \lambda_n u_n$ , then*

$$F^{-1}AF = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

where

$$F = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}$$

is the change of basis matrix of the basis  $u_1, u_2, \dots, u_n$ .

The matrix  $F$  (or the basis  $u_1, u_2, \dots, u_n$ ) diagonalizes the matrix  $A$ .

We call this the *decoupling theorem*, because the transformation taking  $x$  to  $Ax$  is usually very complicated, mixing up the variables  $x_1, x_2, \dots, x_n$  in a tangled mess. But if we can somehow change the variables and make  $A$  into a diagonal matrix, then each of the new variables is just being stretched or squished by a factor  $\lambda_i$ , independently of any of the other variables, so the variables appear “decoupled” from one another.

*Proof.*  $F$  takes  $e$ 's to  $u$ 's,  $A$  scales the  $u$ 's, and then  $F^{-1}$  turns the scaled  $u$ 's back into  $e$ 's. So  $F^{-1}AF e_j = \lambda_j e_j$ , giving the  $j$ -th column of  $F^{-1}AF$ . So

$$F^{-1}AF = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

□

### 12.3 Diagonalize

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}.$$

We save a lot of time if we notice that:

**Theorem 12.2.** *Eigenvectors with different eigenvalues are linearly independent.*

This saves us time because we don't have to check to see if the eigenvectors we come up with are linearly independent, since we generate a basis for each eigenspace, and there are no relations between eigenspaces.

*Proof.* Take a square matrix  $A$ . Pick some eigenvectors, say  $x_1$  with eigenvalue  $\lambda_1$ ,  $x_2$  with eigenvalue  $\lambda_2$ , etc., up to some  $x_p$ . Suppose that all of these eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_p$  are different from one another. If we found a linear relation  $c_1 x_1 = 0$  involving just one vector  $x_1$ , we would divide by  $c_1$  to see that  $x_1 = 0$ . But  $x_1 \neq 0$  (being an eigenvector), so there is no linear relation involving just one eigenvector. Lets suppose we found a linear relation involving just two eigenvectors,  $x_1$  and  $x_2$ , like  $c_1 x_1 + c_2 x_2 = 0$ . We could just replace  $x_1$  by  $c_1 x_1$  and  $x_2$  by  $c_2 x_2$  to arrange a linear relation  $x_1 + x_2 = 0$ . Since  $x_1$  is an eigenvector with eigenvalue  $\lambda_1$ , we know that  $(A - \lambda_1 I) x_1 = 0$ . Apply  $A - \lambda_1 I$  to both sides of our relation to get  $(\lambda_2 - \lambda_1) x_2 = 0$ . Since the eigenvalues are distinct, we can divide by  $\lambda_2 - \lambda_1$  to get  $x_2 = 0$ , again a contradiction. So there are no linear relations involving just two eigenvectors.

Lets imagine a linear relation

$$c_1 x_1 + c_2 x_2 + \dots + c_p x_p = 0,$$

involving any number of eigenvectors, and see why that leads us into a contradiction. If any of the terms are 0, just drop them, so we can assume that there are no 0 terms, i.e. that all coefficients  $c_1, c_2, \dots, c_p$  are nonzero. So we can rescale, replacing  $x_1$  by  $c_1 x_1$ , etc., to arrange that our relation is now

$$x_1 + x_2 + \dots + x_p = 0.$$

Applying  $A - \lambda_1 I$  to our linear relation:

$$\begin{aligned} 0 &= (A - \lambda_1 I)(x_1 + x_2 + \dots + x_p) \\ &= (\lambda_2 - \lambda_1)x_2 + \dots + (\lambda_p - \lambda_1)x_p \end{aligned}$$

a linear relation with fewer terms. Since  $\lambda_1 \neq \lambda_2$ , the coefficient of  $x_2$  won't become 0 in the new linear relation unless it was already 0, so this new linear relation still has nonzero terms. In this way, each linear relation leads to a linear relation with fewer terms, until we get down to one or two terms, which we already saw can't happen. Therefore there are no linear relations among  $x_1, x_2, \dots, x_n$ .  $\square$

**12.4** Diagonalize

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}.$$

**12.5** Diagonalize

$$A = \begin{pmatrix} -1 & 3 & -3 \\ -3 & 5 & -3 \\ -6 & 6 & -4 \end{pmatrix}.$$

**12.6** Prove that a square matrix is diagonalizable (i.e. diagonalized by some matrix) just when it has a basis of eigenvectors.

So  $F$  diagonalizes  $A$  just when the change of coordinates  $y = F^{-1}x$  changes the matrix  $A$  into a diagonal matrix.

**12.7** Give an example of a matrix which is *not* diagonalizable.

**12.8** If  $A$  is diagonalized by  $F$ , say  $F^{-1}AF = \Lambda$  diagonal, then prove that  $A^2$  is also diagonalized by  $F$ . Apply induction to prove that all powers of  $A$  are diagonalized by  $F$ .

**12.9** Use the result of the previous exercise to compute  $A^{100000}$  where

$$A = \begin{pmatrix} -3 & -2 \\ 4 & 3 \end{pmatrix}.$$

**Review problems**

**12.10** Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}.$$

**12.11** Find the matrix  $A$  which has eigenvalues 1 and 3 and corresponding eigenvectors

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix}.$$

**12.12** Imagine that a quarter of all people who are healthy become sick each month, and a quarter die. Imagine that a quarter of all people who are sick die each month, and a quarter become healthy. What happens to the dead people? Write a matrix  $A$  to show how the numbers  $h_n, s_n, d_n$  of healthy, sick and dead people change from month  $n$  to month  $n + 1$ . Diagonalize  $A$ . What happens to the population in the long run? Prove your answer. (Keep in mind that no one is being born in this story.)



**12.13** Let  $A$  be the  $5 \times 5$  matrix all of whose entries are 1.

- Without any calculation, what is the kernel of  $A$ ?
- Use this to diagonalize  $A$ .

**12.14** Lets investigate which  $2 \times 2$  matrices are diagonalizable.

- Prove that every  $2 \times 2$  matrix  $A$  can be written uniquely as

$$A = \begin{pmatrix} p + q & r + s \\ r - s & p - q \end{pmatrix}$$

for some numbers  $p, q, r, s$ .

- Prove that the characteristic polynomial of  $A$  is  $(p - \lambda)^2 + s^2 - q^2 - r^2$ .
- Prove that  $A$  has two different eigenvalues just when  $q^2 + r^2 > s^2$ .
- Prove that any  $2 \times 2$  matrix with two different eigenvalues is diagonalizable.
- Prove that any  $2 \times 2$  matrix with only one eigenvalue is diagonalizable just when it is diagonal.
- Prove that any  $2 \times 2$  matrix with no eigenvalues is not diagonalizable.

**12.15** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.16** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.17** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 3 & -3 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.18** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.19** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} 6 & 8 \\ -4 & -6 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.20** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} -3 & 0 \\ -2 & -1 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.21** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.22** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & -2 \\ -1 & -1 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.23** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.24** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.25** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

**12.26** Find all real eigenvalues of the matrix

$$A = \begin{pmatrix} -\frac{11}{3} & \frac{4}{3} \\ -\frac{4}{3} & -\frac{1}{3} \end{pmatrix}.$$

For each eigenvalue, find a basis of its eigenvectors.

## Summary

Linear algebra has two problems:

- Solving linear equations  $Ax = b$  for the unknown  $x$ . This problem is truly linear. It has a solution  $x$  whenever  $b$  lies in the image, and the solution  $x$  is unique up to adding on vectors from the kernel.
- Find eigenvectors and eigenvalues  $Ax = \lambda x$ . This problem is nonlinear, in fact quadratic, since  $\lambda$  and  $x$  are multiplied by one another. The nonlinear part is finding the eigenvalues  $\lambda$ , which are the roots of the characteristic polynomial  $\det(A - \lambda I)$ . There is an eigenspace of solutions  $x$  for each  $\lambda$ , and finding a basis of each eigenspace is a linear problem. If we get lucky (which *doesn't always happen*), then the eigenvectors might form a basis of  $\mathbb{R}^n$ , diagonalizing  $A$ .

Table 12.1: Invertibility criteria (Strang's nutshell [5]).  $A$  is  $n \times n$ .  $U$  is any matrix obtained from  $A$  by forward elimination.

§	Invertible Just When . . .
5	Gauss–Jordan on $A$ yields $I$ .
5	$U$ is invertible.
5	Pivots lie all the way down the diagonal.
5	$U$ has no zero rows
5	$U$ has $n$ pivots.
5	$Ax = b$ has a solution $x$ for each $b$ .
5	$Ax = b$ has exactly <i>one</i> solution $x$ for <i>each</i> $b$ .
5	$Ax = b$ has exactly <i>one</i> solution $x$ for <i>some</i> $b$ .
5	$Ax = 0$ only for $x = 0$ .
5	$A$ has rank $n$ .
7	$A^t$ is invertible.
7	$\det A \neq 0$ .
9	The columns are linearly independent.
9	The columns form a basis.
9	The rows form a basis.
10	The kernel of $A$ is just the 0 vector.
10	The image of $A$ is all of $\mathbb{R}^n$ .
11	0 is not an eigenvalue of $A$ .

**12.27** Take each of the criteria in table 12.1, and describe an analogous criterion for showing that  $A$  is *not* invertible. For example, instead of  $\det A \neq 0$ , you would write  $\det A = 0$ . Make sure that as many as possible of your criteria express the failure of invertibility in terms of the rank  $r$  of the matrix  $A$ . For

example, instead of turning

$U$  has no zero rows

into

$U$  has a zero row,

you should turn it into

$U$  has  $n - r$  zero rows.

# Orthogonal Linear Algebra



## Chapter 13

### Inner Product

So far, we haven't thought about distances or angles. The elegant algebraic way to describe these geometric notions is in terms of the *inner product*, which measures something like how strongly in agreement two vectors are.

#### Definition and Simplest Properties

The *inner product* (also called the *dot product* or *scalar product*) of two vectors  $x$  and  $y$  in  $\mathbb{R}^n$  is the number

$$\langle x, y \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

The vectors

$$x = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad y = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix},$$

have inner product

$$\begin{aligned} \langle x, y \rangle &= (1)(4) + (0)(5) + (2)(6) \\ &= 16. \end{aligned}$$

In  $\mathbb{R}^n$ ,

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

**13.1** Prove that  $\langle Ae_j, e_i \rangle = A_{ij}$ .

**13.2** Let  $P$  be a permutation matrix. Use the result of problem 13.1 to prove that  $P^{-1} = P^t$ .

Recall that the transpose  $A^t$  of a matrix  $A$  is the matrix with entries  $A^t_{ij} = A_{ji}$ , i.e. with rows and columns switched.

**13.3** Prove that  $\langle x, y \rangle = x^t y$ .

Vectors  $u$  and  $v$  are *perpendicular* if  $\langle u, v \rangle = 0$ . The *length* of a vector  $x$  in  $\mathbb{R}^n$  is  $\|x\| = \sqrt{\langle x, x \rangle}$ . This agrees in the plane with the Pythagorean theorem:

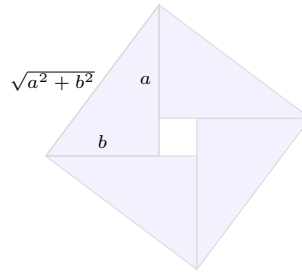


Figure 13.1: The Pythagorean theorem. Rearrange the 4 triangles into 2 rectangles to find the area of all 4 triangles. Add the area of the small white square.

if  $x = \begin{pmatrix} a \\ b \end{pmatrix}$  then we can draw  $x$  as a point of the plane, and the length along  $x$  is  $\sqrt{a^2 + b^2}$ .

**13.4** Prove that for any vectors  $u$  and  $v$ , with  $u \neq 0$ , the vector

$$v - \frac{\langle v, u \rangle}{\langle u, u \rangle} u$$

is perpendicular to  $u$ .

### Review problems

**13.5** How many vectors  $x$  in  $\mathbb{R}^n$  have integer coordinates  $x_1, x_2, \dots, x_n$  and have

- (a)  $\|x\| = 0$ ?
- (b)  $\|x\| = 1$ ?
- (c)  $\|x\| = 2$ ?
- (d)  $\|x\| = 3$ ?

**13.6** a. What is wrong with the clock?



b. At what times of day are the minute and hour hands of a properly functioning clock

- a) perpendicular?
- b) parallel?

(The answer isn't very pretty.)

**13.7** Prove that  $\langle Ax, y \rangle = \langle x, A^t y \rangle$  for vectors  $x$  in  $\mathbb{R}^q$ ,  $y$  in  $\mathbb{R}^p$  and  $A$  any  $p \times q$  matrix.



## Symmetric Matrices

A matrix  $A$  is *symmetric* if  $A^t = A$ .

**13.8** Which of the following are symmetric?

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

**13.9** Prove that an  $n \times n$  matrix is symmetric just when

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

for  $x$  and  $y$  any vectors in  $\mathbb{R}^n$ .

Clearly a symmetric matrix is square.

**13.10** Prove that

- The sum and difference of symmetric matrices is symmetric.
- If  $A$  is a symmetric matrix, then  $3A$  is also a symmetric matrix.

**13.11** Give an example of a pair of symmetric  $2 \times 2$  matrices  $A$  and  $B$  for which  $AB$  is *not* symmetric.

## Review problems

**13.12** For which matrices  $A$  is the matrix

$$B = \begin{pmatrix} 1 & A \\ A & 1 \end{pmatrix}$$

symmetric?

**13.13** If  $A$  is symmetric, and  $F$  an invertible matrix, is  $FAF^{-1}$  symmetric? If not, can you give a  $2 \times 2$  example?

**13.14** If  $A$  and  $B$  are symmetric, is  $AB + BA$  symmetric? Is  $AB - BA$  symmetric?

## Orthogonal Matrices

A matrix  $F$  is *orthogonal* if  $F^t F = I$ . In problem 13.2, you proved that permutation matrices are orthogonal.

**13.15** Which of the following are orthogonal?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Orthogonal matrices are important because they preserve inner products:

**13.16** Prove that a matrix is orthogonal just when

$$\langle Fx, Fy \rangle = \langle x, y \rangle$$

for  $x$  and  $y$  any vectors.

Clearly any orthogonal matrix is square.

**13.17** Prove that

- The product of orthogonal matrices is orthogonal.
- The inverse of an orthogonal matrix is orthogonal.

**13.18** If  $F$  is orthogonal, and  $c$  is a real number, prove that  $cF$  is also orthogonal only when  $c = \pm 1$ .

**13.19** Which diagonal matrices are orthogonal?

**13.20** Prove that the matrices

$$P = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

are orthogonal. Give an example of an orthogonal  $2 \times 2$  matrix *not* of this form.

**13.21** Give an example of a pair of orthogonal  $2 \times 2$  matrices  $A$  and  $B$  for which  $A + B$  is *not* orthogonal.

**13.22** By expanding out the expressions  $\langle x + y, x + y \rangle$  using the properties of inner products, express the inner product  $\langle x, y \rangle$  of two vectors in terms of their lengths. Use this to prove that a matrix  $A$  is orthogonal just when

$$\|Ax\| = \|x\|,$$

for any vector  $x$ .

### Orthonormal Bases

Some bases are much easier to use than others. A basis  $u_1, u_2, \dots, u_n$  is *orthonormal* if

$$\langle u_i, u_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The standard basis is orthonormal. The basis

$$u_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \quad u_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

is orthonormal.

### Why Are Orthonormal Bases Better Than Other Bases?

Take any basis  $u_1, u_2, \dots, u_n$  for  $\mathbb{R}^n$ . Every vector  $x$  in  $\mathbb{R}^n$  can be written as a linear combination

$$x = c_1 u_1 + c_2 u_2 + \dots + c_n u_n.$$

How do you find the coefficients  $c_1, c_2, \dots, c_n$ ? You apply elimination to the matrix

$$\begin{pmatrix} u_1 & u_2 & \dots & u_n & x \end{pmatrix}.$$

This is a big job. But if the basis is orthonormal then you can just read off the coefficients as

$$c_1 = \langle x, u_1 \rangle, c_2 = \langle x, u_2 \rangle, \dots, c_n = \langle x, u_n \rangle.$$

**13.23** Prove that if  $u_1, u_2, \dots, u_n$  is an orthonormal basis for  $\mathbb{R}^n$  and  $x$  is any vector in  $\mathbb{R}^n$  then

$$x = \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2 + \dots + \langle x, u_n \rangle u_n.$$

### How Do We Tell If a Basis is Orthonormal?

**Proposition 13.1.** *A square matrix is orthogonal just when its columns are orthonormal.*

*Proof.* Write the matrix as

$$F = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}.$$

Calculate

$$\begin{aligned} F^t F &= \begin{pmatrix} u_1^t \\ u_2^t \\ \vdots \\ u_n^t \end{pmatrix} \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \\ &= \begin{pmatrix} u_1^t u_1 & u_1^t u_2 & \dots & u_1^t u_n \\ u_2^t u_1 & u_2^t u_2 & \dots & u_2^t u_n \\ \vdots & \vdots & \vdots & \vdots \\ u_n^t u_1 & u_n^t u_2 & \dots & u_n^t u_n \end{pmatrix} \\ &= \begin{pmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \dots & \langle u_1, u_n \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \dots & \langle u_2, u_n \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_n, u_1 \rangle & \langle u_n, u_2 \rangle & \dots & \langle u_n, u_n \rangle \end{pmatrix}. \end{aligned}$$

□

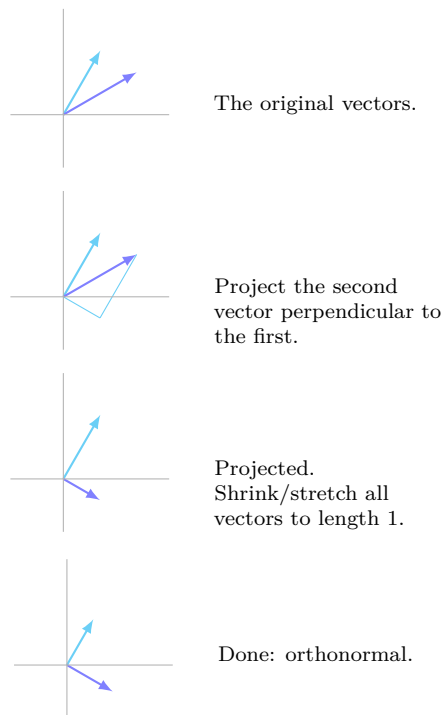


Figure 13.2: The Gram-Schmidt process

**13.24** Is the basis

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

orthonormal? Draw a picture of these two vectors.

