Over a Class of geometrical Transformations.[1]

The rapid development of geometry in our century stands, as is well known, in an intimate dependence on philosophical reflections upon the nature of Cartesian Geometry - reflections, which are expounded in their most universal form by *Plücker* in his oldest works.

For one, who has immersed himself in the spirit of Plücker´s works, there is nothing fundamentally new in the idea, that as element for the geometry of the space can be used any curve that is dependent on three parameters. When none-the-less no one, as far as I know, has realised this thought, the ground must probably be sought in that no advantage that might result from this was seen.

I have been brought to a general study of the said theory by my finding that, through a particularly remarkable transformation, the theory of main tangential curves can be brought back to that of rounded curves.

Following Plücker's trail I discuss the equation system:

$$[F_1\ (x\ y\ z\ X\ Y\ Z) = 0, \quad F_2(x\ y\ z\ X\ Y\ Z) = 0],$$

which in one meaning, later to be explained, defines a general reciprocity between two spaces. When in particular the two equations are linear in relation to each system's variables, a projection is obtained by which to each space's points correspond in the other space the lines of a Plücker Line-complex. The simplest among the class of transformations I obtain in this way is the well-known Ampèreish, which hereby is shown in a new light. In particular I study the aforementioned projection, upon which I found a - as it appears to me - *fundamental relation between the Plücker line geometry and a spatial geometry whose element is the sphere.*

While I was occupied with the present thesis I have been standing in a vivid exchange of thoughts with Plücker's pupil, Dr. Felix Klein, to whom I owe many ideas, more, no doubt, than what by quotation I am able to indicate.

I will also notify that this work has many points of contact with my works over

1 The most important aspects of the present thesis I reported to Christiania Science Association in July and October 1870. One may also compare a note of Mr Klein and me in the Berlin Academy's "Monatsbericht" 15 December, 1870.

the imaginaries of plane geometry. When I am not letting these relations be exposed in my narrative here, it is partly because I consider it incidental, and partly that I don't wish to deviate from the customary language of mathematics.[1]

First Section

Over a new Reciprocity of Space

§ 1.

Reciprocity between two planes or two spaces.

1. The Poncelet-Gergonne reciprocity theory can, as is well known in respect of plane geometry, be derived from the equation:

$$X(a_1x + b_1y + c_1) + Y(a_2x + b_2y + c_2) + (a_3x + b_3y + c_3) = 0 \qquad (1)$$

or by the equivalent:

$$x(a_1X + a_2Y + a_3) + y(b_1X + b_2Y + b_3) + (c_1X + c_2Y + c_3) = 0$$

provided that one interprets (x,y) and (X,Y) as Cartesian point co-ordinates for two planes.

If, namely, one uses the term, *conjugate*, of two points (x,y) and (X,Y), whose co-ordinate values satisfy equation (1), one can say that, to a given point (x,y), conjugate points (X,Y) form a straight line that can be perceived as *corresponding* to the given point.

When all points of a given straight line have a mutual conjugated point in the other plane, their corresponding straight lines go through this common point.

The two planes are thus mapped into each other by equation (1) in such a way that to the points of the one plane correspond the straight lines of the other

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1 Guided by the theories expounded in the present thesis, Mr Klein in a recently published note (Gesellschaft d. Wissensch. zu Göttingen, 4 March 1870) brought the Plücker ideas one step forward in that he showed, that the Plücker line geometry - or by my transformation the corresponding sphere geometry - in a remarkable way manifests itself as an illustration of the metric geometry between four variables.

plane. To points of a given line λ correspond the straight lines that go through λ's image point.

But herein lies just the principle of the Poncelet-Gergonne reciprocity theory.

One considers now in the one plane a polygon whose corners are: $(p_1, p_2 \dots p_n)$, and in the other plane the polygon, whose sides: $(S_1, S_2 \dots S_n)$ correspond to these points. From what we have said follows also that the last-mentioned polygon's corners: $(S_1\ S_2)\ (S_2\ S_3) \dots (S_{n-1}\ S_n)$ are projection points of the given sides: $(p_1\ p_2)\ (p_2\ p_3) \dots (p_{n-1}\ p_n)$, that thus the two polygons stand in a reciprocal relation.

By a limit transition one is brought from here to a consideration of two curves c and C, that correspond to each other in such a way that the tangents of the one project themselves as the points of the other. Two such curves are said to be *reciprocal* relative to equation (1).

2. Plücker[1] has based a generalisation of the above presented theory on the interpretation of the general equation:

$$F[x,y\ \ X,Y] = 0 \tag{2}$$

Those to a given point (x,y) [or (X,Y)] conjugated points (X,Y) [or (x,y)] now form a curve C [or c], which is produced by equation (2), when in the same (x,y) [or (X,Y)] are regarded as parameters, (X,Y) [or (x,y)] on the contrary as running co-ordinates.

By equation (2) the two planes are thus projected into each other in such a way, that to the points of the one plane unambiguously correspond the curves of a certain curve-net in the other.

Quite as before it is understood, that to points of a given curve c [or C] correspond the curves C (or c) that go through the given image point.

To a polygon of curves c $(c_1\ c_2 \dots c_n)$ correspond n points: $(p_1\ p_2 \dots p_n)$ which pairwise lie upon those curves C: $(p_1\ p_2)\ (p_2\ p_3) \dots (p_{n-1}\ p_n)$, whose image points are corners of the given curvilinear polygon. Eventually one is here also brought to a consideration of curves σ and Σ in the two planes, that stand in such a mutual relation to each other, that to the points of the one correspond the

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1 Analytisch geometrische Entwickelungen. T.I. Zweite Abth.

curves c [or C] that envelope the other. In general this reciprocity relation is not complete, though, inasmuch as adjunct forms usually appear.

3. *Plücker*[1] bases the general reciprocity between two spaces on the interpretation of the general equation:

$$F\,(x\ y\ z\ \ X\ Y\ Z) = 0.$$

When F is linear in respect of each system's variables, the Poncelet-Gergonne reciprocity between the two spaces' points and planes is obtained.

In the present thesis and especially in the first section of the same, I aim at studying a new reciprocity of space, which is to be considered side-ordered to Plücker's, and that is defined by the equation system:

$$F_1\,(x\ y\ z\ X\ Y\ Z) = 0$$
$$F_2\,(x\ y\ z\ X\ Y\ Z) = 0,$$

where (x y z) and (X Y Z) are perceived as point co-ordinates of two spaces r and R.

§ 2.

A space curve, that depends upon three parameters can be chosen as the element of the geometry of the space.

4. The transformation of geometric postulates that is founded upon the Poncelet-Gergonne or the Plücker reciprocity can - as Gergonne and Plücker have emphasised - be seen from a higher point of view, which we here want to state, because the same applies to our new reciprocity.

The Cartesian geometry, namely, translates any geometric theorem into an algebraic one and thus of the geometry of the plane renders a faithful representation of the algebra of two variables and likewise of the geometry of space a representation of the algebra of three variable quantities.

Now Plücker in particular has directed attention to the circumstance that to Cartesian analytic geometry is attributed a double conditionality.

Descartes produces a system of values of the variables x and y at a *point* in the plane; he has, as one uses to express it, *chosen the point as the element of the*

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1 Although I am unable to provide a quotation, I believe that it is correct to attribute this reciprocity to Plücker.

geometry of the plane, while with the same justification one could use the straight line or any curve at all depending upon two parameters. Now - as regards the plane - the geometrical transformation that is founded upon the Poncelet-Gergonne reciprocity can be perceived as consisting of a transition from a point to a straight line as element, and likewise the Plücker plane reciprocity in the same sense rests upon the introduction of a curve depending upon two parameters as the element of the geometry of the plane.

Further, Descartes produces a quantity-system (x,y) by the point in the plane whose distance from two given axes equals x and y; *he has among the unlimited manifold of possible co-ordinate systems chosen a definite one.*

The progress that geometry has made in the 19th century depends to a large part upon the fact that these two conditions in Cartesian analytic geometry have been clearly recognised as such, and it is accordingly close at hand to exploit these important facts even more.

5. The in the following presented new theories are founded upon the fact, *that one can choose any space-curve which depends upon three parameters as the element of the geometry of the space.* If, for instance, one remembers that the equations of the straight line in space contain four essential co-ordinates, one realises that the straight lines that meet a given condition may be used as the element of a geometry of the space, which - like the ordinary one - gives a faithful representation of the algebra of three variables.

Hereby, however, a certain line-system - *the Plücker line-complex* - is distinguished, and it is as a consequence of this seen that a certain representation of this kind can have only a limited utility. If, however, it concerns a *study of the space relative to a given line-complex*, it may be particularly suitable to choose the straight lines of this complex as space-element. As is well known, in the metric geometry, the infinitely distanced imaginary circle and as consequence hereof the straight lines that intersect the same are marked out, *and therefore there might a priori be some grouns to suppose that, as regards the treatment of certain metric problems, it might be advantageous to introduce these straight lines as element.*

It is to be emphasised that when we, for instance, have just said that it is possible to choose the straight lines of a line-complex as space-elements, this is something different, something more particular if one so likes, than those ideas that lie as a ground for Plücker's last work: *"Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raum-Element"*. Plücker had already drawn attention to the fact that it is possible to create a representation of an algebra that embraces an arbitrary number of variables in that one namely introduces a figure that depends upon the necessary number of parameters as element. Especially he emphasised[1], that the space-line has four co-ordinates, that by choosing the same as space element one thus obtains a geometry for which the space has *four* dimensions.

§ 3.

Curve complex. New geometric representation of partial differential equations of the first order. The main tangent-curves of a line-complex.

6. *Plücker* has used the expression *line-complex* to denote the collection of the straight lines, that satisfy a given condition, and which thus depend upon three unspecified (undetermined) parameters. By analogy herewith, in the following, by *curve-complex* I understand an arbitrary system of space-curves c, whose equations:

$$f_1\,(x\;y\;z\;\;a\;b\;c) = 0,\;\; f_2\,(x\;y\;z\;a\;b\;c) = 0 \qquad (3)$$

contain *three essential constants.*

On differentiation of (3) with respect to x y z and elimination of a, b, c between the two new and the initial equations a result is obtained of the form:

$$f(x\;y\;z\;\;dx\;dy\;dz) = 0 \qquad (4)$$

If here x, y, z are perceived as parameters, dx, dy, dz on the other hand as direction-cosines, each point in the space defined by (4) is associated with a cone, namely, the collection of tangents to those complex-curves c, that go through the point in question. These cones I call *elementary complex-cones*; further I use the designation: *elementary complex-direction* to denote an arbitrary line-element (dx dy dz) that belongs to a complex-curve c. *The collection of the to a point corresponding elementary complex-directions generate the to the point associated elementary complex-cone.*

To a given system (3) or - as one may also say - to a given curve-complex correspond a definite equation: [f = 0]; *on the other hand [f = 0], through the mentioned operations, can be derived from an unlimited manifold of systems* (3).

1 Geometrie des Raumes. n. 258. (1846).

If namely one chooses an arbitrary relation of the form:

$$\psi[x\ y\ z\ dx\ dy\ dz\ \alpha] = 0,$$

where α denotes a constant, and represents

$$\varphi_1(x\ y\ z\ \alpha\ \beta\ \gamma) = 0, \quad \varphi_2(x\ y\ z\ \alpha\ \beta\ \gamma) = 0$$

the integral of the simultaneous system:

$$f = 0, \quad \psi = 0,$$

it is evident, that also $[\varphi_1 = 0, \ \varphi_2 = 0]$ by differentiation relative to x, y, z and elimination of α, β, γ leads to: (f=0).

Any curve of this new complex: $[\varphi_1 = 0, \ \varphi_2 = 0]$ is enveloped by curves c, inasmuch as its elements are all complex-directions.

7. A partial differential equation of the first order between x, y, z is, according to *Monge*, equivalent to the following problem: to find the general surface which in each of its points touches a cone associated with the point in question and whose general equation in plane co-ordinates is produced by just the given partial differential equation.