**13.25** Prove that a square matrix is orthogonal just when its rows are orthonormal.

### Gram-Schmidt Orthogonalization

The idea: if I start with a basis  $v_1, v_2$  of  $\mathbb{R}^2$  which is *not* orthonormal, I can fix it up (as in figure 13.2).

The formal definition: given any linearly independent vectors  $v_1, v_2, \dots, v_p$

(as input), the output are orthonormal vectors  $u_1, u_2, \dots, u_p$ :

$$\begin{aligned} w_1 &= v_1, \\ w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1, \\ w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2, \\ w_j &= v_j - \sum_{i < j} \frac{\langle v_j, w_i \rangle}{\langle w_i, w_i \rangle} w_i, \\ u_j &= \frac{1}{\sqrt{\langle w_j, w_j \rangle}} w_j. \end{aligned}$$

Each  $w_j$  is just  $v_j$  with all parts “pulled off” that head in the directions of previous  $w_i$ ’s (the directions we are already finished with). At the final step, each  $u_j$  is just  $w_j$  rescaled to unit length. We say that we are orthogonalizing the vectors  $v_1, v_2, \dots, v_p$ .

**13.26** Orthogonalize

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}.$$

**13.27** Orthogonalize

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

**13.28** Orthogonalize

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and then draw pictures explaining the process.

**13.29** Orthogonalize

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

**13.30** Prove that if  $v_1, v_2, \dots, v_p$  are linearly independent vectors, then each step of Gram–Schmidt makes sense (no dividing by zero), and the resulting  $u_1, u_2, \dots, u_p$  are an orthonormal basis for the span of  $v_1, v_2, \dots, v_p$ .

**13.31** Prove that any set of vectors, all of unit length, and perpendicular to one another, is contained in an orthonormal basis.

**13.32** If  $u$  and  $v$  are two vectors in  $\mathbb{R}^n$ , and every vector  $w$  which is perpendicular to  $u$  is perpendicular to  $v$ , then  $v = au$  for some number  $a$ .

### Review problems

**13.33** What happens to a basis when you carry out Gram–Schmidt, if it was already orthonormal to begin with?

**13.34** Orthogonalize

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

**13.35** Orthogonalize

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

**13.36** Orthogonalize

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

**13.37** Orthogonalize

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

**13.38** Orthogonalize

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

**13.39** Orthogonalize

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

**13.40** Orthogonalize

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

**13.41** Orthogonalize

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

**13.42** Orthogonalize

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

**13.43** Orthogonalize

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

**13.44** Orthogonalize

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

**13.45** Orthogonalize

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

**13.46** Orthogonalize

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$





## Chapter 14

### The Spectral Theorem

Finally, our goal: we want to prove that symmetric matrices can be made into diagonal matrices by orthogonal changes of variable.

#### Statement and Proof

**Proposition 14.1** (The Minimum Principle). *Let  $A$  be a symmetric  $n \times n$  matrix. The function*

$$Q(x) = \langle Ax, x \rangle$$

*is a quadratic polynomial function. Restrict  $x$  to lie on the sphere of unit length vectors. Then  $Q(x)$  reaches a minimum among all vectors on that sphere at some vector  $x = u$ . This vector  $u$  is an eigenvector of  $A$ .*

*Proof.* In the appendix to this chapter, we prove that the minimum occurs. So there is some vector  $x = u$  so that  $\langle Ax, x \rangle \geq \langle Au, u \rangle$  for any  $x$  of unit length. Fixing  $u$ , consider the quadratic function

$$H(x) = \langle Ax, x \rangle - \langle Au, u \rangle \langle x, x \rangle.$$

For  $x$  of unit length,  $\langle x, x \rangle = 1$ , so

$$\begin{aligned} H(x) &= \langle Ax, x \rangle - \langle Au, u \rangle \\ &\geq 0. \end{aligned}$$

But if we scale  $x$ , say to  $ax$ , clearly  $H(x)$  is quadratic in  $x$ , so

$$H(ax) = a^2 H(x).$$

So rescaling, we find that  $H(x) \geq 0$  for any vector  $x$ , of any length. Pick  $w$  any vector perpendicular to  $u$ . For any number  $t$ :

$$\begin{aligned} 0 &\leq H(u + tw) \\ &= \langle A(u + tw), u + tw \rangle - \langle Au, u \rangle \langle u + tw, u + tw \rangle \\ &= \langle Au, u \rangle + 2t \langle Au, w \rangle + t^2 \langle Aw, w \rangle - \langle Au, u \rangle (1 + t^2 \langle w, w \rangle) \\ &= 2t \langle Au, w \rangle + t^2 H(w) \\ &= t (2 \langle Au, w \rangle + tH(w)). \end{aligned}$$

First lets try  $t$  positive, so we can divide by  $t$ , and find  $0 \leq 2 \langle Au, w \rangle + tH(w)$ . Let  $t$  to go to zero, to see that  $0 \leq \langle Au, w \rangle$ . Next try  $t$  negative, and divide by  $t$  and then let  $t$  go to zero, and see that  $0 \geq \langle Au, w \rangle$ . Therefore  $\langle Au, w \rangle = 0$ . So every vector  $w$  perpendicular to  $u$  is also perpendicular to  $Au$ . By problem 13.32 on page 120,  $Au$  is a multiple of  $u$ , so  $u$  is an eigenvector.  $\square$

**14.1** If two eigenvectors of a symmetric matrix have different eigenvalues, prove that they are perpendicular.

**Theorem 14.2** (Spectral Theorem). *Each symmetric matrix  $A$  is diagonalized by an orthogonal matrix  $F$ . The columns of  $F$  form an orthonormal basis of eigenvectors. We say that  $F$  orthogonally diagonalizes  $A$ .*

*Proof.* Start with a unit eigenvector  $u_1$ , given by the minimum principle. Take any orthonormal basis  $u_1, u_2, \dots, u_n$  that starts with this vector, and let  $F$  be the matrix with these vectors as columns. Replace  $A$  with  $F^t A F$ . After replacement,  $A$  has  $e_1$  as eigenvector:  $Ae_1 = \lambda e_1$ , so the first column of  $A$  is  $\lambda e_1$ . Because  $A$  is symmetric, we see that

$$A = \begin{pmatrix} \lambda I & 0 \\ 0 & B \end{pmatrix}$$

with  $B$  a smaller symmetric matrix. By induction on the size of matrix, we can orthogonally diagonalize  $B$ .  $\square$

The previous exercise is vital for calculations: to orthogonally diagonalize, find all eigenvalues, and for each eigenvalue  $\lambda$  find an orthonormal basis  $u_1, u_2, \dots$  of eigenvectors of that eigenvalue  $\lambda$ . All of the eigenvectors of all of the other eigenvalues will automatically be perpendicular to  $u_1, u_2, \dots$ , so put together they make an orthonormal basis of  $\mathbb{R}^n$ .

**14.2** Find a matrix  $F$  which orthogonally diagonalizes the matrix

$$A = \begin{pmatrix} 7 & -6 \\ -6 & 12 \end{pmatrix},$$

by finding the eigenvectors  $u_1, u_2$  and eigenvalues  $\lambda_1, \lambda_2$ .

**14.3** Prove that a square matrix is orthogonally diagonalizable just when it is symmetric.

An elegant (but longer) proof of the spectral theorem can be made along the following lines. Once we have used the minimum principle to find one eigenvector  $u_1$ , we can then look among all unit length vectors  $x$  perpendicular to  $u_1$ , and see which of these vectors has smallest value for  $\langle Ax, x \rangle$ . Call that vector  $u_2$ . Look among all unit vectors  $x$  which are perpendicular to both  $u_1$  and  $u_2$  for one which has the smallest value of  $\langle Ax, x \rangle$ , and call it  $u_3$ , etc. This recipe will actually generate the eigenvectors for us, although it isn't easy to use either by hand or by computer.

## Review problems

14.4 Find a matrix  $F$  which orthogonally diagonalizes the matrix

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

14.5 Find a matrix  $F$  which orthogonally diagonalizes the matrix

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

14.6 Find a matrix  $F$  which orthogonally diagonalizes the matrix

$$A = \begin{pmatrix} \frac{7}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{7}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{5}{3} \end{pmatrix}$$

14.7 Find a matrix  $F$  which orthogonally diagonalizes

$$A = \begin{pmatrix} -\frac{7}{25} & 0 & \frac{24}{25} \\ 0 & 1 & 0 \\ \frac{24}{25} & 0 & \frac{7}{25} \end{pmatrix}$$

14.8 Let

$$A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$

What are *all* of the orthogonal matrices  $F$  for which  $F^t A F$  is diagonal with entries increasing as we move down the diagonal?

14.9 If  $A$  is symmetric, prove that  $A^2$  has the same rank as  $A$ .

14.10 If  $A$  is  $n \times n$ , prove that  $A^t A$  has no negative eigenvalues, and its eigenvalues are all positive just when  $A$  is invertible.

## Quadratic Forms

A *quadratic form* is a polynomial in several variables, with all terms being quadratic (like  $x^2$  or  $xy$ ). In particular, no linear or constant terms can appear in a quadratic form.

### Symmetrizing

If we have a quadratic form in variables  $x_1$  and  $x_2$ , we can write it more symmetrically; for example:

$$x_1x_2 = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_1.$$

We just leave alone a term like  $x_1^2$ , so for example

$$x_1^2 + 8x_1x_2 = x_1^2 + 4x_1x_2 + 4x_2x_1.$$

**14.11** Symmetrize:

- $x_2^2$
- $x_1^2 + x_2^2$
- $x_1^2 + 3x_1x_2$
- $x_1(x_1 + x_2)$

### Making a matrix

Pluck out the quadratic terms in the polynomial to make a matrix. For example:

$$ax_1^2 + bx_1x_2 + cx_2^2 = ax_1^2 + \frac{b}{2}x_1x_2 + \frac{b}{2}x_2x_1 + cx_2^2$$

becomes

$$A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}.$$

More generally,  $\sum_{ij} A_{ij}x_ix_j$  becomes  $A = (A_{ij})$ . Because we symmetrized, the matrix is symmetric.

**14.12** Make matrices for

- $x_2^2$
- $x_1^2 + x_2^2$
- $x_1^2 + \frac{3}{2}x_1x_2 + \frac{3}{2}x_2x_1$
- $x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_1$

### Diagonalizing

Diagonalize our matrix, by orthogonal change of variables. Then the same orthogonal change of variables will simplify our quadratic form, turning it into a sum of quadratic forms in one variable each. For example, take the quadratic form

$$23x_1^2 + 72x_1x_2 + 2x_2^2.$$

Symmetrize:

$$23x_1^2 + 36x_1x_2 + 36x_2x_1 + 2x_2^2.$$

The associated matrix is

$$A = \begin{pmatrix} 23 & 36 \\ 36 & 2 \end{pmatrix}.$$

We let the reader check that  $A$  is orthogonally diagonalized by

$$F = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix},$$

so that

$$F^t A F = \begin{pmatrix} -25 & 0 \\ 0 & 50 \end{pmatrix}.$$

We also let the reader check that if we take new variables  $y$ , defined by  $y = F^t x$ , i.e. by  $x = Fy$ , then the same quadratic form is

$$-25 y_1^2 + 50 y_2^2.$$

**Theorem 14.3** (Decoupling Theorem). *Any quadratic form in any number of variables becomes a sum of quadratic forms in one variable each, after a change of variables  $x = Fy$  given by an orthogonal matrix  $F$ .*

The quadratic form is *diagonalized* by the orthogonal matrix.

*Proof.* The problem comes from the mixed terms, like  $x_1 x_2$ . Symmetrize and write a symmetric matrix  $A$  out of the coefficients. Then the quadratic form is  $\sum_{ij} A_{ij} x_i x_j = \langle Ax, x \rangle$ . Diagonalize  $A$  to  $\Lambda = F^t A F$ . Let  $y = F^t x$ . Then  $x = Fy$ , so

$$\begin{aligned} \langle Ax, x \rangle &= \langle AFy, Fy \rangle \\ &= \langle F^t AFy, y \rangle \\ &= \langle \Lambda y, y \rangle \\ &= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2. \end{aligned}$$

□

## Review problems

### 14.13 Diagonalize

- (a)  $3x_1^2 - 2x_1x_2 + 3x_2^2$
- (b)  $9x_1^2 + 18x_1x_2 + 9x_2^2$
- (c)  $11x_1^2 + 6x_1x_2 + 3x_2^2$
- (d)  $3x_1^2 + 4x_1x_2 + 6x_2^2$
- (e)  $8x_1^2 - 12x_1x_2 + 3x_2^2$
- (f)  $10x_1^2 + 6x_1x_2 + 2x_2^2$
- (g)  $4x_1x_2 + 3x_2^2$
- (h)  $6x_1^2 - 12x_1x_2 + 11x_2^2$
- (i)  $10x_1^2 - 12x_1x_2 + 5x_2^2$
- (j)  $-2x_1^2 - 4x_1x_2 + x_2^2$
- (k)  $2x_1^2 - 6x_1x_2 + 10x_2^2$
- (l)  $-2x_1^2 + 6x_1x_2 + 6x_2^2$

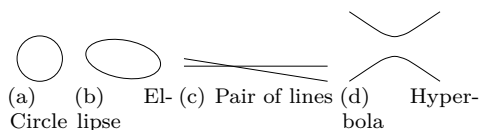


Figure 14.1: Some examples of solutions of quadratic equations

### Application to Quadratic Equations

In the plane, with two variables  $x_1$  and  $x_2$ , a quadratic equation  $Q(x) = c$  (with  $Q(x)$  a quadratic form and  $c$  a constant number) cuts out a circle, ellipse, hyperbola, pair of lines, single line, point, or empty set. The quadratic equation

$$4x_1x_2 + 3x_2^2 = 0$$

involves the quadratic form with matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}.$$

Its eigenvalues are  $\lambda = -1$  and  $\lambda = 4$ . So we can change variables (somehow) to get to

$$-x_1^2 + 4x_2^2 = 0.$$

This is just

$$x_1 = \pm 2x_2,$$

a pair of lines intersecting at a point. Since the change of variables is linear, the original quadratic equation also cuts out a pair of lines intersecting at a point.

The equation

$$x_1^2 + 4x_1x_2 + x_2^2 = 1$$

contains the quadratic form with matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The eigenvalues are  $\lambda = -1$  and  $\lambda = 3$ . So after a linear change of variables, we get

$$-x_1^2 + 3x_2^2 = 1.$$

(The right hand side is a constant, so doesn't change.)

It is well known that an equation of the form

$$ax_1^2 + bx_2^2 = 1$$

with  $a$  and  $b$  of different signs is a hyperbola, while if  $a$  and  $b$  have the same signs then it is an ellipse. So our last example must be a hyperbola. Warning: until you diagonalize the associated matrix, and look at the eigenvalues, you can't easily see what shape a quadratic equation cuts out. You can't just look at whether the coefficients are positive, or anything obvious like that.

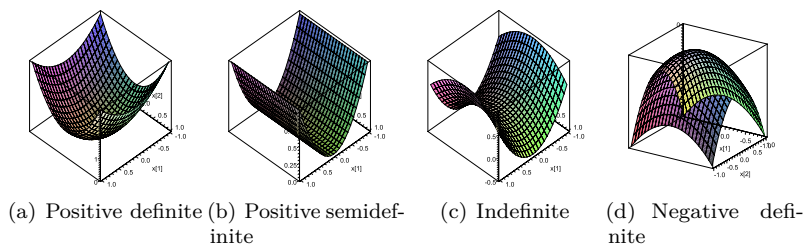


Figure 14.2: Various behaviours of quadratic forms in two variables

## Review problems

**14.14** Just by finding eigenvalues (without finding eigenvectors), determine what geometric shape (ellipse, hyperbola, pair of lines, line, empty set) the following are:

- $5x_1^2 + 2x_2^2 - 4x_1x_2 = 1$
- $3x_1^2 + 8x_1x_2 + 3x_2^2 = 1$
- $x_1^2 - 4x_1x_2 + 4x_2^2 = -1$
- $6x_1x_2 + 8x_2^2 = 1$
- $x_1^2 - 6x_1x_2 + 9x_2^2 = 0$
- $4x_1x_2 - 3x_1^2 - 6x_2^2 = 1$

**14.15** What more can you do to normalize a quadratic form if you allow arbitrary invertible matrices instead of orthogonal ones?

## Positivity

A quadratic form  $Q(x)$  is *positive definite* if  $Q(x) > 0$  except if  $x = 0$ . (Clearly if  $x = 0$  then  $Q(x) = 0$ .) For example,  $Q(x) = x^2$  is a positive definite quadratic form on  $\mathbb{R}$ , while  $Q(x) = x_1^2 + x_2^2$  is positive definite on  $\mathbb{R}^n$ . But it is *not* at all clear whether  $Q(x) = 6x_1^2 - 12x_1x_2 + 11x_2^2$  is positive definite, because it has positive terms and negative ones. As in figure 14.2, we can also define *positive semidefinite* forms ( $Q(x) \geq 0$ ), *negative definite* forms ( $Q(x) < 0$  for  $x \neq 0$ ), and *indefinite* forms (not positive semidefinite or negative semidefinite), but they are less important.

**Lemma 14.4.** *A quadratic form  $Q(x) = \langle x, Ax \rangle$  (with a symmetric matrix  $A$ ) is positive definite just if all of the eigenvalues of  $A$  are positive.*

*Proof.* Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues, and change variables to  $y = F^{-1}x$ , so  $x = Fy$ , to diagonalize the quadratic form:

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2.$$

Suppose that all of the eigenvalues are positive. Clearly this quantity is then positive for nonzero vectors  $y$ , because each term is positive or zero, and at least one of  $y_1, y_2, \dots, y_n$  is not zero, so gives a positive term.

On the other hand, if one of these  $\lambda_j$  is negative, then take  $y = e_j$ , and you get  $Q \leq 0$  but  $x = Fy = Fe_j \neq 0$ .  $\square$

## Review problems

**14.16** Which of the quadratic forms in problem 14.13 on page 127 are positive definite?

## Appendix: Continuous Functions and Maxima

There is one gap in our proof of the spectral theorem for symmetric matrices: we need to know that a quadratic function on the sphere in  $\mathbb{R}^n$  has a maximum. This appendix gives the proof. Warning: students are *not* required or expected to work through this appendix, which is advanced and is included only for completeness. A sequence of numbers  $x_1, x_2, \dots$  *converges* to a number  $x$  if, in order to make  $x_j$  stay as close as we like to  $x$ , we have only to ensure that  $j$  is kept large enough. A sequence of points

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \dots$$

in  $\mathbb{R}^2$  *converges* to a point

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

if  $x_1, x_2, \dots$  converges to  $x$  and  $y_1, y_2, \dots$  converges to  $y$ . Similarly, a sequence of points in  $\mathbb{R}^n$  converges if all of the coordinates of those points converge.