Lagrange and *Monge* have led this problem back to the determination of a definite curve-complex - the so called *characteristic curves* - inasmuch as they have shown, that one always gets an integral surface by adjoining to a surface a collection of characteristic curves each of which intersects the nearest preceding one.

One may note that the equation:

$$f(x\ y\ z\ \ dx\ \ dy\ \ dz) = 0,$$

which the characteristic curves, according to the aforesaid determine, is to be considered as equivalent to the partial differential equation itself, inasmuch as both these equations are the analytic definition of the same three-fold infinity of cones.

8. *A general geometric interpretation of partial differential equations of the first order between x y z is obtained by showing that the task: finding the* *general surface which at all its points has a three-point contact with a curve of a given curve-complex - whereby it is implied, however, that the said curve is not in its whole extension residing upon the surface - finds its analytic expression in a partial differential equation of the first order. When, further:*

$$f(x\ y\ z\ \ dx\ dy\ dz) = 0$$

is the equation, that the characteristic Curves determine, any curve-complex whose equations satisfy ($f = 0$) will stand in the said geometrical relation to the given partial differential equation.

One considers that a complex of curves c is given, which satisfies ($f = 0$) and analytically expresses the requirement that a surface [$z= F(x\ y)$] at each of its points has a three-point contact with a curve c, *without, however, excluding the possibility of an even more intimate contact.* It is easy to see that to determine z a partial differential equation of the second order ($\delta_2 = 0$) is obtained.[1] But any surface which is generated by infinitely many c, apparently satisfies ($\delta_2 = 0$), and hence its general integral with two arbitrary functions is known. By analytical deliberations of great simplicity - albeit formally of some breadth - I intend to show that the partial differential equation of the first order ($\delta_1 = 0$), that corresponds to ($f = 0$), satisfies ($\delta_2 = 0$). When now apparently ($\delta_1 = 0$) in general is not included in the aforementioned integral, ($\delta_1 = 0$) is a *singular* integral of ($\delta_2 = 0$).

The equation: [$f(x\ y\ z\ \ dx\ dy\ dz) = 0$] gives by differentiation:

$$f'_x dx + f'_y dy + f'_z dz + f'_{dx} d^2x + f'_{dy} d^2y + f'_{dz} d^2z = 0, \qquad (6)$$

whereby ($dx\ dy\ dz\ \ d^2x\ d^2y\ \ d^2z$) are to be regarded as belonging to an arbitrary curve, that satisfies: ($f = 0$). In particular (6) is valid for [$\delta_1 = 0$]'s characteristic curves, and in that we denote these by an index, we obtain:

$$f'_{x_1} dx_1 + \dots f'_{dx_1} d^2x_1 + \dots = 0.$$

Now remarking that any curve that touches one of ($\delta_1 = 0$)'s integral

1 ($\delta_2 = 0$) has the form: [$A(rt - s^2) + Br + Cs + Dt + E = 0$]. One may compare with a dissertation by Boole in Crelles Journal. Bd. 61.

surfaces: (U = 0) satisfies the equation:

$$\frac{dU}{dx} dx + \frac{dU}{dy} dy + \frac{dU}{dz} = 0, \tag{7}$$

that further any curve, that by (U = 0) has a three-point contact, in addition fulfills the relation:

$$\frac{d^2U}{dx^2}(dx)^2 + \ldots \left(\frac{dU}{dx}\right) d^2x \ldots = 0, \tag{8}$$

it is seen, that any characteristic curve, that lies upon (U = 0), satisfies (7) as well as (8).

But at each of its points (U = 0) touches the associated cone of the system: (f = 0), and thus apply the equations:

$$f'_{dx} = \rho \frac{dU}{dx'}, \quad f'_{dy} = \rho \frac{dU}{dy'}, \quad f'_{dz} = \rho \frac{dU}{dz'},$$

in which ρ denotes an undetermined proportionality factor. Thus the accentuated equation (8) transforms into the following

$$\rho\left[\frac{d^2U}{dx_1^2}(dx_1)^2 + \ldots\right] + \left[f_{dx_1} d^2x_1 + \ldots\right] = 0.$$

But we know, that:

$$f'_{x_1} dx_1 + \ldots + f'_{x_1} d^2x_1 + \ldots = 0$$

and hence is:

$$\rho\left[\frac{d^2U}{dx_1^2} + \ldots\right] = f_{x_1}\left[d\,x_1 + \ldots\right]$$

or by exclusion of the now unneccesary index:

$$\rho\left[\frac{d^2U}{dx^2}\,d\,x^2\ +\ldots\right] = f_x\,dx + \ldots$$

Now, however,

$$\rho\left[\frac{dU}{dx}\,d^2x + \frac{dU}{dy}\,d^2y\ +\ \frac{dU}{dz}\,d^2z\ \right] = \left[f_{dx}\,d^2x + \ldots\right].$$

and thus the equation:

$$\rho\left[\frac{dU}{dx}\,d^2x + \frac{dU}{dy}\,d^2y\ +\ \frac{dU}{dz}\,d^2z\ +\ \frac{d^2U}{dx^2}(dx)^2 + \ldots\ \right] =$$

$$= f_x\,dx + f_y\,dy + f_z\,dz + f_{dx}\,d^2x + f_{dy}\,d^2y + f_{dz}\,d^2z,$$

whose right and left parts thus simultaneously vanish.

Our expansions show, that any curve, that satisfies ($f = 0$), and that touches one upon ($U = 0$) lying characteristic curve with the said surface has a three-point contact; ($\delta_1 = 0$) is thus a singular integral of ($\delta_2 = 0$).

Finally we show that ($\delta_2 = 0$) does not allow any other singular integral.

On an integral surface I of ($\delta_2 = 0$), every point is namely associated with a direction - the respective, three-point-contacting c's tangent. If it is now implied, that I is not generated by a manifold c, so goes through each point of I two converging c, that both touch upon the surface in the point in question. But in consequence, I in each of its points is contacted by the corresponding elementary complex-cone; I satisfies the equation: ($\delta_1 = 0$).