Any sequence of increasing real numbers, all of which are bounded from above by some large enough number, must converge to something. This fact is a property of real numbers which we cannot prove without giving an explicit and precise definition of the real numbers; see Spivak [4] for the complete story. We will just assume that this fact is true.

A function  $f(x)$  of any number of variables  $x_1, x_2, \dots, x_n$  (writing  $x$  for

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

as a point of  $\mathbb{R}^n$ ) is *continuous* if, in order to get  $f(y)$  to stay as close to  $f(x)$  as you like, you have only to ensure that  $y$  is kept close enough to  $x$ .

If two numbers are close, their sums, products and differences are clearly close. The reader should try to prove:

**Lemma 14.5.** *The function  $f(x) = x_1$  is continuous. Constant functions are continuous. The sum, difference and product of continuous functions is continuous.*



**Corollary 14.6.** *Any polynomial function in any finite number of variables is continuous.*

*Proof.* Induction on the degree and number of terms of the polynomial.  $\square$

A *ball* in  $\mathbb{R}^n$  is a set  $B$  consisting of all points closer than some distance to some chosen point (called the *center* of the ball). The distance is called the *radius*. A *closed ball* includes also the points of distance equal to the radius (an apple with the skin), while an *open ball* does not include any such points, only including points of distance less than the radius (an apple without the skin).

A set  $S \subset \mathbb{R}^n$  is *bounded* if it lies in a ball (open or closed).

A set  $S \subset \mathbb{R}^n$  is called *closed* if every point  $x$  of  $\mathbb{R}^n$  not belonging to  $S$  can be surrounded by an open ball not belonging to  $S$ .

A *closed box* is the set of points  $x = (x_1, \dots, x_n)$  for which each  $x_j$  lies in some chosen interval,  $a_j \leq x_j \leq b_j$ . An open box is the same but with  $a_j < x_j < b_j$ .

**Lemma 14.7.** *A closed ball is a closed set, as is a closed box.*

*Proof.* Given a closed ball, say of radius  $r$ , take any point  $p$  not belonging to it, say of distance  $R$  from the center, and draw an open ball of radius  $R - r$  about  $p$ . By the triangle inequality, no point in the open ball lies in the closed ball.

Given a closed box, and a point  $p$  not belonging to it, there must be some coordinate of  $p$  which does not satisfy the inequalities defining the closed box. For example, suppose that the box is cut out by inequalities including  $a_1 \leq x_1 \leq b_1$ , and  $p$  fails to satisfy these bounds because  $p_1 > b_1$ . Then every point  $q$  closer to  $p$  than  $p_1 - b_1$  will still fail:  $q_1 > b_1$ . So then a ball of radius  $p_1 - b_1$  around  $p$  will not overlap the closed box.  $\square$

**Theorem 14.8.** *Every infinite sequence of points in a closed, bounded set has a convergent subsequence.*

*Proof.* Suppose that the set is a box. Cover the box with a finite number of small closed boxes (perhaps overlapping). There are infinitely many points  $x_1, x_2, \dots$ , and only finitely many of the small boxes, so there must be infinitely many  $x_j$  lying in the same small box.

Similarly, subdivide that small box into much smaller closed boxes. Repeating, we find a sequence of closed boxes, like Russian dolls, each contained entirely in the previous one, with infinitely many  $x_j$  in each. We get to choose how small the boxes are going to be at each step, so let's make them get much smaller at each step, with side lengths decreasing as rapidly as we like. Pick out one of these  $x_i$  points, call it  $y_j$ , from the  $j$ -th nested box, as the point with the smallest possible coordinates among all points of that box. The sequence of points  $y_1, y_2, \dots$  must converge, since all of the coordinates of the point  $y_j$  are constrained by the box  $y_j$  lies in, and each coordinate only increases with  $j$ .

If we face a closed, bounded set  $S$ , which is *not* a closed box, then find a closed box  $B$  containing it, and repeat the argument above. The problem is to ensure that the limit  $x$  of the sequence constructed belongs to the set  $S$ . Even if not, it certainly belongs to  $B$ . Since  $S$  is closed, if  $x$  does not belong to  $S$ , then there must be an open ball around  $x$  not containing any points of  $S$ . But that open ball can not contain any of the points in the nested boxes, and therefore  $x$  cannot be their limit.  $\square$

**Theorem 14.9.** *Every continuous function  $f$  on a closed, bounded set attains a maximum and a minimum.*

*Proof.* For the moment, let's suppose that our closed, bounded set is just a closed box. Suppose that  $f$  has no maximum. So the values of  $f$  can get larger and larger, but never peak. Let  $M$  be the smallest positive number so that  $f$  never exceeds  $M$ ; if there is no such number let  $M = \infty$ . By definition,  $f$  gets as close to  $M$  as we like (which, if  $M = \infty$ , means simply that  $f$  gets as large as we like), but never reaches  $M$ . Let  $x_1, x_2, x_3, \dots$  be any points of the closed bounded set on which  $f(x_j)$  approaches  $M$ . Taking a subsequence, we find  $x_j$  approaching a limit point  $x$ , and by continuity  $f(x_j)$  must approach  $f(x)$ , so  $f(x) = M$ .

So every continuous function on a closed bounded set has a maximum. If  $f$  is a continuous function on a closed bounded set, then  $-f$  is too, and has a maximum, so  $f$  has a minimum.  $\square$

## Chapter 15

### Complex Vectors

The entire story so far can be retold with a cast of complex numbers instead of real numbers. Most of this is straightforward. But there turns out to be an important twist in the complex theory of the inner product. The minimum principle doesn't make any sense in the setting of complex numbers, and the spectral theorem as it was stated just isn't true any more for complex matrices. Moreover, the natural notion of inner product itself is quite different for complex vectors—this new notion leads directly to the complex spectral theorem.

#### Complex Numbers

A *complex number* is a pair  $(x, y)$  of real numbers. Write  $1$  to mean the pair  $(1, 0)$ , and  $i$  to mean  $(0, 1)$ . Addition is defined by the rule

$$(x, y) + (X, Y) = (x + X, y + Y),$$

subtraction by

$$(x, y) - (X, Y) = (x - X, y - Y),$$

and multiplication by

$$(x, y)(X, Y) = (xX - yY, xY + yX).$$

We henceforth write any pair as  $x + iy$ . We call  $x$  the *real part* and  $y$  the *imaginary part*. When working with complex numbers, we draw them as points of the  $xy$ -plane, which we call the *complex plane*. Complex numbers are associative, commutative and distributive, and every nonzero complex number  $z = x + iy$  has a reciprocal:

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

You can easily check all of this, but you may assume it if you prefer.

#### Polar Coordinates

Trigonometry tells us that any point  $(x, y)$  of the plane can be written as  $x = r \cos \theta, y = r \sin \theta$  in polar coordinates. Therefore

$$x + iy = r \cos \theta + ir \sin \theta,$$

The number  $r$  is called the *modulus* of the complex number (written  $|z|$  if  $z$  is the complex number). The angle  $\theta$  is called the *argument* of the complex number (written  $\arg z$  if  $z$  is the complex number). The modulus is the distance from 0, or the length if think of  $(x, y)$  as a vector.

**Theorem 15.1** (de Moivre). *Under multiplication of complex numbers, moduli multiply, while arguments add. Under division of complex numbers, moduli divide, while arguments subtract.*

*Proof.* Let  $z$  and  $w$  be two complex numbers. Write them as  $z = r \cos \alpha + ir \sin \alpha$  and  $w = \rho \cos \beta + i\rho \sin \beta$ . Then calculate

$$zw = r\rho [(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)].$$

A trigonometric identity:

$$\begin{aligned} \cos \alpha \cos \beta - \sin \alpha \sin \beta &= \cos(\alpha + \beta) \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta &= \sin(\alpha + \beta). \end{aligned}$$

Division is similar. □

**15.1** Explain why every complex number has a square root.

The *conjugate*  $\bar{z}$  of a complex number  $z = x + iy$  is the number  $\bar{z} = x - iy$ .

**15.2** If  $z$  is a complex number, prove that  $|z|^2 = z\bar{z}$

We write  $\mathbb{C}$  for the set of complex numbers, and  $\mathbb{C}^n$  for the set of vectors

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

with each of  $z_1, z_2, \dots, z_n$  a complex number. We won't go through the effort of translating the theorems above into complex linear algebra, except to say that all of the results before chapter 13 (on inner products) are still true for complex matrices, with identical proofs.

**Review problems**

**15.3** Draw the point  $z = -\frac{1}{2} + \frac{1}{3}i$  on the plane. Draw  $\bar{z}$ ,  $2z$ ,  $z^2$ ,  $\frac{1}{z}$ .

**15.4** Draw  $w = 1 + 2i$  and  $z = 2 + i$ , and draw  $z + w$ ,  $zw$ ,  $\frac{z}{w}$ .

**15.5** The *unit disk* in the complex plane is the set of complex numbers of modulus less than 1. The *unit circle* is the set of complex numbers of modulus 1. Draw the unit circle. Pick  $z$  a nonzero complex number, and consider its integer powers  $z^n$  ( $n$  ranging over the integers). Prove that either infinitely many of these powers lie inside the unit disk, or all of them lie on the unit circle.

**15.6** Let

$$z_k = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right).$$

Use de Moivre's theorem to show that  $z_k^n = 1$ . These are the so-called  $n^{\text{th}}$  roots of 1. Why are they all different for  $k = 0, 1, \dots, n - 1$ ? Draw the 3<sup>rd</sup> roots of 1 and (in another colour) the 4<sup>th</sup> roots of 1.

**15.7** With pictures and words, explain what you know about

- $z + \bar{z}$ ,
- $z^2$  if  $|z| > 1$ ,
- $\bar{z}$  if  $|z| = 1$ ,
- $zw$  if  $|z| = |w| = 1$ .

## Complex Linear Algebra

The main differences between real and complex linear algebra are (1) eigenvalues and (2) inner products. We will consider inner products soon, but first lets consider eigenvalues.

**Theorem 15.2.** *Every square complex matrix has a complex eigenvalue.*

*Proof.* The eigenvalues are the roots of the characteristic polynomial  $\det(A - \lambda I)$ ; a polynomial with complex number coefficients in a complex variable  $\lambda$ . The existence of a complex number root of any complex polynomial is proven in the appendix.  $\square$

It may be that there are not very many eigenvectors.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

as a complex matrix still only has one eigenvalue, and (up to rescaling) one eigenvector. Two linearly independent eigenvectors are just what we need to diagonalize a  $2 \times 2$  matrix. Clearly we cannot diagonalize  $A$ . So complex numbers don't resolve all of the subtleties.

**15.8** Find the (complex) eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and write down a matrix  $F$  which diagonalizes  $A$ . Moral of the story: even a matrix like  $A$ , which has only real number entries, can have complex number eigenvalues, and complex eigenvectors.

### Hermitian Inner Product

The spectral theorem for symmetric matrices breaks down:

**15.9** Prove that the symmetric complex matrix

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

is not diagonalizable.

We will find a complex spectral theorem, but with a different concept replacing symmetric matrices.

The equation  $|z|^2 = z\bar{z}$  is very important. Think of a complex number as if it were a vector in the plane:

$$\begin{aligned} z &= x + iy \\ &= \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Then  $|z|^2 = z\bar{z}$  is the squared length  $x^2 + y^2$ .

The *Hermitian inner product* of two vectors  $z$  and  $w$  in  $\mathbb{C}^n$  is the complex number

$$\langle z, w \rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n.$$

The curious bars on top of the  $w$  terms allow us to write  $\|z\|^2 = \langle z, z \rangle$ , just as we would for real vectors. Warning: the Hermitian inner product  $\langle z, w \rangle$  is a complex number, not the real number we had in inner products before.

**15.10** Compute  $\langle z, w \rangle$ ,  $\|z\|$  and  $\|w\|$  for

$$z = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad w = \begin{pmatrix} i \\ 2 + 2i \end{pmatrix},$$

**15.11** Prove that

- $\langle w, z \rangle = \overline{\langle z, w \rangle}$
- $\langle cz, w \rangle = c \langle z, w \rangle$
- $\langle z + w, u \rangle = \langle z, u \rangle + \langle w, u \rangle$
- $\langle z, z \rangle \geq 0$
- $\langle z, z \rangle = 0$  just when  $z = 0$

for  $z, w$  and  $u$  any complex vectors in  $\mathbb{C}^n$  and  $c$  and complex number.

## Adjoint of a Matrix

The *adjoint*  $A^*$  of a matrix  $A$  is the matrix whose entries are  $A_{ij}^* = \bar{A}_{ji}$  (the conjugate of the transpose). Note that  $(A^*)^* = A$ .

**15.12** Prove that

$$\langle Az, w \rangle = \langle z, A^*w \rangle$$

for any vectors  $z$  and  $w$  (if one side is defined, then they both are and they are equal).

**15.13** Prove that if some matrices  $A$  and  $B$  satisfy  $\langle Az, w \rangle = \langle z, Bw \rangle$  for any vectors  $z$  and  $w$  for which this is defined, then  $B = A^*$ .

## Self-adjoint Matrices

A complex matrix  $A$  is *self-adjoint* if  $A = A^*$ . This is the complex analogue of a symmetric matrix. Clearly self-adjoint matrices are square.

**15.14** Prove that sums and differences of self-adjoint matrices are self-adjoint, and that any real multiple of a self-adjoint matrix is self-adjoint.

The matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are self-adjoint.

**15.15** Which diagonal matrices are self-adjoint?

**15.16** Prove that the eigenvalues of a self-adjoint matrix are real numbers.

## Review problems

**15.17** A matrix  $A$  is called *skew-adjoint* if  $A^* = -A$ . Prove that a matrix  $A$  is skew-adjoint just when  $iA$  is self-adjoint, and vice versa.

## Unitary Matrices

A complex matrix  $A$  is *unitary* if  $A^* = A^{-1}$ . This is the complex analogue of an orthogonal matrix. Clearly every unitary matrix is square.

**15.18** Prove that a matrix is unitary just when

$$\langle Az, Aw \rangle = \langle z, w \rangle$$

for any vectors  $z$  and  $w$ .

**15.19** Which diagonal matrices are unitary?

**15.20** If  $A$  is a real orthogonal matrix, prove that  $A$  is also unitary.

## Review problems

**15.21** Prove that the eigenvalues of a unitary matrix are complex numbers of modulus 1.

## Orthonormal Bases

An *orthonormal basis* of  $\mathbb{C}^n$  is a complex basis  $u_1, u_2, \dots, u_n$  for which

$$\langle u_p, u_q \rangle = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

It may be helpful to use letters like  $p$  and  $q$  as subscripts, rather than  $i, j$ , to avoid confusion with the complex number  $i$ .

**15.22** Prove that any complex basis  $v_1, v_2, \dots, v_n$  determines a orthonormal basis  $u_1, u_2, \dots, u_n$  by the complex Gram-Schmidt process:

$$w_p = v_p - \sum_{q < p} \langle v_p, u_q \rangle u_q$$
$$u_p = \frac{w_p}{\|w_p\|}.$$

**15.23** Apply the complex Gram-Schmidt process to find a orthonormal basis for the basis

$$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

## Normal Matrices

A complex matrix  $A$  is *normal* if  $AA^* = A^*A$ .

**15.24** Prove that self-adjoint, skew-adjoint and unitary matrices are normal.

**15.25** Which diagonal matrices are normal?

**15.26** If  $A$  is normal, and  $c$  is a constant, prove that  $A + cI$  is also normal.

**Lemma 15.3.** *If  $A$  is normal, and  $Az = 0$  for some vector  $z$ , then  $A^*z = 0$ .*

*Proof.* Suppose that  $Az = 0$ . Then

$$\begin{aligned} 0 &= \|Az\|^2, \\ &= \langle Az, Az \rangle, \\ &= \langle z, A^*Az \rangle, \\ &= \langle z, AA^*z \rangle, \\ &= \langle A^*z, A^*z \rangle, \\ &= \|A^*z\|^2. \end{aligned}$$



So  $A^*z = 0$ . □

**Lemma 15.4.** *If  $A$  is normal, then every eigenvector  $z$  of  $A$  with eigenvalue  $\lambda$  is also an eigenvector of  $A^*$ , but with eigenvalue  $\bar{\lambda}$ .*

*Proof.* Let  $B = A - \lambda I$ . Then  $Bz = 0$ . Moreover,  $B$  is normal since  $A$  is. By the previous lemma,  $B^*z = 0$ , so  $(A^* - \bar{\lambda}I)z = 0$ . □

### The Spectral Theorem for Normal Matrices

At last, the complex version of our main theorem: normal matrices are diagonal after change of variables by a unitary matrix.

**Theorem 15.5.** *A square matrix is normal just when it is unitarily diagonalizable.*

*Proof.* Let  $A$  be unitarily diagonalizable. So  $F^*AF = \Lambda$  is diagonal, for some unitary matrix  $F$ . Then  $A = F\Lambda F^*$ , and one easily checks that  $AA^* = A^*A$ .

Let  $A$  be a normal matrix. Pick any eigenvector  $u_1$  of  $A$ , say with eigenvalue  $\lambda$ . Scale  $u_1$  to be a unit vector. Pick unit vectors  $u_2, u_3, \dots, u_n$  so that  $u_1, u_2, u_3, \dots, u_n$  is an orthonormal basis. Let  $F$  be the associated unitary change of basis matrix,

$$F = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}.$$

Replace  $A$  by  $F^*AF$ . After replacement  $A$  is still normal, and  $Ae_1 = \lambda e_1$ ; the first column of  $A$  is  $\lambda e_1$ . So

$$A = \begin{pmatrix} \lambda & B \\ 0 & C \end{pmatrix}$$

for some smaller matrices  $B$  and  $C$ . By lemma 15.4,  $A^*e_1 = \bar{\lambda}e_1$ , so the first column of  $A^*$  is  $\bar{\lambda}e_1$ , and so  $B = 0$ . Moreover,  $C$  is also normal, so by induction we can unitarily diagonalize  $C$ , and therefore  $A$ . □

**Corollary 15.6.** *Self-adjoint, skew-adjoint and unitary matrices are unitarily diagonalizable.*

**15.27** Let

$$A = \begin{pmatrix} \frac{7}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{7}{2} \end{pmatrix}.$$

- a. Is  $A$  self-adjoint or skew-adjoint?
- b. Find the eigenvalues and eigenvectors of  $A$ .
- c. Find a unitary matrix  $F$  which diagonalizes  $A$ , and unitarily diagonalize  $A$ .