9. *Corollary. The determination of the most general surface that in each of its points has an - not upon the surface lying - main tangent, belonging to a given line-complex, depends upon the solution of a partial differential equation of the first order, whose characteristic curves are enveloped by the complex's lines. The said curves appear in this case as main tangent-curves on the integral surfaces.*

We present an independent geometrical proof of this corollary.

The partial differential equation, whose characteristics are enveloped by a given line-complex's lines, is according to Monge's theory the analytic expression of the following problem: to find the most general surface which at each of its points touches the complex-cone corresponding to the point. But when a curve's tangents belong to a line-complex, the same osculation-plane is the tangent plane to the corresponding complex-cone, and thus our characteristic curves' osculation plane is the tangent plane for all integral surfaces which contain the curve at hand. Here a couple of further remarks are required, which, however, may be a repetition of what we have said before.

Any line-complex determines, according to the above-mentioned, a complex of curves that are enveloped by the line-complex's lines, and which have the property to be main tangent-curves on any surface that is generated by a system of these curves, each of which intersects the preceding. *This complex of curves we in the following designate the line-complex's main tangent-curves.*

I owe Mr. Klein the acknowledgement, that the congruence of straight lines, that Plücker calls a line-complexe's *singular lines*, belong to the said curve-complex. If the given complex is formed by a surface's tangents [or by the straight lines that intersect a curve], then all the lines of the line-complex are singular lines and hence also main tangent-curves.

§ 4

The equation system : $F_1(x\ y\ z\ \ X\ Y\ Z) = 0$, $F_2(x\ y\ z\ \ X\ Y\ Z) = 0$, determines a reciprocity between two spaces.[1]

10. We begin a study of the spatial reciprocity determined by the equations:

$$\left.\begin{aligned} F_1(x\ y\ z\ \ X\ Y\ Z) = 0 \\ F_2(x\ y\ z\ \ X\ Y\ Z) = 0 \end{aligned}\right\} \qquad (9)$$

when in the same (x y z) and (X Y Z) are perceived as point co-ordinates of two spaces r and R.[2]

1 One compares this paragraph with § 1.

2 Things, that belong to the space r, we as a rule denote by small letters; on the other hand versals are used for everything belonging to R.

When the term conjugated is used of two points, whose co-ordinate values (x y z) and (X Y Z) satisfy the relations (9), one can say, that the to a given point (x y z) conjugated points (X Y Z) generate a curve C, which is formulated by (9), when in the same (x y z) are regarded as parameters, (X Y Z) on the other hand as running co-ordinates.

To the curves of the space r, thus unambiguously correspond the curves C of a definite curve-complex in R, and likewise appears in r a complex of curves c, which stand in the same relation to R's points.

A curve c's points have a mutual conjugated point in R, and in consequence their corresponding cuirves C go through this mutual point.

The two spaces are thus mapped by the equation system (9) into each other in such a way, that to each space's points unambiguously correspond in the other the curves of a definite complex. When a point describes a complex-curve, the complex-curve corresponding to the point turns round[1] the image-point of the intersected one.

11. It is now possible to show, that the equations (9) determine a general reciprocity between figures in the two spaces and especially between curves that are enveloped by complex-curves c and C.

When two curves of the one complex have a mutual-point - which is obviously not so in general - their imagepoints lie upon a complex-curve. Note in particular, that two endlessly close-lying complex-curves, which intersect each other, project themselves as two points whose infinitesimal connection-line is an elementary complex-direction.

One now considers a curve σ in r, that is enveloped by curves c, and all curves C that correspond to σ's points. Two consecutive of these C would, after what we have just said, intersect each other, and thus their collection determines an envelope-curve Σ.

It is further apparent, that when a point runs through Σ, the corresponding c will envelop a curve σ^1, and it can be shown that σ^1 is precisely the originally given curve σ.

One may namely consider on the one hand a curved polygon formed by

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1 The term, "turns round" is unfortunate inasmuch as, of course, a turn associated by a change of form is meant.

complex-curves (c_1 c_2 c_3...c_n), whose corners are (c_1 c_2) (c_2 c_3)...(c_{n-1} c_n) - and on the other hand the imagepoints of the curves c: (P_1 P_2...P_n), which obviously pairwise: (P_1 P_2) (P_2 P_3)...(P_{n-1} P_n) lie upon complex-curves c, those, namely, which correspond to the corners of the given polygon. The new polygon in R and the given one thus stand in a fully reciprocal relation to each other.

By a limit transition one obtains in the two spaces curves, which are enveloped by complex-curves c and C, and which stand in such a mutual relation to each other, that to the points of the one correspond the complex-curves that envelope the other.

A curve enveloped by complex-curves is thus projected in a double sense as another, likewise by complex-curves enveloped curve, which we say is the rendered *reciprocal* relative to the equation system (9).

One may also notice that elementary complex-directions (dx dy dz) (dX dY dZ) arrange themselves pairwise as reciprocals, and that thus two rounded lines enveloped by complex-curves, that touch upon each other, are transformed in the other space as curves that stand in the same mutual relation.

12. Also between other space-forms equations (9) determine a correspondence, which, however, in general is not a complete reciprocity.

A given surface f's points are, namely, projected in R as a double infinity of curves C; as a curve-congruence, whose focus-surface[1] *is F. Likewise correspond to F's points a congruence of curves c, whose focus-surface, as we will later see, contains f as reducible part.*

The elementary complex-cones whose apex-points lie upon the surface f, intersect the corresponding tangent-planes of this in n straight lines - by n is understood the said complex-cones' order - and thus in each point of f determine n elementary complex-directions. The continuous succession of these directions forms an f n-fold enveloping curve set, which is all enveloped by complex-curves c. *The geometric locus for this curve-collection's reciprocal curves, or, as we may also say, collection of image-points of the c, which touch upon f, forms the focus-surface F.*

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1 In analogy with the terminology used for line-congruences, I understand by this curve-congruence's focus-surface: the geometric locus for intersection-points between infinitesimally close-lying curves C. If the curve-congruence is thought of as defined by a partial linear differential equation, its focus-surface is just what in general one calls the differential equation's singular integral.

In order to prove this, one remembers, that two infinitesimally close-lying, each other intersecting curves C project themselves as two points, whose infinitesimal connection-line is an elementary complex-direction. Now there go from a point p_0 on f, n complex-directions, and thus p_0's image-point C_0 is intersected at n points of the adjacent C, which belong to our curve-congruence earlier considered - in those n points, namely, which correspond to the n complex-curves x, which touch the surface f in the point p_0. F's points are thus the image of the c that touch f.

When now f has a general location in the space r, a c, that touches f in a point, will in general not have more contacts with the same. But all these c form a congruence in which each c touches the focus system in N points - by N is understood the order of the elementary complex-cones in R -, and thus, as said above, our congruence's focus-system decomposes in f and a surface φ, which is touched by each c in (N-1) points.

If thus the correspondence determined by equations (9) between surfaces in r and R is to be a complete reciprocity, it is necessary and sufficient that n and N both equal 1. *In general, the reciprocity-relation is incomplete inasmuch as analogous operations on the one hand transform f in F, and on the other, F in the collection of f and φ.*

The above deliberations are also valid, when f, and as a consequence hereof, F are surface-elements; if f is infinitesimal in one direction alone, the same is the case with F.

One considers finally a curve k, which is not enveloped by complex-curves c, together with the surface F, that is generated by all C, which correspond to k's points. The points of a c are transformed to the through C's image-point going curves c, and thus correspond to F's points the collection of curves c, that intersect k. *The interrelation between k and F is thus a double one.*

Equations (9) which map the two spaces into each other, transform the according to the above-mentione given spaceforms to new ones that stand in a reciprocal relation to the given ones, and can thus serve to transform geometrical theorems and problems. For a special form of the equations (9) we will make important uses of this transformation-principle later on.

§ 5

13. Legendre has given a general method to - in the language of modern

1 One compares also: Plücker, Geometrie des Raumes. § 2. (1846).

geometry - transform a partial differential equation between point co-ordinates x y z into a differential equation between plane-co-ordinates t, u, v, or - as one may also say - between point-co-ordinates t, u, v, of a space, that stands in a reciprocal relation to the given one.

When the curves c are introduced as elements of the space r, it is in a similar way possible to transform a partial differential equation between x, y, z into a differential equation between the new space-element's co-ordinates X Y Z, whereby one may also interpret X, Y, Z as point-co-ordinates of the space R, - a notion that will prevail in our exposition.

Hence, given an arbitrary partial differential equation of the first order between x, y, z and all surfaces ψ that generate a so called *"integral complet"* of the same, one should bear in mind, that any other integral surface f can be represented as envelope of single-infinitely many ψ.

One considers further in the space R all surfaces Ψ and Φ which correspond to the surfaces ψ and f. We will soon show, that any F is the envelope-surface of single-infinitely many Ψ, that thus the surfaces F satisfy a partial differential equation of the first order, for which all Ψ form an "integral complet".

Two given surfaces in r, which possess a mutual surface-element, namely project themselves in R as surfaces which touch each other, and likewise surfaces that possess infinitely many mutual surface-elements are transformed in surfaces, which like the given one touch each other along a curve.

This provided, one considers an integral surface f_0 and all single-infinitely many ψ_0, that touch the same along a characteristic curve, and finally the corresponding F_0 and Ψ_0. It is clear that F_0 is touched by each Ψ_0 along a curve, and F_0 thus is the envelope-surface of all Ψ_0.

14. A particular interest is offered by the fact that the *partial differential equation,* that is transformed, is precisely that, *which is determined by the complex-curves c* (compare § 3); in that case it can be shown, that the corresponding differential equation between X, Y, Z is decomposed into two equations, of which one *is just that which corresponds to the complex-curves C.*

Consider an integral surface by the given differential equation between x y z, and all to the surface f's points corresponding complex-cones. These cones after § 4 in each point of f determine n complex-directions, of which *in casu* two coincide; thus the in § 4 on the surface f considered collection of curves, which are enveloped by complex-curves c, decompose into f's characteristic curves and a curve-system, that covers f (n-2)fold.

The curve-congruence in the space R corresponding to f's points thus has a focus-system, which is decomposed into two surfaces, of which the one - that we are calling Φ - is touched by each c in two coinciding points, while (n-2) contact-points fall upon the other. *The surfaces Φ thus satisfy the partial differential equation that, after the theorem in § 3, are determined by the complex-curves C.*

Now noting, that Φ is the geometric locus for the reciprocal curves of f's characteristic curves, it is seen that the two integral surfaces f_1 and f_2, which touch each other along a characteristic curve k, are transformed into two surfaces Φ_1 and Φ_2,which touch each other along k's *reciprocal* curve; k is namely enveloped by complex-curves c.

The characteristic curves for the two partial differential equations which, after § 3, are determined by the curve-complexes c and C, are reciprocal curves relative to the equation-system (9).

15. The theorem just stated gives the following general method for the transformation of partial differential equations of the first order.

One determines after the customary method the equation:

$$f(x\ y\ z\ dx\ dy\ dz) = 0,$$

which the given partial differential equation's characteristics satisfy, and choose an arbitrary relation of the form:

$$\psi(x\ y\ z\ dx\ dy\ dz X) = 0,$$

where X denotes a constant. The simultaneous system:

$$f = 0,\ \psi = 0$$

be integrated in the form:

$$F_1(x\ y\ z\ X\ Y\ Z) = 0.\ \ F_2(x\ y\ z\ X\ Y\ Z),$$

where Y and Z are the constants introduced by the integration.

By differentiation and elimination one obtains a relation of the form:

$$F_3(X\ Y\ Z\ dX\ dY\ dZ) = 0,$$

which we regard as an equation for the characteristic curves of a definite partial differential equation:

$$F_4\left(X\ Y\ Z \frac{dZ}{dX}\ \frac{dZ}{dY}\right) = 0.$$

Our earlier expansions show that ($F_4 = 0$), that is derived of ($F_3 = 0$) according to the ordinary rules, and the given partial differential equation stand in such a mutual interrelation, that if the one can be integrated, so is the other also open for treatment.