## Appendix: The Fundamental Theorem of Algebra

There was one missing ingredient in the proof of the complex spectral theorem: we need to know that complex polynomials have zeros. This gap is filled in here. Warning: students are *not* required or expected to work through this appendix, which is advanced and is included only for completeness.

**Lemma 15.7.** *Take*

$$p(z) = a_0 + a_1z + \cdots + a_nz^n$$

*any nonconstant polynomial. In order to keep  $p(z)$  at as large a modulus as we like, we only have to keep  $z$  at a large enough modulus.*

*Proof.* We can assume that  $a_n \neq 0$ . Write

$$\frac{p(z)}{z^n} = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n.$$

All of the terms get as small as we like, for  $z$  of large modulus, except the last one. So for large enough  $z$ ,  $p(z)/z^n$  is close to  $a_n$ . Since  $z^n$  has large modulus,  $p(z)$  must as well.  $\square$

**Corollary 15.8.** *For any polynomial  $p(z)$ , there must be a point  $z = z_0$  at which  $p(z)$  has smallest modulus.*

*Proof.* By lemma 15.7, if we choose a large enough disk containing  $z_0$  then  $p(z)$  has large modulus at each point  $z$  around the edge of that disk. Making the disk even larger if need be, we can ensure that the modulus all around the edge is larger than at some chosen point inside the disk. By theorem 14.9 on page 132, there is a point of the disk where  $p(z)$  has minimum modulus among all points of that disk. The minimum can't be on the edge. Moreover, the modulus stays large as we move past the edge. Thus any minimum modulus point in that large disk is a minimum modulus point among all points of the plane.  $\square$

**Lemma 15.9.** *The modulus  $|p(z)|$  of any nonconstant polynomial function reaches a minimum just where  $p(z)$  reaches zero.*

*Proof.* Take any point  $z_0$ . Suppose that  $p(z_0) \neq 0$ , and let's find a reason why  $z_0$  is *not* a minimum modulus point. Replace  $p(z)$  by  $p(z - z_0)$  if needed, to arrange that  $z_0 = 0$ . Write out

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n.$$

It might happen that  $a_1 = 0$ , and maybe  $a_2$  too. So write

$$p(z) = a_0 + a_kz^k + \cdots + a_nz^n,$$

writing down only the nonzero terms, in increasing order of their power of  $z$ .

Clearly  $a_0 \neq 0$  because  $p(0) \neq 0$ . We can divide by  $a_0$  if we wish, which alters modulus only by a positive factor, so let's assume that  $a_0 = 1$ . We can rotate the  $z$  variable, and rescale it, which rotates and scales each coefficient. Thereby arrange  $a_k = -1$ , so

$$p(z) = 1 - z^k + \dots + a_n z^n.$$

Calculate

$$\begin{aligned} |p(z)|^2 &= p(z)\overline{p(z)} \\ &= 1 - z^k - \bar{z}^k + \dots, \end{aligned}$$

where the dots indicate terms involving more  $z$  and  $\bar{z}$  factors. Write  $z = r \cos \theta + ir \sin \theta$ . De Moivre's theorem gives

$$|p(z)|^2 = 1 - 2r^k \cos k\theta + r^{k+1}(\dots).$$

The error term  $(\dots)$  is some (probably very complicated) polynomial in  $r$  with (complicated) coefficients involving  $\cos \theta$  and  $\sin \theta$ . We don't need to work it out. We only need to know that it is bounded for  $z$  near enough to 0, which is clear whatever the terms involved are. For  $r > 0$  sufficiently small,

$$2 - r(\dots) > 0.$$

Multiplying by  $-r^k$ ,

$$-2r^k + r^{k+1}(\dots) < 0.$$

Therefore  $|p(z)|^2$  gets even smaller at the point  $z = r$  than at  $z = 0$ .  $\square$

**Corollary 15.10.** *Every nonconstant complex polynomial has a root.*

**Theorem 15.11** (Fundamental Theorem of Algebra). *Every nonconstant complex polynomial  $p(z)$  can be factored into linear factors. More specifically,*

$$p(z) = c(z - z_1)^{d_1} (z - z_2)^{d_2} \dots (z - z_k)^{d_k}$$

where  $c$  is a constant,  $z_1, z_2, \dots, z_k$  are the roots of  $p(z)$ , and  $d_1, d_2, \dots, d_k$  are positive integers, with sum  $d_1 + d_2 + \dots + d_k$  equal to the degree of  $p(z)$ .

*Proof.* We have a root, say  $z_1$ . Therefore  $p(z)/(z - z_1)$  is a polynomial, and we apply induction.  $\square$



## Hints

1.1.

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Swap rows 1 and 3.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to row 4.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

Swap rows 2 and 4.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Move the pivot ↘.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Add  $-(\text{row } 3)$  to row 4.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot ↘.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot →.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**1.2.**

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Swap rows 1 and 3.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to row 4.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

Swap rows 2 and 4.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Add  $-(\text{row } 3)$  to row 4.

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**1.3.**

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Swap rows 3 and 4.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



1.4.

$$\begin{aligned}x_1 &= 7 - x_4 \\x_2 &= 3 - \frac{x_4}{2} \\x_3 &= -3 + \frac{x_4}{2}.\end{aligned}$$

1.5. Forward eliminate:

$$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 4 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \end{pmatrix}$$

Swap rows 1 and 2.

$$\begin{pmatrix} 4 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 4 & 3 & 3 & 4 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to row 3.

$$\begin{pmatrix} 4 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 2 & 2 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 4 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 2 & 2 \end{pmatrix}$$

Add  $-2(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 4 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 4 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 4 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 4 & -1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Back substitute:

Scale row 2 by  $\frac{1}{2}$ .

$$\begin{pmatrix} 4 & -1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Add row 2 to row 1.

$$\begin{pmatrix} 4 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Scale row 1 by  $\frac{1}{4}$ .

$$\begin{pmatrix} 1 & 0 & \frac{3}{8} & \frac{5}{8} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = -3/8 x_3 + 5/8$$

$$x_2 = -1/2 x_3 + 1/2$$

**1.6.**

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

1.7.

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

1.10.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Swap rows 1 and 4.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 4.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

Swap rows 3 and 4.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot ↘.

$$\begin{pmatrix} \boxed{1} & 1 & 0 & 1 & 1 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{0} & 0 \end{pmatrix}$$

Move the pivot →.

$$\begin{pmatrix} \boxed{1} & 1 & 0 & 1 & 1 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{0} \end{pmatrix}$$

Move the pivot →.

$$\begin{pmatrix} \boxed{1} & 1 & 0 & 1 & 1 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**1.11.**

$$\begin{pmatrix} \boxed{1} & 3 & 2 & 6 \\ 2 & 5 & 4 & 1 \\ 3 & 8 & 6 & 7 \end{pmatrix}$$

Add  $-2(\text{row } 1)$  to row 2,  $-3(\text{row } 1)$  to row 3.

$$\begin{pmatrix} \boxed{1} & 3 & 2 & 6 \\ 0 & -1 & 0 & -11 \\ 0 & -1 & 0 & -11 \end{pmatrix}$$

Move the pivot ↘.

$$\begin{pmatrix} \boxed{1} & 3 & 2 & 6 \\ 0 & \boxed{-1} & 0 & -11 \\ 0 & -1 & 0 & -11 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 3.

$$\begin{pmatrix} \boxed{1} & 3 & 2 & 6 \\ 0 & \boxed{-1} & 0 & -11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} \boxed{1} & 3 & 2 & 6 \\ 0 & \boxed{-1} & 0 & -11 \\ 0 & 0 & \boxed{0} & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} \boxed{1} & 3 & 2 & 6 \\ 0 & \boxed{-1} & 0 & -11 \\ 0 & 0 & 0 & \boxed{0} \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} \boxed{1} & 3 & 2 & 6 \\ 0 & \boxed{-1} & 0 & -11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

### 1.12.

Scale row 3 by  $-1$ .

$$\begin{pmatrix} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Add  $-(\text{row } 3)$  to row 1.

$$\begin{pmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 1.

$$\begin{pmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**1.13.**

Scale row 2 by  $-\frac{1}{2}$ .

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 1.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Scale row 1 by  $-1$ .

$I$

**1.14.**

Scale row 2 by  $-1$ .

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

**1.15.**

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

**1.16.**  $I$

**1.17.** Forward eliminate:

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ -1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to row 2.

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 3.

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Add  $-\frac{1}{2}(\text{row } 3)$  to row 4.

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

Back substitute:

Scale row 4 by  $\frac{2}{3}$ .

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Add  $-(\text{row } 4)$  to row 1, row 4 to row 2,  $-(\text{row } 4)$  to row 3.

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Scale row 3 by  $\frac{1}{2}$ .

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Add  $-(\text{row } 3)$  to row 1.

$$\begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Add  $-(\text{row } 2)$  to row 1.

$$\begin{pmatrix} -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Scale row 1 by  $-1$ .

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

There are no solutions.

**1.18.** Forward eliminate:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 7 & 11 & 12 \\ 0 & 1 & 1 & 4 & 3 \end{pmatrix}$$

Add  $-2(\text{row } 1)$  to row 2.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 4 & 3 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 4 & 3 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & \boxed{1} & 1 & 3 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{pmatrix}$$

Back substitute:

Add  $-4(\text{row } 3)$  to row 1,  $-3(\text{row } 3)$  to row 2.

$$\begin{pmatrix} \boxed{1} & 2 & 3 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{pmatrix}$$

Add  $-2(\text{row } 2)$  to row 1.

$$\begin{pmatrix} \boxed{1} & 0 & 1 & 0 & 3 \\ 0 & \boxed{1} & 1 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{pmatrix}$$

$$x_1 = -x_3 + 3$$

$$x_2 = -x_3 - 1$$

$$x_4 = 1$$

**1.19.** Forward eliminate:

$$\begin{pmatrix} \boxed{-2} & 1 & 1 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & -2 & 0 \end{pmatrix}$$

Add  $\frac{1}{2}(\text{row } 1)$  to row 2,  $\frac{1}{2}(\text{row } 1)$  to row 3,  $\frac{1}{2}(\text{row } 1)$  to row 4.

$$\begin{pmatrix} \boxed{-2} & 1 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} & 0 \end{pmatrix}$$

Add row 2 to row 3, row 2 to row 4.

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

Swap rows 3 and 4.

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

Back substitute:

Scale row 4 by  $\frac{1}{3}$ .

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Add  $-(\text{row } 4)$  to row 1,  $-\frac{3}{2}(\text{row } 4)$  to row 2.

$$\begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Scale row 3 by  $\frac{1}{3}$ .

$$\begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Add  $-(\text{row } 3)$  to row 1,  $-\frac{3}{2}(\text{row } 3)$  to row 2.

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Scale row 2 by  $-\frac{2}{3}$ .

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 1.

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Scale row 1 by  $-\frac{1}{2}$ .

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

$$x_4 = 0$$

**1.21.** You could try:

- One solution:  $x_1 = 0$ .
- No solutions:  $x_1 = 1, x_2 = 1, x_1 + x_2 = 0$ .
- Infinitely many solutions:  $x_1 + x_2 = 0$ .

**1.23.**



**1.28.** (a)=(2), (b)=(4), (c)=(1), (d)=(5), (e)=(3)

**2.2.**

- All coordinates of each vertex are  $\pm 1$ .
- The vertices of a regular octahedron lie in the centers of the faces of a cube.
- Try an equilateral triangle in the plane first. This should lead you to the points

$$\begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix}$$

with an even number of minus signs.



**2.22.**

$$\begin{aligned}(c(AB))_{ij} &= c(AB)_{ij} \\ &= c \sum_k A_{ik} B_{kj} \\ &= \sum_k c A_{ik} B_{kj} \\ &= \sum_k (cA)_{ik} B_{kj} \\ &= ((cA)B)_{ij}.\end{aligned}$$

**2.23.**

$$\begin{aligned}((AB)C)_{ij} &= \sum_k (AB)_{ik} C_{kj} \\ &= \sum_k \sum_\ell A_{i\ell} B_{\ell k} C_{kj}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(A(BC))_{ij} &= \sum_k A_{ik} (BC)_{kj} \\ &= \sum_k \sum_\ell A_{ik} B_{k\ell} C_{\ell j}.\end{aligned}$$

Since  $k$  and  $\ell$  are just used to add up, we can change their names to anything we like. In particular, the resulting sums won't change if we rename  $k$  to  $\ell$  and  $\ell$  to  $k$ . Moreover, we can carry out the sums in any order. (You still have to show that each side is defined just when the other is.)

**2.24.**

$$\begin{aligned}(A(B+C))_{ij} &= \sum_k A_{ik} (B+C)_{kj} \\ &= \sum_k A_{ik} (B_{kj} + C_{kj}) \\ &= \sum_k A_{ik} B_{kj} + \sum_k A_{ik} C_{kj} \\ &= (AB)_{ij} + (AC)_{ij}.\end{aligned}$$

**3.2.** One proof:

$$\begin{aligned}(IA)_{ij} &= \sum_k I_{ik} A_{kj} \\ &= A_{ij}\end{aligned}$$

because  $I_{ik} = 1$  just when  $k = i$ .

A hint for a different proof (without using  $\sum$ ): if  $A$  is  $1 \times 1$ , then the result is clear. Now suppose that we have proven the result already for all matrices of some size smaller than  $p \times q$ , but that we face a matrix  $A$  which is  $p \times q$ . Then split  $A$  into blocks, in any way you like, say as

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

and write out

$$IA = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

and calculate out the result, using the fact that since  $P, Q, R$  and  $S$  are smaller matrices, we can pretend that have already checked the result for them.

**3.3.**  $IB = BI = I$  but  $IB = B$ .

**3.4.**  $IB = BI = I$  but  $IB = B$ .

**3.6.**

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

**3.7.** Row  $j$ . All rows except row  $j$ .

**3.8.** One proof:

$$Ae_1 = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so running your fingers along the rows of  $A$  and column of  $e_1$ :

$$\begin{aligned} &= \begin{pmatrix} A_{11} \cdot 1 + A_{12} \cdot 0 + \cdots + A_{1n} \cdot 0 \\ A_{21} \cdot 1 + A_{22} \cdot 0 + \cdots + A_{2n} \cdot 0 \\ \vdots \\ A_{n1} \cdot 1 + A_{n2} \cdot 0 + \cdots + A_{nn} \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{n1} \end{pmatrix} \end{aligned}$$

Another proof:  $e_1$  has entries  $(e_1)_i = 1$  if  $i = 1$  and  $(e_1)_i = 0$  if  $i \neq 1$ . So  $Ae_1$  has entries  $(Ae_1)_i = \sum_k A_{ik} (e_1)_k$ . Each term is zero except if  $k = 1$ , in which case it is  $A_{i1}$ . But the entries of the first column of  $A$  are  $A_{11}, A_{21}, \dots, A_{n1}$ .



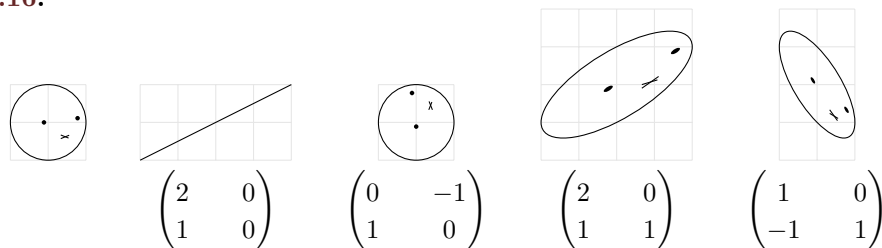
**3.10.** Write  $x = \sum x_j e_j$ , and multiply both sides by  $A$ .

**3.11.** Use the fact that the columns of  $AB$  are  $A$  times columns of  $B$ .

**3.12.** You could try

$$A = \begin{pmatrix} I_3 & 0 \end{pmatrix}, B = \begin{pmatrix} I_3 \\ 0 \end{pmatrix}.$$

**3.16.**



**3.17.** Mostly yes. You only have to try to figure out where  $e_1$  goes to, and where  $e_2$  goes to. The vector  $e_1$  is close to her left eye. The vector  $e_2$  is close to the top of her head, which is not really marked by anything, so harder to follow. But you can't figure out what matrix gives the straight line segment. Why?

**3.19.**  $C = CI = C(AB) = (CA)B = IB = B$ .

**3.23.**

$$\begin{aligned} B^{-1}A^{-1}(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I. \end{aligned}$$

and similarly multiplying out  $(AB)B^{-1}A^{-1}$ .

**3.24.** By definition,  $AA^{-1} = A^{-1}A = I$ . But these equations say exactly that  $A$  is the inverse of  $A^{-1}$ .

**3.25.** Multiply both sides of the equation  $Ax = 0$  by  $A^{-1}$ .

**3.26.** Multiply both sides of  $AB = I$  by  $A^{-1}$  to find  $B = A^{-1}$ . Therefore  $BA = I$ , and so  $A = B^{-1}$ .

**3.28.** If  $x = y$  then clearly  $Ax = Ay$ . If  $Ax = Ay$ , then multiply both sides by  $A^{-1}$ .

**3.29.**

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

**3.30.** See figure 1 on the next page.

**3.31.**

$$S = \begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}.$$

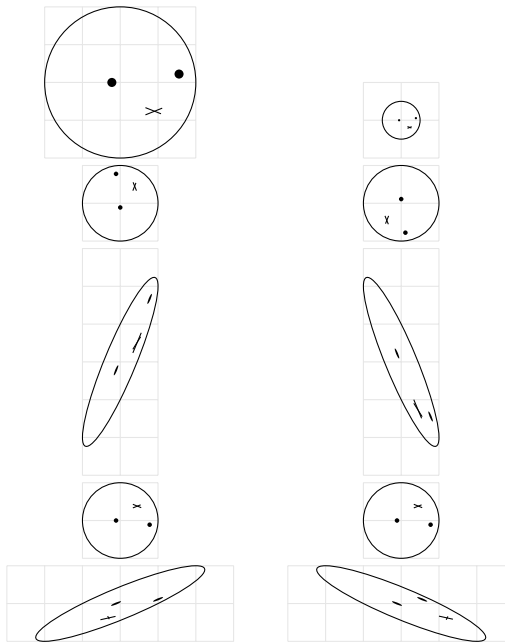


Figure 1: Images coming from some matrices, and from their inverses

### 3.32.

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

**4.1.** Each column has to be a column of the identity matrix, since it has all 0's except for a single 1. But all of the columns have to be different, in order that no two columns have 1's in the same row. So they are different columns of the identity matrix. If some column of the identity matrix doesn't show up anywhere in our matrix, say the third column for example, then the third row has only 0's. So every column of the identity matrix shows up, precisely once, scrambled in some permutation.