One may draw from this general conclusions on the reduction in degree of partial differential equations of the first order, defined by a complex of curves, the order of which is given.

Thus, for example, any partial differential equation of the first order, that is defined by a line-complex (§ 3), may be transformed into a partial differential equation of the second degree.[1]

Likewise, any partial differential equation, defined by a cone-intersection-complex, can be transformed into a differential equation of the 30th degree.[2]

§ 6.

Over the most general transformation that turns surfaces that touch each other into similar surfaces.

16. In the study of partial differential equations, an important role is played by transformations that can be expressed in the form:

$$X = F_1(x\ y\ z\ p\ q),\ Y = F_2(x\ y\ z\ p\ q),\ Z = F_3(x\ y\ z\ p\ q).$$

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1 This reduction is due to the fact that each line of a line-congruence touches the focus-system in 2 points (§ 4, 12).

2 The number 30 comes up as product of 6 and (6-1); 6 is the number of points, in which the focus-system of a focus-intersection-congruence is touched by each focus-intersection.

By p and q one as usual understands the partially derived: dz/dx, dz/dy; likewise P and Q denote dZ/dX and dZ/dY.

In the following we would consider the instance where the functions F_1, F_2 and F_3 are chosen in such a way that P and Q also only depend of (x y z p q):

$$P = F_4(x\ y\ z\ p\ q); \quad Q = F_5(x\ y\ z\ p\ q).$$

In that we imply that, from the above 5 equations, a relation between (X Y Z P Q) cannot be derived, note also that each separate quantity (x y z p q) can be expressed as a function of (X Y Z P Q).

When x y z and X Y Z are perceived as point co-ordinates for r and R, one can say, that by a transformation of this kind is defined *a correspondence between the two spaces' surface-elements, and nota bene the most general.* We will show, that *these transformations fall into two distinct, side-ordered classes, of which the one*[1] *corresponds to the Plücker reciprocity, while the other corresponds to the by me propounded reciprocity.*

By elimination of p, q, P and Q between the five equations:

$$X = F_1,\ Y = F_2,\ Z = F_3,\ P = F_4,\ Q = F_5$$

two essentially different situations may occur. Either only an equation between (x y z X Y Z) is obtained, or two relations exist between these quantities. (The existence of *three* mutually independent equations between the two spaces' point-co-ordinates requires the transformation in question to be a *point*-transformation.)

But it is known, that the equation:

$$F(x\ y\ z\ X\ Y\ Z) = 0$$

always defines a reciprocal correspondence between the two spaces' surface-elements; and likewise I have in the foregoing shown, that the equation-system:

$$F_1(x\ y\ z\ X\ Y\ Z) = 0,\ F_2(x\ y\ z\ X\ Y\ Z) = 0$$

always determines a transformation, that turns surfaces, that touch each other in like surfaces.

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1 Compare: Du Bois-Reymond, Partielle Differential-Gleichungen. 75 - 81.

Hereby my proposition is proved.

At this point I will draw attention to the fact that these transformations possess the remarkable ability to project an arbitrary differential equation of the form: $[A(rt - s^2) + Br + Cs + Dt + E = 0]$, in which A, B, C, D depend only on x, y, z, p, q into an equation of the same form. Inasmuch as the given equation satisfies a general first integral, the same is of course also the case with the new equation. (Compare *Boole's* thesis in Crelle's Journal Bd. 61).

Second Section

The Plücker Line-Geometry can be transformed into a Sphere-Geometry

§ 7.

The two curve-complexes are line-complexes.

17. When we imply that the equations, that map the two spaces into each other, are linear in relation to any system variables:

$$(10)\quad \begin{cases} 0 = X(a_1x+b_1y+c_1z+d_1) + Y(a_2x+b_2y+c_2z+d_2) + Z(a_3x+b_3y+c_3z+d_3) + (a_4 + \dots) \\ 0 = X(\alpha_1x+\beta_1y+\gamma_1z+\delta_1) + Y(\alpha_2x+\beta_2y+\gamma_2z+\delta_2) + Z(\alpha_3x+\beta_3y+\gamma_3z+\delta_3) + (\alpha_4x+\beta_4y+\gamma_4z+\delta_4). \end{cases}$$

the points conjugate to a given point in the other space obviously generate a straight line. The two curve-Complexes are Plücker line-complexes[1], and in consequence the equations (10) determine a correspondence between r and R, that possesses the following characteristic properties:

a) *To each space's points correspond unambiguously the lines of a line-complex in the other.*

b) *When a point describes a complex-line, the corresponding line in the other space turns around the intersected's image-point.*

c) *Curves that are enveloped by the two complexes' lines, arrange themselves together pairwise as reciprocals in such a way that the tangents*

1 Regarding the theory of line-complexes I assume as known: 1) *Plücker*, Neue Geometrie des Raumes, gegründet auf etc....1868-69; 2) *Klein*, Zur Theori der Complexe.....math. Annalen. Bd. II.

of each correspond to the points of the other.

d) To a surface f in the space r is associated a surface F in R for two reasons. On the one hand, F is the focus-surface of the line-congruence whose image is f; on the other F's points correspond to those of f's tangents which belong to the line-complexes in r.

e) On the surfaces f and F just mentioned all curves arrange themselves together pairwise, conjugated in such a way, that to one upon f [or F] lying curve's points correspond in the other space a line-surface, which contains the conjugated curves and after the same touches F [or f].

f) To a curve upon f, which is enveloped by the line-complexes lines, corresponds as conjugated a likewise by complex-lines enveloped curve on F, and these curves are reciprocal curves, in the sense stated under (c).

Any one of the equations (10) determines an an-harmonic correspondence between points and planes in the two spaces, and thus each of our line-complexes may be defined as collections of an-harmonically corresponding planes' intersection-lines - or as an-harmonically corresponding points' intersection-lines. But the complex of the second degree here defined is according to Mr. *Reye* identical to the line-system that initially *Binet* has considered as the collection of a material body's stationary revolution-axes and that later on numerous mathematicians, especially *Chasles* and *Reye*, have studied.

When the constants in equations (10) are particularised, the two complexes can either get a special status - they may for example coincide, the case of which Mr. Reye has treated in his *"Geometrie des Lages, 1868"*, second part, in that he simultaneously set forth the equations obeying (a) and (b) - or they may themselves be particularised. Without entering a discussion of all the possible special-varieties, I wish to stress the two most important degenerations:[1]

Both complexes can be *special, linear*. This case leads to the well-known *Ampere's* transformation, which can thus be regarded as dependant upon the fact that one introduces as space-element, instead of the point, the collection of straight lines which intersect a given line.

The one complex may degenerate into the collection of straight lines, that

1 Lie, "Repräsentation der Imaginæren etc. Christiania Vidensk.-Selskab 1869. Februar og August". The in the mentioned dissertation's §§ 17 and 27-29 treated spatial transformation is identical to the one, I treat in the present Paragraph. In § 25 I explicitly stress the first of the degenerations here reported.

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intersect a given cone-section. In that case, the other complex is a general linear complex. I here wish to mention, that Mr *Nöether* (Götting. Nachr. 1869) has just reported a projection of the linear complex in a point-space, which is identical to the one we consider here. The fundamental notion for us: that *each* space contains a complex, whose lines are mapped as the other space's curves, is not expressed in Mr. Nöether's brief note. - It is this degeneration that we wish to study in the following, under the condition that the fundamental cone-intersection is the infinitely distanced imaginary circle.

18. We have found that the two curve-complexes are line-complexes, when the transformation-equations are linear in respect of any variable system, and we are hereby led to investigate whether this sufficient condition is necessary.

When the one complex is a general line-complex, the corresponding curve-complex's elementary complex-cones must be decomposed in cones of the second degree. The proof (§4, 12) of this lies in the fact that a line-congruence's lines touch the focus-surface in *two* points. When the one complex is a special line-complex, the corresponding curve-complex's elementary complex-cones in the other space are decomposed into plane bundles.

Thus, when both complexes would be line-complexes, the elementary complex-cones in both spaces must decompose into cones of the 2nd or 1st degree. But when a line-complex's cones always decompose, the complex itself is reducible,[1] and thus it is shown that, when two line-complexes are transformed into each other in the previously stated way, either both must be of 2nd degree, or the one a special complex of 2nd degree and the other linear, or both special linear complexes. All these three cases are represented by the equation-system (10), and we wish indicate *that (10) defines the most general mutual transformation of two line-complexes.*

When, namely, both complexes are of 2nd degree, it can be shown, that the singularity-surface cannot be a *curved* surface.

From each point of the surface in question emanate two plane bundles, whose lines project in the other space as a straight line's points. It follows that all lines of one bundle correspond to one and the same point in the other space.

But the collection of lines which do not have an independent mapping cannot

1 I don't know of any proof of this proposition, which, however, is reported to me as certain.

form a complex, at the most a congruence or a number of congruences. When, however, the collection of plane ray-bundles which emanate from all points of a *curved* surface necessarily form a complex, our proposition, that the singularity-surface cannot be a *curved* surface, is proved.

When two 2nd degree complexes are transformed into each other - in which case neither of them can be a special complex - in both the singularity-surface must consist only of planes, and in consequence both systems are such as those Binet first examined.

When a 2nd degree complex and a linear complex are transformed into each other, two cases are conceivable: the 2nd degree complex could be formed by lines, which intersect a conesection - this occasion according to the aforesaid actually exists -; the 2nd degree complex could consist of all a 2nd degree surface's tangents. I have through considerations, that have something in common with those I use in § 12, proved that this second case does not exist; because I could in that event, from the fact that a linear complex can be turned into itself by a three-fold infinity of linear transformations permutable between themselves deduce, that the same must be the case with the 2nd degree surface, which, however, is not how the matter stands.

§ 8.

Reciprocity between a linear complex and the colletion of straight lines, which intersect the endlessly distanced imaginary circle.

19. In the following we subject to a closer study the equation system:

$$\left.\begin{aligned} -\frac{\lambda}{2B} Zz &= x - \frac{1}{2A}(x + iY) \\ \frac{1}{2B}(x - iY)z &= y - \frac{1}{2\lambda A} Z, \end{aligned}\right| \; i = \sqrt{-1} \quad \Bigg| \qquad (11)$$

which is linear relative to both variable systems, and which after § 7 determines a correspondence between two line-complexes. First we wish to seek these complexes' equations in the Plücker line co-ordinates.

Plücker writes the straight line's equations in the form:

$$rz = x - \rho, \;\; sz = y - \sigma,$$

where he considers the five quantities: r, ρ, s, σ, (rσ - sρ) as line-co-ordinates. The equations (11) thus reproduce, provided that one perceives X, Y, Z as parameters in the same, the system of straight lines, whose co-ordinates satisfy the relations:

$$r = -\frac{\lambda}{2B} Z, \quad \rho = \frac{1}{2A}(x + iY),$$

$$s = \frac{1}{2B}(x - iY), \quad \sigma = \frac{1}{2\lambda A} Z,$$

which by elimination of X, Y and Z give as our complex's equation:

$$\lambda^2 A\sigma + Br = 0 \qquad (12).$$

The line-complex in the space r is thus a linear complex and it is a general linear complex which - as one may notice - contains the xy-planes endlessly distanced straight line.