**4.4.**  $Se_j$  must be  $e_j$  with multiples of rows added only to later rows, so column  $j$  of  $S$  has zeros above row  $j$  and 1 on row  $j$ . Hence  $S_{1j} = S_{2j} = \dots = S_{j-1j} = 0$  and  $S_{jj} = 1$ .

**4.5.** Proof (1): Multiplying by  $R$  adds entries only to lower entries, preserving the 1's on and 0's above the diagonal.

Proof (2): A matrix  $R$  is strictly lower triangular just when  $R_{ij} = 0$  unless  $i \geq j$  and  $R_{ij} = 1$  for  $i = j$ .

$$(RS)_{ij} = \sum_k R_{ik} S_{kj}.$$

But this sum is all 0's unless we find  $i \geq k$  and  $k \geq j$ , so we need  $i \geq j$ . If  $i = j$ , then only the term  $k = i = j$  makes a contribution, which is  $R_{ii}S_{ii} = 1$ .

Proof (3): Obvious for  $1 \times 1$  matrices. Suppose that we have proven the result for all matrices of size smaller than  $n \times n$ . If  $R$  and  $S$  are  $n \times n$ , write

$$R = \begin{pmatrix} 1 & 0 \\ a & B \end{pmatrix}, S = \begin{pmatrix} 1 & 0 \\ p & Q \end{pmatrix}$$

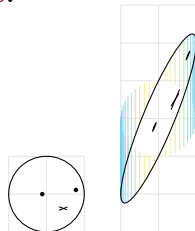
where  $a$  and  $p$  are columns, and  $B$  and  $Q$  are strictly lower triangular. Then

$$RS = \begin{pmatrix} 1 & 0 \\ a + Bp & BQ \end{pmatrix}$$

which is strictly lower triangular because  $B$  and  $Q$  are strictly lower triangular of smaller size.

**4.6.** Suppose that we want to make a matrix  $S$  which is  $p \times q$ . Start with the identity matrix. Multiply it by the elementary matrix with  $S_{1q}$  in row 1, column  $q$ . We get one element into place. Keep going, first getting things set up properly along the bottom row.

**4.8.**



**4.10.** If

$$D = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix},$$

then

$$D^{-1} = \begin{pmatrix} t_1^{-1} & & & \\ & t_2^{-1} & & \\ & & \ddots & \\ & & & t_n^{-1} \end{pmatrix}.$$

If one of these  $t_i$  is 0, then there can't be an inverse, because  $De_i = 0$ , so if  $D$  has an inverse, then  $D^{-1}De_i = e_i$ , but also  $D^{-1}De_i = D^{-1}0 = 0$ .

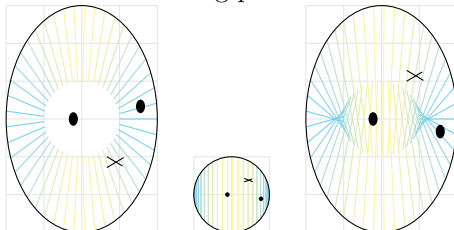
**4.12.**

$$\begin{pmatrix} ad & 0 & 0 \\ 0 & be & 0 \\ 0 & 0 & cf \end{pmatrix}$$

4.13. The original picture is



The three resulting pictures are:



4.14.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

4.15.  $P^{99} = P$  swaps rows 1 and 2.

$$P^{99} = P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.16.  $S^{101}$  adds  $202(\text{row } 1)$  to row 3.

$$S^{101} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 202 & 0 & 1 \end{pmatrix}.$$

4.24. Any number between 0 and 3.

4.25.

a.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

b. Impossible: can only use 1 pivot in each row, so at most 3 pivots binding up variables, so must have at least 2 left over free variables (or at least one free variable, if the last column is a column of constants).

c.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4.28. There is at most one pivot in each row. There are more columns than rows. So there must be a pivotless column: a free variable for the equation  $Ax = 0$ .

5.1. Forward eliminate:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Swap rows 1 and 2.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to row 3.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

Swap rows 2 and 3.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Back substitute:

Add  $-(\text{row } 3)$  to row 2.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 1.

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 1 & 2 & -1 \\ 0 & \boxed{1} & 0 & -1 & -1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 0 & 0 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

**5.2.** Forward eliminate:

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add  $-(\text{row } 1)$  to row 2,  $-(\text{row } 1)$  to row 3.

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -3 & -1 & 1 & 0 \\ 0 & -2 & -3 & -1 & 0 & 1 \end{array} \right)$$

Move the pivot  $\searrow$ .

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{0} & -3 & -1 & 1 & 0 \\ 0 & -2 & -3 & -1 & 0 & 1 \end{array} \right)$$

Swap rows 2 and 3.

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -3 & -1 & 0 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array} \right)$$

Move the pivot  $\searrow$ .

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -3 & -1 & 0 & 1 \\ 0 & 0 & \boxed{-3} & -1 & 1 & 0 \end{array} \right)$$

Back substitute:

Scale row 3 by  $-\frac{1}{3}$ .

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{-2} & -3 & -1 & 0 & 1 \\ 0 & 0 & \boxed{1} & \frac{1}{3} & -\frac{1}{3} & 0 \end{array} \right)$$

Add  $-3(\text{row } 3)$  to row 1,  $3(\text{row } 3)$  to row 2.

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 0 & 1 & 0 \\ 0 & \boxed{-2} & 0 & 0 & -1 & 1 \\ 0 & 0 & \boxed{1} & \frac{1}{3} & -\frac{1}{3} & 0 \end{array} \right)$$

Scale row 2 by  $-\frac{1}{2}$ .

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \boxed{1} & \frac{1}{3} & -\frac{1}{3} & 0 \end{array} \right)$$

Add  $-2(\text{row } 2)$  to row 1.

$$\left( \begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \boxed{1} & \frac{1}{3} & -\frac{1}{3} & 0 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}.$$

**5.3.** Not invertible.

**5.4.**

$$A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ \frac{3}{2} & 1 & -\frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

**5.6.** Yes, invertible.

**5.7.** No, not invertible.

**5.8.** If  $A$  is invertible,  $A^{-1}Ax = x = 0$ . On the other hand, if  $Ax = 0$  holds only for  $x = 0$ , then the same is true of  $Ux = 0$ , after Gauss–Jordan elimination. Any column of  $U$  with no pivot gives a free variable, so there must be a pivot in each column, going straight down the diagonal, so  $U$  is invertible.

**5.9.** Lets use theorem 5.4. Is there a solution  $x$  to the equation  $Ax = b$  for every choice of  $b$ ? Yes: try  $x = Bb$ . So  $A$  is invertible. Multiply  $AB = I$  on both sides by  $A^{-1}$  to get  $B = A^{-1}$ .

**5.10.** We have already seen that if  $A$  and  $B$  are both invertible, then  $(AB)^{-1} = B^{-1}A^{-1}$ . Suppose that  $AB$  is invertible. Then  $(AB)(AB)^{-1} = I$  so  $A(B(AB)^{-1}) = I$ . Therefore  $A$  is invertible. Multiply on the left by  $A^{-1}$ :  $B(AB)^{-1} = A^{-1}$ . Multiply by  $A$  on the right:  $B(AB)^{-1}A = I$ . So  $B$  is invertible.

**5.11.** Yes

**5.12.** Forward elimination:

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 15 \end{pmatrix}$$

Add  $-1(\text{row } 1)$  to row 2.

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{0} & -2 \\ 0 & 1 & 15 \end{pmatrix}$$

Swap rows 2 and 3

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 15 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 15 \\ 0 & 0 & \boxed{-2} \end{pmatrix}$$

The matrix is invertible, so the equations have a unique solution.

**5.15.** You could try

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

which has forward elimination

$$\begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{2} & 1 \\ 0 & 0 & \boxed{-\frac{1}{2}} \end{pmatrix},$$

while its transpose  $A^t$  has forward elimination

$$\begin{pmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}.$$



**5.16.**  $Ux = Vb$  just when  $VAx = Vb$  just when  $Ax = b$ . So to solve  $Ax = b$  we need  $b$  to solve  $Ux = Vb$ . The last two rows of  $Ux = Vb$  give the conditions above. If those are satisfied, we can then drop those rows, and start solving the first two rows with the pivots.

**6.1.** If  $a \neq 0$ , use it as a pivot, and forward elimination yields

$$\begin{pmatrix} \boxed{a} & b \\ 0 & \boxed{d - \frac{bc}{a}} \end{pmatrix}.$$

Therefore if  $a \neq 0$ , then  $A$  is invertible just when  $d - \frac{bc}{a} \neq 0$ . Multiplying by  $a$ , we see that  $A$  is invertible just when  $ad - bc \neq 0$ .

What if  $a = 0$ ? We try to swap. Forward elimination yields

$$\begin{pmatrix} \boxed{c} & d \\ 0 & \boxed{b} \end{pmatrix}.$$

Invertibility (when  $a = 0$ ) is just precisely both  $b$  and  $c$  not vanishing. But (when  $a = 0$ )  $ad - bc = -bc$  vanishes just when  $b$  or  $c$  does, so just when invertibility fails.

**6.2.**

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= a \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} - c \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= ad - bc. \end{aligned}$$

**6.3.**

$$+3(-3) - 1(1) = -10$$

**6.4.**

$$+1 \det \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} - 0 \det \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + 1 \det \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = -2$$

**6.10.**  $1 \cdot 4 \cdot 6 = 24$

**6.11.** For  $1 \times 1$  matrices  $U$ , the results are obvious:

$$U = (u), \quad U^{-1} = \left(\frac{1}{u}\right).$$

Suppose that  $U$  is  $n \times n$  and assume that we have already checked that all smaller invertible upper triangular matrices. Split into blocks, in any manner at all

$$U = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with  $A$  and  $C$  square and upper triangular. You will have to find a way to see that  $A$  and  $C$  are invertible. Once you do that, check that

$$U^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}.$$

We see by induction that (1)  $U^{-1}$  is upper triangular, (2) the diagonal entries of  $U^{-1}$  are the reciprocals of the diagonal entries of  $U$ , and (3) we can compute the entries  $U^{-1}$  inductively in terms of the entries of  $U$ .

**6.13.** Swapping changes the sign of the determinant. But swapping doesn't change the matrix, so it doesn't change the sign of the determinant. Therefore the determinant is 0.

**6.14.** You can use

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

for example.

**6.15.** The determinant goes up by 6 times.

**6.16.** A  $2 \times 2$  determinant looks like

$$\begin{array}{|c|c|} \hline \text{yellow} & \text{gray} \\ \hline \text{gray} & \text{yellow} \\ \hline \end{array} - \begin{array}{|c|c|} \hline \text{gray} & \text{yellow} \\ \hline \text{yellow} & \text{gray} \\ \hline \end{array} = A_{11}A_{22} - A_{21}A_{12}.$$

**7.1.**

$$\begin{pmatrix} \boxed{2} & 5 & 5 \\ 2 & 5 & 7 \\ 2 & 6 & 11 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to row 2,  $-(\text{row } 1)$  to row 3.

$$\begin{pmatrix} \boxed{2} & 5 & 5 \\ 0 & 0 & 2 \\ 0 & 1 & 6 \end{pmatrix}$$

Make a new pivot  $\searrow$ .

$$\begin{pmatrix} \boxed{2} & 5 & 5 \\ 0 & \boxed{0} & 2 \\ 0 & 1 & 6 \end{pmatrix}$$

Swap rows 2 and 3

$$\begin{pmatrix} \boxed{2} & 5 & 5 \\ 0 & \boxed{1} & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

Make a new pivot  $\searrow$ .

$$\begin{pmatrix} 2 & 5 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

So  $\det A = -(2)(1)(2)$ : the minus sign because of one row swap.

**7.2.** 0 because the second row is a multiple of the first.

**7.3.** From the fast formula,  $\det A \neq 0$  just when there is a pivot in each column.

**7.4.** Gauss–Jordan elimination with one row swap yields

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

so

$$\det \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = -1$$

**7.5.** Gauss–Jordan elimination with one row swap yields

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

so

$$\det \begin{pmatrix} 0 & 2 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix} = 6$$

**7.6.** Gauss–Jordan elimination with no row swaps yields

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

so

$$\det \begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 0 \\ 2 & -1 & -1 \end{pmatrix} = 0$$

**7.7.**

$$+0 \det \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix} - 0 \det \begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix} + 2 \det \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = -4$$

**7.8.**

$$+2 \det \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} + 0 \det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = -14$$

**7.9.**

$$+0 \det \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - 0 \det \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} + 0 \det \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = 0$$

**7.10.** Rescaling that row rescales the determinant. But rescaling that row doesn't change anything, so it must not change the determinant. The determinant must not change when scaled, so must be 0.

**7.12.** To see that  $\det A = 12$ , we compute

$$\begin{pmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \\ 3 & 5 & 2 \end{pmatrix}$$

Swap rows 1 and 2

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 2 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to row 3.

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{pmatrix}$$

Make a new pivot  $\searrow$ .

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{pmatrix}$$

Add  $-2(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

Make a new pivot  $\searrow$ .

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

**7.13.** If  $L_{11} = 0$ , then we have a zero row, so  $\det = 0$ , and the result is obviously true. If  $L_{11} \neq 0$ , then use it as a pivot to kill everything underneath it:

$$\begin{pmatrix} L_{11} & & & & & \\ 0 & L_{22} & & & & \\ 0 & L_{32} & L_{33} & & & \\ 0 & L_{42} & L_{43} & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & L_{n2} & L_{n3} & \dots & L_{n(n-1)} & L_{nn} \end{pmatrix}$$

Proceed by induction.

**7.15.** Here is one proof: the  $i$ -th row of  $A^t$  is obvious the transpose of the  $i$ -th column of  $A$ :  $e_i^t A^t = (Ae_i)^t$ . Writing out any vector  $x$  as  $x = x_1e_1 + x_2e_2 + \dots + x_n e_n$ , and adding up, we see that  $x^t A^t = (Ax)^t$  for any vector  $x$ . Apply this to a column of  $B$ , say  $x = Be_i$ .

$$\begin{aligned} e_i^t B^t A^t &= (Be_i)^t A^t \\ &= (ABe_i)^t \\ &= e_i^t (AB)^t. \end{aligned}$$

Here is another proof, using lots of indices:

$$\begin{aligned} (AB)_{ij}^t &= (AB)_{ji} \\ &= \sum_k A_{jk} B_{ki} \\ &= \sum_k B_{ki} A_{jk} \\ &= \sum_k B_{ik}^t A_{kj}^t \\ &= (B^t A^t)_{ij}. \end{aligned}$$

**7.16.** It is the inverse permutation; see problem 4.2 on page 31.

**7.17.** 4: expand down the third column.

**7.18.** For  $k = 1$ ,  $A^1 = A$ , so obvious. By induction,

$$\begin{aligned}\det(A^{k+1}) &= \det(A \cdot A^k) \\ &= (\det A) (\det A^k) \\ &= (\det A) (\det A)^k \\ &= (\det A)^{k+1}.\end{aligned}$$

**7.19.**  $\det(A^{2222444466668888}) = (\det A)^{2222444466668888} = (-1)^{2222444466668888} = 1$  (an even number of minus signs).

**7.20.**  $AA^{-1} = I$  so  $\det A \det(A^{-1}) = 1$ .

**7.21.**

- By expanding down any column.
- By expanding across any row.
- By forward elimination, and then taking the product of the diagonal entries. (The fastest way for a big matrix.)

**7.22.** One, because the determinant of the coefficients is  $1 \cdot 2 \cdot 3 = 6$ .

**8.1.**

$$\begin{aligned}x_1 + 2x_2 + x_3 &= 1 \\ x_2 + x_3 &= 0 \\ -x_1 + x_2 &= -2\end{aligned}$$

**8.3.** Yes.

**8.4.** Lets call these vectors  $x_1, x_2, x_3$ . Make the matrix

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}.$$

Apply forward elimination:

$$A = \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Add  $-1(\text{row } 1)$  to row 3.

$$\begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

Swap rows 2 and 3

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

If we add any vector  $y$ , we can't add another pivot, so every vector  $y$  is a linear combination of  $x_1, x_2, x_3$ . Therefore the span is all of  $\mathbb{R}^3$ .

**8.5.** Yes. Forward eliminate:

$$\begin{pmatrix} 0 & 2 & -1 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

Swap rows 1 and 2.

$$\begin{pmatrix} -1 & 1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -1 & 1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 3.

$$\begin{pmatrix} -1 & 1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -1 & 1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

There is no pivot in the final column, so the final column is a linear combination of earlier columns.

**8.6.** No. Forward eliminate:

$$\begin{pmatrix} 2 & 2 & 4 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & 2 & 4 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Add row 2 to row 3.

$$\begin{pmatrix} 2 & 2 & 4 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & 2 & 4 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & 2 & 4 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

There is a pivot in the final column, so the final column is linearly independent of earlier columns.

**8.9.**  $x_1 + x_2 + 2x_3 = 0$

**8.11.** You can rescale and add as many times as you need to, forming any linear combination.

**8.12.** No: it doesn't contain 0.

**8.13.** Yes

**8.14.** Obviously every vector in a subspace is a linear combination of vectors in the subspace:  $x = 1 \cdot x$ . So the subspace lies inside the span of its vectors. Conversely, every linear combination of vectors in a subspace belongs to the



subspace, so the span of the vectors in the subspace lies in the subspace. Therefore a subspace is its own span.

**8.15.**  $\{0\}$  and  $\mathbb{R}$ .

**8.16.**

- Not always. Take the  $x$  and  $y$  axes in the  $(x, y)$  plane.
- Yes.

**8.17.** No: it contains

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

but doesn't contain

$$2x = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

**8.18.**

- no
- yes
- yes
- no

**8.20.** The lines through 0.

**9.2.** Put the standard basis into the columns of a matrix, and you have the identity matrix. Look: there is a pivot in each column.

**9.4.** Try adding a vector to the set. If you can't then you are done: a basis. If you can, then keep going. If you end up with more than  $n$  vectors, then use theorem 9.2.

**9.5.** Put them into the columns of matrix  $A$ . You find  $\det A = 0$ , so these vectors are linearly dependent.

**9.6.** Put them into the columns of a matrix, and apply forward elimination to find pivots:

$$\begin{pmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

A square matrix, a pivot in every column, so a basis.

**9.10.** Write  $A$  as columns

$$A = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}.$$

The equation  $Ax = 0$  is the equation  $x_1u_1 + x_2u_2 + \dots + x_nu_n = 0$ , which imposes a linear relation. Therefore  $u_1, u_2, \dots, u_n$  are linearly independent just when  $Ax = 0$  has  $x = 0$  as its only solution.

**9.11.**

$$F^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F^{-1}AF = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

**9.12.**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$
$$F^{-1} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, F^{-1}AF = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

**9.14.** Expand out  $x = x_1e_1 + x_2e_2 + \cdots + x_qe_q$  to give

$$Ax = x_1(Ae_1) + x_2(Ae_2) + \cdots + x_q(Ae_q).$$

So  $Ax = 0$  just when  $x_1, x_2, \dots, x_q$  give a linear relation among the columns of  $A$ .

**9.15.** No

**9.17.** Yes

**9.19.** Let

$$F = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$
$$G = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

and let  $A = GF^{-1}$ . But why is there only one such matrix?

**9.21.** The idea is that  $e_2 - e_3 = (e_1 - e_3) - (e_1 - e_2)$ , etc. So consider the vectors  $e_1 - e_2, e_1 - e_3, \dots, e_1 - e_{n-1}$ . Clearly if  $i = 1$ , then  $e_i - e_j$  is one of these vectors. But if  $i \neq 1$ , then

$$e_i - e_j = (-1)(e_1 - e_i) + (e_1 - e_j).$$

So the vectors  $e_1 - e_2, e_1 - e_3, \dots, e_1 - e_{n-1}$  span the subspace. Clearly these vectors are linearly independent, because each one has a nonzero entry just at a spot where all of the others have a zero entry. Alternatively, to see linear independence, any linear relation among them:

$$0 = c_2(e_1 - e_2) + c_3(e_1 - e_3) + \cdots + c_n(e_1 - e_n)$$
$$= (c_2 + c_3 + \cdots + c_n)e_1 + c_2e_2 + c_3e_3 + \cdots + c_n e_n$$

determines a linear relation among the standard basis vectors, forcing  $0 = c_2 = c_3 = \cdots = c_n$ . The subspace is actually the set of vectors  $x$  for which  $x_1 + x_2 + \cdots + x_n = 0$ .

**9.22.** You could try

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Depending on whether you swap row 1 with row 2 or with row 3, forward elimination yields

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

**10.3.** The reduced echelon form is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The basis for the kernel is

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

**10.6.** Remove zero rows.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & -2 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

**10.7.** Remove zero rows.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

**10.8.** The reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 0 & \frac{5}{2} \\ 0 & 0 & \boxed{1} & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} -2 \\ \frac{5}{2} \\ -\frac{5}{2} \\ -1 \\ 1 \end{pmatrix}$$

**10.9.**

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

**10.10.** The only vector in the kernel is 0.

**10.11.** 1, 1, 2, 1, 1, 0

**10.13.** The kernel of  $B$  consists in the vectors

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for which  $Bv = 0$ . This is just asking for  $Ax + Ay + Az = 0$ , i.e. for  $A(x + y + z) = 0$ . We can pick  $x$  and  $y$  arbitrarily, and pick an arbitrary vector  $w$  in the kernel of  $A$ , and set  $z = w - (x + y)$ . In particular,  $B$  has kernel of dimension  $2q + k$  where  $k$  is the dimension of the kernel of  $A$ .

**10.14.**

$$\begin{aligned} y &= Ax \\ &= A \sum_j x_j e_j \\ &= \sum_j x_j A e_j \end{aligned}$$

so a linear combination (with coefficients  $x_1, x_2, \dots, x_q$ ) of the columns  $Ae_j$  of  $A$ . Therefore the vectors of the form  $y = Ax$  are precisely the linear combinations of the columns of  $A$ .

**10.16.** It is the horizontal plane in  $\mathbb{R}^3$ , the  $xy$ -plane in  $xyz$  coordinates.

**10.18.** If  $A$  is tall, then  $A^t$  is short, so there are nonzero vectors  $c$  so that  $A^t c = 0$ , i.e.  $c^t A = 0$ . Since  $c \neq 0$ , there must be some entry  $c_i \neq 0$ . If  $Ax = e_i$ , we find  $0 = c^t Ax = c^t e_i = c_i$  a contradiction. So  $e_i$  is not in the image.

**10.19.** Forward eliminate:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 2 & 2 & -1 \end{pmatrix}$$

Add row 1 to row 2,  $-2(\text{row } 1)$  to row 3.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 4 & -1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 4 & -1 \end{pmatrix}$$

Add  $4(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$

The pivot columns are columns 1,2,3. So the columns:

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

form a basis.

**10.20.** Forward eliminate:

$$\begin{pmatrix} 2 & -6 & -6 \\ -6 & 18 & 18 \\ 0 & 0 & 0 \end{pmatrix}$$

Add 3(row 1) to row 2.

$$\begin{pmatrix} 2 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The only pivot column is column 1. So the column:

$$\begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix}$$

is a basis.

**10.21.** Forward eliminate:

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & -1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

Swap rows 1 and 2.

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

Add  $-\frac{1}{2}$ (row 1) to row 3.

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & 0 \\ 0 & \frac{3}{2} & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & 0 \\ 0 & \frac{3}{2} & 1 \end{pmatrix}$$

Add  $-\frac{3}{4}$ (row 2) to row 3.

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The pivot columns are columns 1,2,3. So the columns:

$$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

form a basis.

**10.22.** Forward eliminate:

$$\begin{pmatrix} 2 & -1 & 3 \\ -6 & 3 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

Add 3(row 1) to row 2.

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The only pivot column is column 1. So the column:

$$\begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix}$$

is a basis.

**10.23.** Forward eliminate:

$$\begin{pmatrix} -3 & 1 & 0 \\ 3 & -1 & 0 \\ 3 & -1 & 0 \end{pmatrix}$$

Add row 1 to row 2, row 1 to row 3.

$$\begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The only pivot column is column 1. So the column:

$$\begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$$

is a basis.

**10.24.** Forward eliminate:

$$\begin{pmatrix} 2 & 4 & 0 \\ -6 & -12 & 0 \\ 4 & 8 & 0 \end{pmatrix}$$

Add 3(row 1) to row 2,  $-2$ (row 1) to row 3.

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The only pivot column is column 1. So the column:

$$\begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$$

is a basis.

**10.25.** Forward eliminate:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

Add  $-(\text{row } 1)$  to row 2, row 1 to row 3.

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The pivot columns are columns 1,2,3. So the columns:

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

form a basis.

**10.26.** Forward eliminate:

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$

Add row 1 to row 2,  $-(\text{row } 1)$  to row 3.

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

Swap rows 2 and 3.

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

The pivot columns are columns 1,2,3. So the columns:

$$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

form a basis.

**10.27.** Forward eliminate:

$$\begin{pmatrix} -1 & 4 & 4 \\ 3 & -12 & -12 \\ 3 & -12 & -12 \end{pmatrix}$$

Add 3(row 1) to row 2, 3(row 1) to row 3.

$$\begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The only pivot column is column 1. So the column:

$$\begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$$

is a basis.

**10.28.** Forward eliminate:

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

Swap rows 1 and 2.

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

The pivot columns are columns 1,2,3. So the columns:

$$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

form a basis.

**10.29.** Forward eliminate:

$$\begin{pmatrix} 2 & -6 & -4 \\ -6 & 18 & 12 \\ 0 & 0 & 0 \end{pmatrix}$$

Add  $3(\text{row } 1)$  to row 2.

$$\begin{pmatrix} 2 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 2 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Move the pivot  $\rightarrow$ .

$$\begin{pmatrix} 2 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The only pivot column is column 1. So the column:

$$\begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix}$$

is a basis.

**10.30.** Forward eliminate:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Add  $-2(\text{row } 1)$  to row 2,  $-(\text{row } 1)$  to row 3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Add  $-(\text{row } 2)$  to row 3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Move the pivot  $\searrow$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

The pivot columns are columns 1,2,3. So the columns:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

form a basis.

**10.31.**  $Ax = 0$  implies that  $Bx = CAx = 0$ . Conversely,  $Bx = 0$  implies that  $Ax = C^{-1}Bx = 0$ .

**10.32.** Linear relations among vectors pass through  $C$  and through  $C^{-1}$ .

**10.33.** Compute echelon form:

$$\begin{pmatrix} 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

Swap rows 1 and 2

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

Add  $-1/2(\text{row } 2)$  to row 3.



Add  $-1(\text{row } 2)$  to row 4.

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the rank is 3, the number of pivots. There is 1 pivotless column. The image is 3-dimensional, while the kernel is 1-dimensional.

**10.34.** You could try

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The image of  $A$  is the span of the columns, so the span of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

while the image of  $B$  is the span of its columns, so the span of

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which are clearly not the same subspaces, but both one dimensional.

**10.35.** The rank of  $A^t$  is the number of linearly independent columns of  $A^t$  (rows of  $A$ ). Let  $U$  be the forward elimination of  $A$ . When we compute forward elimination, we add rows to other rows, and swap rows, so the rows of  $U$  are linear combinations of the rows of  $A$ , and vice versa. Thus the rank of  $A^t$  is the number of linearly independent rows of  $U$ . None of the zero rows of  $U$  count toward this number, while pivot rows are clearly linearly independent. Therefore the rank of  $A^t$  is the number of pivots, so equal to the rank of  $A$ .

**10.42.** If they both have solutions, then  $A^t y = 0$  so  $y^t A = 0$ , so  $y^t A x = 0$ . But  $y^t A x = y^t b$ , so  $b^t y = 0$ , a contradiction.

On the other hand, if neither has a solution, then  $b$  is not in the image of  $A$ . So  $b$  is not a linear combination of the columns of  $A$ , and the matrix  $M = (A \ b)$  has rank 1 higher than the matrix  $A$ . Therefore the matrix

$$M^t = \begin{pmatrix} A^t \\ b^t \end{pmatrix}$$

also has rank one higher than  $A^t$ . So the dimension of the kernel of  $M^t$  is one lower than the dimension of the kernel of  $A^t$ , and therefore there is a vector  $y$  in the kernel of  $A^t$  but not in the kernel of  $M^t$ .

**10.45.**

$$A + B = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

so you can use the previous exercise.

**10.46.** (a) only

**10.47.** The rank of  $AB$  is the dimension of the image. But  $(AB)x = A(Bx)$ , so every vector in the image of  $AB$  lies in the image of  $A$ . The clever bit: on the other hand, the rank of  $AB$  is the same as the rank of  $AB^t$ , as shown in problem 10.35 on page 89. But  $AB^t = B^t A^t$ , so the rank is at most the rank of  $B^t$ , which is the rank of  $B$ .

**11.2.** The determinant  $\det(A - \lambda I)$  is the determinant of an upper (lower) triangular matrix if  $A$  is upper (lower) triangular, with diagonal entries.

**11.3.** You could try

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has eigenvalues  $\lambda = 1$  and  $\lambda = -1$ .

**11.4.** You could try

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, A + B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Their eigenvalues are: for  $A$ ,  $\lambda = 1$ ; for  $B$ ,  $\lambda = 1$ , for  $A + B$ ,  $\lambda = 1$  or  $\lambda = 3$ .

**11.5.** Suppose that  $A$  is an  $n \times n$  matrix. The determinant of a matrix  $A$  is a sum of terms, each one linear in any of the entries of  $A$  which appear in it. The characteristic polynomial  $\det(A - \lambda I)$  therefore involves at most  $n$  terms with a  $\lambda$  in them, coming from the  $n$  diagonal entries of  $A - \lambda I$ , so a polynomial in  $\lambda$  of degree at most  $n$ .

**11.6.**  $(-1)^n \lambda^n$

**11.7.**  $A : 0, 5, B : -1, 3, C : 0, 1, D : 0, 0, 2$

**11.8.**  $\det A = \det A^t$  for any square matrix  $A$ , so  $I^t = I$  and  $\det(A - \lambda I) = \det(A^t - \lambda I)$ .

**11.9.**

$$\begin{aligned} \det(F^{-1}AF - \lambda I) &= \det(F^{-1}(A - \lambda I)F) \\ &= \det(F^{-1}) \det(A - \lambda I) \det(F) \\ &= \det(A - \lambda I). \end{aligned}$$

**11.11.** If  $A$  is invertible, then

$$\begin{aligned} \det(AB - \lambda I) &= \det(A^{-1}(AB - \lambda I)A) \\ &= \det(BA - \lambda I). \end{aligned}$$

If  $B$  is invertible, the same trick works. But if neither  $A$  nor  $B$  is invertible, we have to work harder. Pick  $\alpha$  any number which is *not* an eigenvalue of  $A$ . Then  $A - \alpha I$  is invertible, so  $(A - \alpha I)B$  and  $B(A - \alpha I)$  have the same eigenvalues. Therefore

$$\det((A - \alpha I)B - \lambda I) = \det(B(A - \alpha I) - \lambda I)$$

as polynomials in  $\alpha$ . Now we can plug in  $\alpha = 0$ .

**11.13.** 0,3

**11.16.**

- a.  $s_n(A) = 1$
- b.  $s_0(A) = \det(A - 0) = \det A$
- c. The expression  $\det A$  is a sum of terms, each a product of precisely  $n$  entries of  $A$ , as we have seen. So  $\det(A - \lambda I)$  is a sum of terms, each involving some  $A$  entries and some  $\lambda$ 's, with a total of  $n$  factors in each term.
- d. If  $A$  is upper triangular, or lower triangular, then the result is obvious. But each term of  $s_{n-1}(A)$  involves precisely one entry of  $A$ , hence linear in those entries. So  $s_{n-1}(A)$  is a linear function of  $A$ , i.e.  $s_{n-1}(A + B) = s_{n-1}(A) + s_{n-1}(B)$ . We can write any matrix as a sum of lower triangular and an upper triangular.
- e. Follows immediately from problem 11.9.
- f. Clearly  $F^{-1}AF e_j = F^{-1}A u_j = 0$  for  $j > r$ . So we get zeros just where we need them, to be able to write

$$F^{-1}AF = \begin{pmatrix} P & 0 \\ Q & 0 \end{pmatrix}.$$

$P$  must have rank  $r$ , since there is nowhere else to put the  $r$  pivots, so  $P$  is invertible. Since  $A$  has rank  $r$ , so must  $F^{-1}AF$ .

g.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} P - \lambda I & 0 \\ Q & -\lambda I \end{pmatrix} \\ &= \det(P - \lambda I) (-\lambda)^{n-r} \end{aligned}$$

has no terms of degree less than  $n - r$  in  $\lambda$ .

h. You could try

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

**11.17.** The  $\lambda = 3$ -eigenvectors are multiples of

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

**11.20.**

$$\lambda = 1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

11.21.

$$\lambda = -2 \quad \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$$
$$\lambda = 3 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

11.22.

$$\lambda = 1 \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\lambda = 2 \quad \begin{pmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{pmatrix}$$
$$\lambda = 3 \quad \begin{pmatrix} 0 \\ \frac{3}{2} \\ 1 \end{pmatrix}$$

11.27. Permutation matrices are orthogonal, so  $P^t = P^{-1}$ . Alternately: any permutation  $P$  must generate a finite number of permutations  $P, P^2, P^3, \dots$ , so eventually we must find a loop:  $P^m = P^n$  for some integers  $m > n$ . Being invertible, we see that  $P^k = I$  for some integer  $k$ .

- (a)  $\pm 1$  since  $P^t = P^{-1}$ , so  $\det P = \det P^t = \frac{1}{\det P}$ . Alternately:  $P^k = I$  so  $(\det P)^k = 1$  so  $\det P = \pm 1$ .
- (b) The trace is the number of 1's that stay on the diagonal when we permute the columns of  $I$  to make up  $P$ , i.e. the number of fixed columns. This number can be any value between 1 and  $n$ , except for  $n - 1$ , since you can cyclically permute any number of elements, not fixing any, except that you can't permute one element without fixing it.
- (c)  $P^t = P^{-1}$  so  $P$  is the transpose of a permutation matrix. Therefore the rows can be permuted to yield the identity matrix. So all pivots equal 1.
- (d) Pick an eigenvector  $v$  so that  $Pv = \lambda v$ . But then  $\langle v, v \rangle = \langle P^t P v, v \rangle = \langle P v, P v \rangle = \lambda^2 \langle v, v \rangle$ . So  $\lambda = \pm 1$ . Alternately: any eigenvalue will satisfy  $\lambda^k = 1$ , i.e.  $\lambda = \pm 1$ .

11.28.

$$\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

11.29.

$$\begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix}$$

11.30.

$$\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

11.31.

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

11.32.

$$\begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$$

11.33.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

11.34.

$$\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$$

11.35.

$$\begin{pmatrix} 2 & 0 \\ 5 & -3 \end{pmatrix}$$

11.36.

$$\begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix}$$

11.37.

$$\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

11.38.

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

11.39.

$$\begin{pmatrix} 2 & 0 \\ \frac{3}{2} & -1 \end{pmatrix}$$

11.40.

$$\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$$

11.41.

$$\begin{pmatrix} -3 & 0 \\ 1 & -1 \end{pmatrix}$$

**11.42.**

$$\begin{pmatrix} -1 & 6 \\ 0 & 2 \end{pmatrix}$$

**12.1.** The kernel of any matrix is a subspace.

**12.2.**  $Ax = \lambda x$  so

$$\begin{aligned} ABx &= BAx \\ &= B\lambda x \\ &= \lambda Bx. \end{aligned}$$

**12.3.** Depending on the order in which you write down your eigenvalues, you could get:

$$\begin{aligned} F &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ F^{-1} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ F^{-1}AF &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{aligned}$$

**12.4.** Again, it depends on the order you choose to write the eigenvalues, and the order in which you write down basis vectors for each eigenspace. You could get:

$$\begin{aligned} \lambda = -3 & \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ \lambda = -2 & \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \lambda = 0 & \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

$$F = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} 1 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -2 & -1 \end{pmatrix}$$

$$F^{-1}AF = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**12.5.** You could get

$$\lambda = 2 \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = -4 \quad \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$F = \begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 1 & 0 & 1 \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 1 & -1 & 1 \end{pmatrix}$$

$$F^{-1}AF = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

**12.6.** Suppose we have a matrix  $F$  for which

$$F^{-1}AF = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Call this diagonal matrix  $\Lambda$ . Therefore  $AF = F\Lambda$ . Lets check that the columns of  $F$  are eigenvectors. We need only see that  $F\Lambda$  is just  $F$  with columns scaled by the diagonal entries of  $\Lambda$ .

**12.7.** You could try

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

because it has only one eigenvector,

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(up to rescaling), with eigenvalue  $\lambda = 0$ . Therefore there is no basis of eigenvectors.

**12.9.** You could get

$$\begin{aligned} \lambda = -1 & \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \lambda = 1 & \quad \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} F &= \begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix} \\ F^{-1} &= \begin{pmatrix} -2 & -1 \\ 2 & 2 \end{pmatrix} \\ F^{-1}AF &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Let  $A = F^{-1}AF$ . So  $A = FAF^{-1}$ . Clearly  $A^{100000} = 1$ , so  $A^{100000} = FAF^{100000}F^{-1} = 1$ .

**12.11.**

$$A = \frac{1}{5} \begin{pmatrix} 13 & -4 \\ -4 & 7 \end{pmatrix}$$

**12.12.**

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}$$



$$\lambda = \frac{3}{4} \quad v_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$\lambda = 1 \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = \frac{1}{4} \quad v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$F = \begin{pmatrix} -\frac{1}{2} & 0 & -1 \\ -\frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$F^{-1}AF = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Consider the coordinates of the vector

$$\begin{pmatrix} h_0 \\ s_0 \\ d_0 \end{pmatrix}.$$

Numbers of people can't be negative. Clearly the vector must be a linear combination of eigenvectors, with a positive coefficient for the  $\lambda = 1$  eigenvector:

$$\begin{pmatrix} h_0 \\ s_0 \\ d_0 \end{pmatrix} = a_1 v_1 + a_2 v_2 + a_3 v_3,$$

with  $a_2 > 0$ . Over time, the numbers develop according to

$$\begin{aligned} \begin{pmatrix} h_n \\ s_n \\ d_n \end{pmatrix} &= A \begin{pmatrix} h_{n-1} \\ s_{n-1} \\ d_{n-1} \end{pmatrix} \\ &= A^n \begin{pmatrix} h_0 \\ s_0 \\ d_0 \end{pmatrix} \\ &= \left(\frac{3}{4}\right)^n a_1 v_1 + a_2 v_2 + \left(\frac{1}{4}\right)^n a_3 v_3. \end{aligned}$$

Since the other eigenvalues are smaller than 1, their powers become very small, and their components in the resulting vector gradually decay away. Therefore the result becomes every closer to

$$a_2 v_2 = \begin{pmatrix} 0 \\ 0 \\ a_2 \end{pmatrix},$$

everybody dead. So everyone dies, in an exponential decay of population. (It should be obvious, because we didn't allow any births in our model.)

**12.15.**

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 + \lambda \\ &= \lambda(\lambda + 1) \end{aligned}$$

$$\lambda = -1$$

$$A - \lambda = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & \frac{1}{2} \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & -\frac{1}{2} \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$\lambda = 0$$

$$A - \lambda = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 0 & \boxed{1} \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 0 & \boxed{1} \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 0 \\ 0 & \boxed{1} \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = -1 \quad \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



**12.16.**

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 1)(\lambda - 2) \end{aligned}$$

$$\lambda = 1$$

$$A - \lambda = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & 2 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & -2 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\lambda = 2$$

$$A - \lambda = \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & 0 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & 0 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = 1 \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



**12.17.**

$$\det(A - \lambda I) = \lambda^2 + 3\lambda$$

$$= \lambda(\lambda + 3)$$

$$\lambda = -3$$

$$A - \lambda = \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & 0 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & 0 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = 0$$

$$A - \lambda = \begin{pmatrix} 0 & 0 \\ 3 & -3 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & -1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & -1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -3 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



**12.18.**

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 + 2\lambda \\ &= \lambda(\lambda + 2) \end{aligned}$$

$$\lambda = -2$$

$$A - \lambda = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & -1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = 0$

$$A - \lambda = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & 1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & -1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda = -2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



**12.19.**

$$\begin{aligned}\det(A - \lambda I) &= \lambda^2 - 4 \\ &= (\lambda - 2)(\lambda + 2)\end{aligned}$$

$$\lambda = -2$$

$$A - \lambda = \begin{pmatrix} 8 & 8 \\ -4 & -4 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & 1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & 1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & -1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & -1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda = 2$$

$$A - \lambda = \begin{pmatrix} 4 & 8 \\ -4 & -8 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & 2 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & 2 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & -2 \end{pmatrix}$$



Pad with rows from the identity matrix, to get 1's down the diagonal.

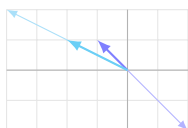
$$\begin{pmatrix} \boxed{1} & -2 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\lambda = -2 \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$



**12.20.**

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 + 4\lambda + 3 \\ &= (\lambda + 3)(\lambda + 1) \end{aligned}$$

$$\lambda = -3$$

$$A - \lambda = \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & -1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & -1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & 1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1$$

$$A - \lambda = \begin{pmatrix} -2 & 0 \\ -2 & 0 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & 0 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & 0 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

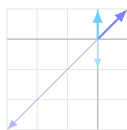
$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = -3 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



**12.21.**

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 + \lambda \\ &= \lambda(\lambda + 1) \end{aligned}$$

$$\lambda = -1$$

$$A - \lambda = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 0 & \boxed{1} \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 0 & \boxed{1} \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 0 \\ 0 & \boxed{1} \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda = 0$

$$A - \lambda = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & -1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & -1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & 1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



**12.22.**

$$\det(A - \lambda I) = \lambda^2 + \lambda - 2$$

$$= (\lambda + 2)(\lambda - 1)$$

$$\lambda = -2$$

$$A - \lambda = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & -1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & -1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & 1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 1$$

$$A - \lambda = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & 2 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & 2 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & -2 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\lambda = -2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 1 \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$



**12.23.**

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 1)(\lambda - 2) \end{aligned}$$

$$\lambda = 1$$

$$A - \lambda = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 0 & 1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 0 & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda = 2$

$$A - \lambda = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Gauss-Jordan elimination yields

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & 1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & -1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda = 1 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



**12.24.**

$$\begin{aligned}\det(A - \lambda I) &= \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1)\end{aligned}$$

$$\lambda = -1$$

$$A - \lambda = \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & -\frac{1}{2} \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & \frac{1}{2} \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\lambda = 1$$

$$A - \lambda = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & -1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & -1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1 \quad \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\lambda = 1 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



**12.25.**

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 - \lambda \\ &= \lambda(\lambda - 1) \end{aligned}$$

$\lambda = 0$

$$A - \lambda = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & -1 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & 1 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$$\lambda = 1$$

$$A - \lambda = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} \boxed{1} & 0 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} \boxed{1} & 0 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



**12.26.**

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 + 4\lambda + 3 \\ &= (\lambda + 3)(\lambda + 1) \end{aligned}$$

$$\lambda = -3$$

$$A - \lambda = \begin{pmatrix} -\frac{2}{3} & \frac{4}{3} \\ -\frac{4}{3} & \frac{8}{3} \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} \boxed{1} & -2 \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & -2 \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & 2 \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda = -1$$

$$A - \lambda = \begin{pmatrix} -\frac{8}{3} & \frac{4}{3} \\ -\frac{4}{3} & \frac{2}{3} \end{pmatrix}.$$

Gauss–Jordan elimination yields

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

Remove zero rows.

$$\begin{pmatrix} 1 & -\frac{1}{2} \end{pmatrix}$$

Change signs of the entries after each pivot.

$$\begin{pmatrix} 1 & \frac{1}{2} \end{pmatrix}$$

Pad with rows from the identity matrix, to get 1's down the diagonal.

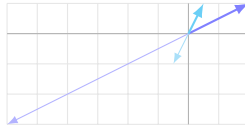
$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

Keep the pivotless columns.

$$\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\lambda = -3 \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda = -1 \quad \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$



12.27. See table 1.

Table 1: Invertibility criteria.  $A$  is  $n \times n$  of rank  $r$ .  $U$  is any matrix obtained from  $A$  by forward elimination.

§	Invertible	Not invertible
5	Gauss–Jordan on $A$ yields 1.	Gauss–Jordan on $A$ yields a matrix with $n - r$ zero rows.
5	$U$ is invertible.	$U$ has $n - r$ zero rows.
5	Pivots lie on the diagonal.	Some pivot lies above the diagonal, and all pivots after it.
5	$U$ has no zero rows	$U$ has $n - r$ zero rows.
5	$U$ has $n$ pivots.	$U$ has $r < n$ pivots.
5	$Ax = b$ has a solution $x$ for each $b$ .	$Ax = b$ has no solution for some $b$ , $n - r$ dimensions worth for other $b$ .
5	$Ax = b$ has exactly <i>one</i> solution $x$ for <i>each</i> $b$ .	$Ax = b$ has no solution for some $b$ , many for other $b$ .
5	$Ax = b$ has exactly <i>one</i> solution $x$ for <i>some</i> $b$ .	$Ax = b$ has no solution for some $b$ , many for other $b$ .
5	$Ax = 0$ only for $x = 0$ .	$Ax = 0$ for many $x$ .
5	$A$ has rank $n$ .	$A$ has rank $r < n$ .
7	$A^t$ is invertible.	$A^t$ is not invertible.
7	$\det A \neq 0$ .	Every square block larger than $r \times r$ has $\det = 0$ .
9	The columns are linearly independent.	The $n - r$ pivotless columns are linear combinations of the $r$ pivot columns.
9	The columns form a basis.	Each of the $n - r$ pivotless columns is a linear combination of earlier pivot columns.
9	The rows form a basis.	One row is a linear combination of earlier rows.
10	The kernel of $A$ is just the 0 vector.	The kernel has positive dimension $n - r$ .
10	The image of $A$ is all of $\mathbb{R}^n$ .	The image has positive dimension $r$ .

§	Invertible	Not invertible
11	0 is not an eigenvalue of $A$ .	The $\lambda = 0$ eigenspace has positive dimension $n - r$ .

13.1.

$$\begin{aligned}\langle Ae_j, e_i \rangle &= \langle A_{kj}e_k, e_i \rangle \\ &= A_{ij}.\end{aligned}$$

13.2.

$$\begin{aligned}P_{ij} &= \langle Pe_i, e_j \rangle \\ &= \langle e_{p(i)}, e_j \rangle \\ &= \begin{cases} 1, & \text{if } p(i) = j, \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1, & \text{if } i = p^{-1}(j), \\ 0, & \text{otherwise.} \end{cases} \\ &= \langle e_i, e_{p^{-1}(j)} \rangle \\ &= \langle e_i, P^{-1}e_j \rangle \\ &= P_{ji}^{-1}.\end{aligned}$$

13.3.

$$\begin{aligned}x^t y &= \sum_k x_k^t y_k \\ &= \sum_k x_k y_k.\end{aligned}$$

13.4.

$$\begin{aligned}\left\langle v - \frac{\langle v, u \rangle}{\|u\|^2} u, u \right\rangle &= \langle v, u \rangle - \frac{\langle v, u \rangle}{\|u\|^2} \langle u, u \rangle \\ &= \langle v, u \rangle - \langle v, u \rangle \\ &= 0.\end{aligned}$$

13.5.

- (a) One:  $x = 0$ .
- (b)  $2n$ : one  $x_i$  is  $\pm 1$ , all other  $x_j$  are 0.
- (c)  $2n + 16\binom{n}{4}$ : one  $x_i$  is  $\pm 2$ , all other  $x_j$  are 0, or 4  $x_i$ 's are  $\pm 1$  and all other are 0.
- (d)  $2n + 8(n-2)\binom{n}{2} + 2^6(n-5)\binom{n}{5} + 2^9\binom{n}{9}$ :

- a. one  $\pm 3$  or
- b. two  $\pm 2$ 's and one  $\pm 1$  or
- c. one  $\pm 2$  and five  $\pm 1$ 's or
- d. nine  $\pm 1$ 's.

**13.6.** The hour hand starts at angle  $\pi/2$ , and completes a revolution every 12 hours. So the hour hand is at an angle of

$$\theta = \frac{\pi}{2} - \frac{2\pi}{12}t,$$

after  $t$  hours. The minute hand, if we measure time in hours, revolves every hour, so has angle

$$\theta = \frac{\pi}{2} - 2\pi t.$$

- a.  $t = \frac{3}{11}(2k + 1)$ , any integer  $k$
- b.  $t = \frac{6k}{11}$ , any integer  $k$ .

**13.11.** You could try:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, AB = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

**13.12.** For  $A$  symmetric.

**13.19.** Those which have  $\pm 1$  in each diagonal entry.

**13.20.**

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is not of that form.

**13.21.** You could try  $A = B = I$ .

**13.22.**

$$\langle x, y \rangle = \frac{1}{2} \left( \|x + y\|^2 - \|x\|^2 - \|y\|^2 \right).$$

Preserve the right hand side, and you must preserve the left hand side.

**13.25.** The rows of  $A$  are orthonormal just when  $A^t$  is orthogonal, which occurs just when  $A = (A^t)^{-1}$ , which occurs just when  $A^t = A^{-1}$ , which occurs just when  $A$  is orthogonal.

**13.26.** See table 2 on the next page and table 3 on the following page.

**13.27.**

$$u_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{pmatrix}, u_3 = \begin{pmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}.$$

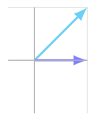
**13.28.** The pictures should look like

$$\begin{aligned}
w_1 &= v_1 \\
&= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
&= \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{(2)(1) + (0)(-1) + (-2)(0)}{(1)(1) + (-1)(-1) + (0)(0)} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}
\end{aligned}$$

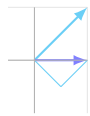
Table 2: Orthogonalizing vectors: the projections

$$\begin{aligned}
u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\
&= \frac{1}{\sqrt{(1)(1) + (-1)(-1) + (0)(0)}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \sqrt{2} \\ -\frac{1}{2} \sqrt{2} \\ 0 \end{pmatrix} \\
u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\
&= \frac{1}{\sqrt{(1)(1) + (1)(1) + (-2)(-2)}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{6} \sqrt{6} \\ \frac{1}{6} \sqrt{6} \\ -\frac{1}{3} \sqrt{6} \end{pmatrix}
\end{aligned}$$

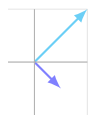
Table 3: Orthogonalizing vectors: rescaling



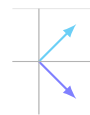
The original vectors.



Project the second vector perpendicular to the first.



Projected.  
Shrink/stretch all vectors to length 1.



Done: orthonormal.

**13.29.**

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{2\sqrt{6}} \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

Notice that  $v_1, v_2, v_3$  did not give a basis, so when we try to compute  $u_3$ , we run into trouble.

**13.30.** The only problem that can come up is division by zero. But that happens only when we divide by a length  $\|w_j\|$ . If this length is 0, then  $w_j$  is zero, so

$$v_j = \sum_i \langle v_j, u_i \rangle u_i.$$

But each  $u_i$  is a linear combination of vectors  $v_1, v_2, \dots, v_i$ , so this is a linear dependence.

**13.32.** If  $v$  is perpendicular to  $u$ , then set  $w = v$ , see that  $v$  is perpendicular to  $v$ , so  $v = 0$ . Otherwise, if  $v$  is not perpendicular to  $u$ , then  $w = v - \frac{\langle v, u \rangle}{\|u\|^2} u$  is perpendicular to  $u$  and  $v$ , and therefore perpendicular to any linear combination of  $u$  and  $v$ . In particular, since  $w$  is a linear combination of  $u$  and  $v$ ,  $w$  must be perpendicular to  $w$ , so  $w = 0$ . So

$$v = \frac{\langle v, u \rangle}{\|u\|^2} u.$$

**13.33.** Nothing.

**13.34.**

$$\begin{aligned}w_1 &= v_1 \\&= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\&= \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \frac{(5)(1) + (3)(-1)}{(1)(1) + (-1)(-1)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\&= \begin{pmatrix} 4 \\ 4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\&= \frac{1}{\sqrt{(1)(1) + (-1)(-1)}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\&= \begin{pmatrix} \frac{1}{2} \sqrt{2} \\ -\frac{1}{2} \sqrt{2} \end{pmatrix} \\u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\&= \frac{1}{\sqrt{(4)(4) + (4)(4)}} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\&= \begin{pmatrix} \frac{1}{2} \sqrt{2} \\ \frac{1}{2} \sqrt{2} \end{pmatrix}\end{aligned}$$

See figure 2 on the next page.



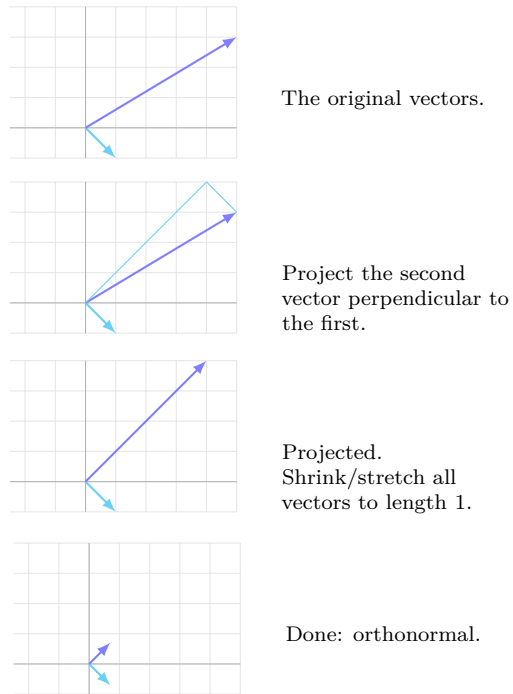


Figure 2: Orthogonalizing vectors in the plane

13.35.

$$\begin{aligned}
 w_1 &= v_1 \\
 &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\
 w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
 &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \frac{(0)(-1) + (2)(-1)}{(-1)(-1) + (-1)(-1)} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} -1 \\ 1 \end{pmatrix}
 \end{aligned}$$

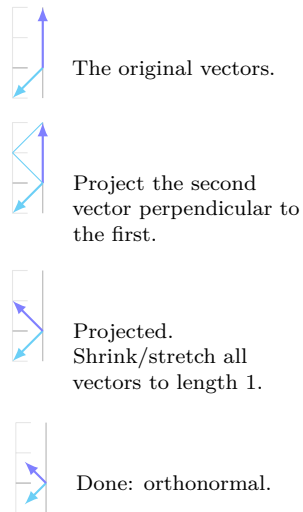


Figure 3: Orthogonalizing vectors in the plane

$$\begin{aligned}
 u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\
 &= \frac{1}{\sqrt{(-1)(-1) + (-1)(-1)}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix} \\
 u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\
 &= \frac{1}{\sqrt{(-1)(-1) + (1)(1)}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}
 \end{aligned}$$

See figure 3.

13.36.

$$\begin{aligned}w_1 &= v_1 \\&= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\&= \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{(-1)(-1) + (2)(1)}{(-1)(-1) + (1)(1)} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\&= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\&= \frac{1}{\sqrt{(-1)(-1) + (1)(1)}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\&= \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} \\u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\&= \frac{1}{\sqrt{(\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2})}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\&= \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\end{aligned}$$

See figure 4 on the next page.



The original vectors.



Project the second vector perpendicular to the first.



Projected.  
Shrink/stretch all vectors to length 1.



Done: orthonormal.

Figure 4: Orthogonalizing vectors in the plane

13.37.

$$\begin{aligned}w_1 &= v_1 \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{(-1)(2) + (1)(-1)}{(2)(2) + (-1)(-1)} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \end{pmatrix}\end{aligned}$$

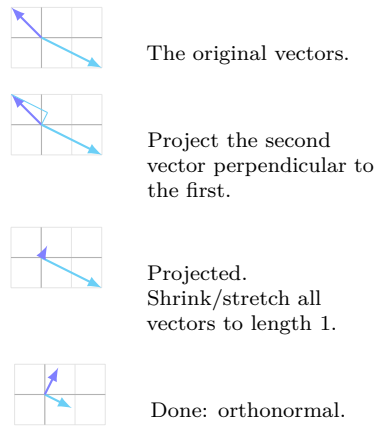


Figure 5: Orthogonalizing vectors in the plane

$$\begin{aligned}
 u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\
 &= \frac{1}{\sqrt{(2)(2) + (-1)(-1)}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \\
 u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\
 &= \frac{1}{\sqrt{(\frac{1}{5})(\frac{1}{5}) + (\frac{2}{5})(\frac{2}{5})}} \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}
 \end{aligned}$$

See figure 5.

13.38.

$$\begin{aligned}w_1 &= v_1 \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \frac{(0)(1) + (2)(2)}{(1)(1) + (2)(2)} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\ &= \frac{1}{\sqrt{(1)(1) + (2)(2)}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5} \sqrt{5} \\ \frac{2}{5} \sqrt{5} \end{pmatrix} \\ u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\ &= \frac{1}{\sqrt{(-\frac{4}{5})(-\frac{4}{5}) + (\frac{2}{5})(\frac{2}{5})}} \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{5} \sqrt{5} \\ \frac{1}{5} \sqrt{5} \end{pmatrix}\end{aligned}$$

See figure 6 on the next page.



The original vectors.



Project the second vector perpendicular to the first.



Projected.  
Shrink/stretch all vectors to length 1.



Done: orthonormal.

Figure 6: Orthogonalizing vectors in the plane

**13.39.**

$$\begin{aligned}w_1 &= v_1 \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \frac{(2)(1) + (0)(1)}{(1)(1) + (1)(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}\end{aligned}$$

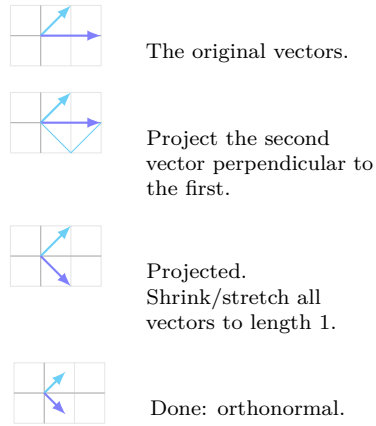


Figure 7: Orthogonalizing vectors in the plane

$$\begin{aligned}
 u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\
 &= \frac{1}{\sqrt{(1)(1) + (1)(1)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2} \sqrt{2} \\ \frac{1}{2} \sqrt{2} \end{pmatrix} \\
 u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\
 &= \frac{1}{\sqrt{(1)(1) + (-1)(-1)}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2} \sqrt{2} \\ -\frac{1}{2} \sqrt{2} \end{pmatrix}
 \end{aligned}$$

See figure 7.



13.40.

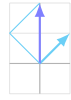
$$\begin{aligned}w_1 &= v_1 \\&= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\&= \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \frac{(0)(1) + (2)(1)}{(1)(1) + (1)(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\&= \begin{pmatrix} -1 \\ 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\&= \frac{1}{\sqrt{(1)(1) + (1)(1)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\&= \begin{pmatrix} \frac{1}{2} \sqrt{2} \\ \frac{1}{2} \sqrt{2} \end{pmatrix} \\u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\&= \frac{1}{\sqrt{(-1)(-1) + (1)(1)}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\&= \begin{pmatrix} -\frac{1}{2} \sqrt{2} \\ \frac{1}{2} \sqrt{2} \end{pmatrix}\end{aligned}$$

See figure 8 on the next page.



The original vectors.



Project the second vector perpendicular to the first.



Projected.  
Shrink/stretch all vectors to length 1.



Done: orthonormal.

Figure 8: Orthogonalizing vectors in the plane

13.41.

$$\begin{aligned}w_1 &= v_1 \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ &= \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{(-1)(1) + (2)(1)}{(1)(1) + (1)(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{pmatrix}\end{aligned}$$

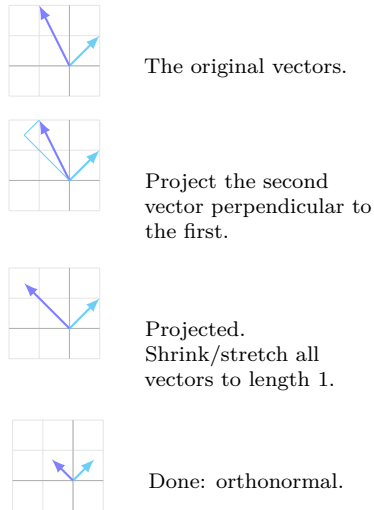


Figure 9: Orthogonalizing vectors in the plane

$$\begin{aligned}
 u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\
 &= \frac{1}{\sqrt{(1)(1) + (1)(1)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2} \sqrt{2} \\ \frac{1}{2} \sqrt{2} \end{pmatrix} \\
 u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\
 &= \frac{1}{\sqrt{(-\frac{3}{2})(-\frac{3}{2}) + (\frac{3}{2})(\frac{3}{2})}} \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{2} \sqrt{2} \\ \frac{1}{2} \sqrt{2} \end{pmatrix}
 \end{aligned}$$

See figure 9.

13.42.

$$\begin{aligned}w_1 &= v_1 \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{(-1)(0) + (1)(1)}{(0)(0) + (1)(1)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\ &= \frac{1}{\sqrt{(0)(0) + (1)(1)}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\ &= \frac{1}{\sqrt{(-1)(-1) + (0)(0)}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}\end{aligned}$$

See figure 10 on the next page.

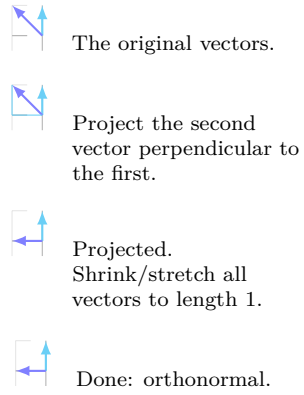


Figure 10: Orthogonalizing vectors in the plane

**13.43.**

$$\begin{aligned}
 w_1 &= v_1 \\
 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
 w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
 &= \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{(-1)(2) + (2)(1)}{(2)(2) + (1)(1)} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} -1 \\ 2 \end{pmatrix}
 \end{aligned}$$

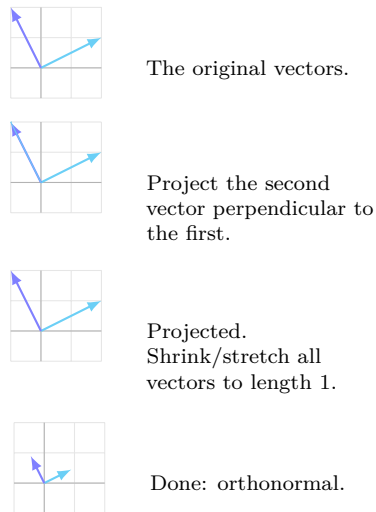


Figure 11: Orthogonalizing vectors in the plane

$$\begin{aligned}
 u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\
 &= \frac{1}{\sqrt{(2)(2) + (1)(1)}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \\
 u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\
 &= \frac{1}{\sqrt{(-1)(-1) + (2)(2)}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}
 \end{aligned}$$

See figure 11.

13.44.

$$\begin{aligned}w_1 &= v_1 \\&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\&= \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{(1)(-1) + (-1)(0)}{(-1)(-1) + (0)(0)} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\&= \begin{pmatrix} 0 \\ -1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\&= \frac{1}{\sqrt{(-1)(-1) + (0)(0)}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\&= \frac{1}{\sqrt{(0)(0) + (-1)(-1)}} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\&= \begin{pmatrix} 0 \\ -1 \end{pmatrix}\end{aligned}$$

See figure 12 on the next page.



The original vectors.



Project the second vector perpendicular to the first.



Projected.  
Shrink/stretch all vectors to length 1.



Done: orthonormal.

Figure 12: Orthogonalizing vectors in the plane

**13.45.**

$$\begin{aligned}w_1 &= v_1 \\&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\&= \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \frac{(0)(2) + (-1)(-1)}{(2)(2) + (-1)(-1)} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\&= \begin{pmatrix} -\frac{2}{5} \\ -\frac{4}{5} \end{pmatrix}\end{aligned}$$



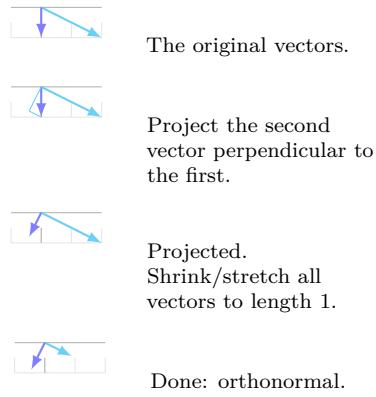


Figure 13: Orthogonalizing vectors in the plane

$$\begin{aligned}
 u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\
 &= \frac{1}{\sqrt{(2)(2) + (-1)(-1)}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \\
 u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\
 &= \frac{1}{\sqrt{(-\frac{2}{5})(-\frac{2}{5}) + (-\frac{4}{5})(-\frac{4}{5})}} \begin{pmatrix} -\frac{2}{5} \\ -\frac{4}{5} \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{5}\sqrt{5} \\ -\frac{2}{5}\sqrt{5} \end{pmatrix}
 \end{aligned}$$

See figure 13.

**13.46.**

$$\begin{aligned}w_1 &= v_1 \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \frac{(-1)(1) + (0)(1)}{(1)(1) + (1)(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}u_1 &= \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} w_1 \\ &= \frac{1}{\sqrt{(1)(1) + (1)(1)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} \\ u_2 &= \frac{1}{\sqrt{\langle w_2, w_2 \rangle}} w_2 \\ &= \frac{1}{\sqrt{(-\frac{1}{2})(-\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2})}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\end{aligned}$$

See figure 14 on the next page.

**14.1.** If  $Au = \lambda u$  and  $Av = \mu v$ , then

$$\begin{aligned}\langle Au, v \rangle &= \lambda \langle u, v \rangle \\ &= \langle u, Av \rangle \\ &= \mu \langle u, v \rangle.\end{aligned}$$

So  $\langle u, v \rangle = 0$ .

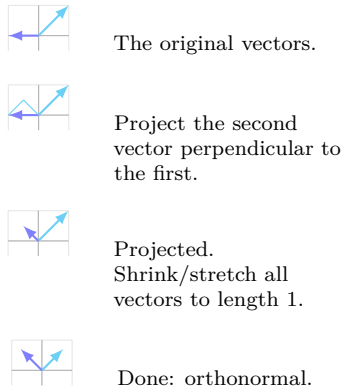


Figure 14: Orthogonalizing vectors in the plane

14.2.

$$\lambda = 16 \quad \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$$

$$\lambda = 3 \quad \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

Gram-Schmidt the eigenvectors, and

$$F = \begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}$$

$$F^{-1}AF = \begin{pmatrix} 16 & 0 \\ 0 & 3 \end{pmatrix}$$

Each eigenvector comes from a different eigenvalue, so they are already perpendicular—you only have to rescale them to have unit length.

14.4.

$$\det(A - \lambda I) = \lambda^2 - 5\lambda$$

$$= \lambda(\lambda - 5)$$

$$\lambda = 5 \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

After orthogonalizing the eigenvectors,

$$F = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$F^{-1}AF = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$$

**14.5.**

$$\det(A - \lambda I) = \lambda^3 - 8\lambda^2 + 15\lambda$$

$$= \lambda(\lambda - 3)(\lambda - 5)$$

$$\lambda = 3 \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = 5 \quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

After orthogonalizing the eigenvectors,

$$F = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 1 & 0 & 0 \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{pmatrix}$$

$$F^{-1}AF = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

**14.6.**

$$\det(A - \lambda I) = \lambda^3 - 4\lambda^2 + 5\lambda - 2$$

$$= (\lambda - 2)(\lambda - 1)^2$$

$$\lambda = 1 \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

After orthogonalizing the eigenvectors,

$$F = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$F^{-1}AF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

**14.7.**

$$\lambda = -1 \quad \begin{pmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = 1 \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{3}{4} \\ 0 \\ 1 \end{pmatrix}$$

After orthogonalizing the eigenvectors,

$$F = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$F^{-1}AF = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**14.8.** Such an  $F$  must preserve the eigenspaces, so preserve the span of  $e_1$ , the span of  $e_2$ , and the span of  $e_3$ . Therefore

$$F = \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}.$$

**14.11.** You only change  $x_1x_2$  terms:

- $x_2^2$
- $x_1^2 + x_2^2$
- $x_1^2 + \frac{3}{2}x_1x_2 + \frac{3}{2}x_2x_1$
- $x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_1$

**14.12.**

a.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

b.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

c.

$$A = \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{pmatrix}$$

d.

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

**14.14.** Look at the eigenvalues of the symmetric matrix, to get started.

- ellipse
- hyperbola
- pair of lines
- hyperbola
- line
- empty set

**14.15.** First, by orthogonal transformations, you can get your quadratic form to look like

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.$$

Then you can get every eigenvalue  $\lambda_j$  to be 0, 1 or  $-1$ , by rescaling the associated variable  $x_j$ . Then you can permute the order of the variables. So you can get

$$\pm x_1^2 + \pm x_2^2 + \cdots + \pm x_s^2,$$

for some  $s$  between 1 and  $n$ .

**15.1.** Take the complex number with half as much argument, and square root as much modulus.

**15.10.**  $\langle z, w \rangle = 1(-i) + i(2 - 2i) = 2 + i$

**15.16.** Since  $A$  is self-adjoint,

$$\langle Az, z \rangle = \langle z, Az \rangle.$$

If we pick  $z$  an eigenvector, with eigenvalue  $\lambda$ , then the left side becomes

$$\langle Az, z \rangle = \lambda \langle z, z \rangle,$$

and the right side becomes

$$\langle z, Az \rangle = \bar{\lambda} \langle z, z \rangle.$$

Since  $\langle z, z \rangle = \|z\|^2 \neq 0$ , we find  $\lambda = \bar{\lambda}$ .

**15.21.** For an eigenvector  $z$ , with eigenvalue  $\lambda$ ,

$$\langle z, z \rangle = \langle Az, Az \rangle$$

(because  $A$  is unitary)

$$\begin{aligned} &= \langle \lambda z, \lambda z \rangle \\ &= \lambda \bar{\lambda} \langle z, z \rangle \\ &= |\lambda|^2 \langle z, z \rangle. \end{aligned}$$

Since  $z \neq 0$ , we can divide  $\langle z, z \rangle$  off of both sides.

**15.23.**

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{1+2i}{\sqrt{10}} \\ \frac{2-i}{\sqrt{10}} \end{pmatrix}.$$

**15.27.**

a.  $A$  is self-adjoint.

b.

$$\begin{aligned} \lambda = 3 & \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \\ \lambda = 4 & \quad \begin{pmatrix} \frac{1}{2}(1+i) \\ \frac{1}{2}(1-i) \end{pmatrix} \end{aligned}$$

c.

$$\begin{aligned} F &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2}(1+i) \\ \frac{i}{\sqrt{2}} & \frac{1}{2}(1-i) \end{pmatrix} \\ F^*AF &= \begin{pmatrix} 3 & \\ & 4 \end{pmatrix}. \end{aligned}$$





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## List of Notation

$\mathbb{R}^n$	The space of all vectors with $n$ real number entries	15
$p \times q$	Matrix with $p$ rows and $q$ columns	15
$0$	Any matrix whose entries are all zeroes	17
$\Sigma$	Sum	20
$I$	The identity matrix	23
$I_n$	The $n \times n$ identity matrix	23
$e_i$	The $i$ -th standard basis vector (also the $i$ -th column of the identity matrix)	23
$A^{-1}$	The inverse of a square matrix $A$	25
$\det$	The determinant of a square matrix	45
$A^t$	The transpose of a matrix $A$	55
$\dim$	Dimension of a subspace	78
$\ker A$	The kernel of a matrix $A$	81
$\text{im } A$	The image of a matrix $A$	85
$\langle x, y \rangle$	Inner product of two vectors	113
$\ x\ $	The length of a vector	113
$ z $	Modulus (length) of a complex number	134
$\arg z$	Argument (angle) of a complex number	134
$\mathbb{C}$	The set of all complex numbers	134
$\mathbb{C}^n$	The set of all complex vectors with $n$ entries	134
$\langle z, w \rangle$	Hermitian inner product	136
$A^*$	Adjoint of a complex matrix or complex linear map $A$	137



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