To determine the line-complexes in R, one replaces the system (11) by the equivalent:

$$\left(\frac{\lambda A}{2B} z - \frac{B}{2\lambda Az}\right) Z = X - \left(Ax + B\frac{y}{z}\right)$$

$$\frac{1}{i}\left(\frac{\lambda A}{2B} z + \frac{B}{2\lambda Az}\right) Z = Y - \frac{1}{i}\left(Ax - B\frac{y}{z}\right),$$

which, by combination with the equations of the straight line in R:

$$RZ = X - P; \; sZ = Y - \Sigma \qquad (13)$$

give:

$$R = \frac{\lambda A}{2B} z - \frac{B}{2\lambda Az} Z, \quad P = Ax + B\frac{y}{z},$$

$$S = \frac{1}{i}\left(\frac{\lambda A}{2B} z + \frac{B}{2\lambda Az}\right), \quad \Sigma = \frac{1}{i}\left(Ax - B\frac{y}{z}\right),$$

and thus is found as equation of the line-complexes in R:

$$R^2 + S^2 + 1 = 0 \tag{14}$$

According to (13), however:

$$R = {dX}/_{dZ}\ ,\ S = {dY}/_{dZ}\ ,$$

and as a consequence, (14) can also be written in the form:

$$dX^2 + dY^2 + dZ^2 = 0. \tag{15}$$

The line-complexes in R are thus formed by the imaginary straight lines, whose length equals zero, or as one may also say, of those lines that intersect the endlessly distanced imaginary circle.

The equations (11) transform the two spaces into each other in such a way that to r's points correspond in R the imaginary straight lines whose length equals zero, while R's points transform as the lines of the linear complex (12).

One sees that, when a point runs through a line of this linear complex, the corresponding straight line in R describes an infinitesimal sphere - a point-sphere.

20. According to the general theory for reciprocal curves, as expounded in § 4, one can, when a curve is known, by simple operations find the image-curve that is enveloped by the other complex's lines. Now *Lagrange* has engaged himself with the most general determination of space-curves whose length equals zero, whose tangents thus possess the same property. He has found these curves' general equation, and thus it is by the aforesaid *also possible to specify general formulas for the curves whose tangents belong to a linear complex.*

In order not to depart from our aim we will not enter here into a closer consideration of the simple geometrical relations that occur between reciprocal curves in the two spaces.[1]

Our earlier expositions of the correspondence between surfaces in the two spaces are now somewhat modified thereby, in that all congruences of straight

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1 When the given curve of length equalling zero has an apex, the corresponding curve in the linear complex has a stationary tangent. On the whole, stationary tangents occur as ordinary singularities, when curves are perceived as line-generations, that is, as enveloped by a given line-complex's lines.

lines, which intersect the endlessly distanced circle possess a mutual focus-curve - namely, that circle - and that furthermore a line-congruence's straight lines only touch the focus-surface in two points.

Because if one imagines a surface F given in R, and that f is the geometric locus for the points in r that correspond to F's tangents of length zero, then also, inversely, F is the *complete* geometric locus for the imagepoints of the straight lines in the linear complex (12), touching f.

On the other hand, the case stands as in the general instance, when a surface φ of general location in r is given, inasmuch as the straight lines of the linear complex (12) which touch φ, in addition envelope another surface ψ, φ's so called reciprocal polarity relative to (12).

The above mentioned line-system transforms in R as a surface Φ, that obviously is the focus-surface for two congruences - firstly for the collection of straight lines, of length zero that correspond to φ's points - secondly for the other collection of the lines that stand in the same relation to ψ's points.

Φ's tangents of length equal to zero thus decompose into two systems, or as one can also say: Φ's geodetic curves of length equalling zero form two distinct sets.

En passant we note that the determination of *the curves that are enveloped by the straight lines of a congruence belonging to a linear complex, according to our general theories can be traced back to the searching out on the image-surface F of the geodetic curves, whose length equals zero.* For these curves are reciprocal between each other (17, f) relative to the equation system (11).

21. In the following we will make use of the ensuing theorems a few times:

a. A surface F of n^{th} order, which contains the endlessly distanced imaginary circle as p-double line is the image of a congruence, whose order, and in consequence also class, equals (n - p).[1]

An imaginary line of length equal to zero namely intersects F in (n - p) points

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1 I will at this occasion express an, as it seems, nowhere explicitly articulated, but none-the-less for any one mathematician, who deals with line-geometry, well-known lemma: *For a congruence that belongs to a linear complex, the order is always equal to the class.*

which lie in the finite space, and thus are always given (n - p) lines of the image-congruence, that run through a given point - or that lie in a given plane in the space r.

b. A curve C of n^{th} order, which intersects the infinitely distanced circle in p points, is projected in r as a linesurface of the $(2n - p)^{th}$ order.

A straight line of the linear complex (12) namely intersects the mentioned linesurface in as many points as the number of - not infinitely distanced - mutual-points between the curve C and an infinitesimal sphere.

§ 9.

The Plücker line geometry can be transformed into a sphere-geometry.

22. In this paragraph we establish a *fundamental relation that takes place between the Plücker line-geometry and a geometry whose elements are the space's spheres.*

Because equations (11) transform the space r's straight lines into the space R's spheres, and that for a double rendition (12).

On the one hand the straight lines of the complex l_2, which intersect a given line l_1, and thus likewise the same's reciprocal polarity l_2 relative to (12), transform according to an earlier lemma (21,b) as a sphere's points; on the other hand the lines l_1 and l_2's points are transformed into this sphere's rectilinear generatrices.

By the following analytic expositions one can find the relations, that take place between l_1 and l_2's line-co-ordinates X', Y', Z' and radius H'.

When

$$\rho z = x - r, \quad \sigma z = y - s$$

are the line l_1 [or l_2's] equation, and it is remembered that the linear complex (12)'s straight lines can be expressed by:

$$-\frac{\lambda}{2B} Zz = x - \frac{1}{2A}(X + iY)$$
$$\frac{1}{2B}(X - iY)\,z = y - \frac{1}{2\lambda A} Z,$$

it is seen, that one must eliminate x y z between these four lines in order to subject the just mentioned lines to the condition of intersecting l_1. One hereby finds the following relation:

$$[Z-(A\sigma\lambda-B/_{\lambda}\, r)]^2 + [X-(A\rho+Bs)]^2 + [Y- i(Bs-A\rho)]^2 =$$

$$[A\lambda\sigma+B/_{\lambda}\, r]^2 \qquad (16)$$

between these linear parameters (X, Y, Z) or, as one may also say, between the imagepoints' co-ordinates.

The immediate interpretation is that this equation confirms what we have said above, and in addition yields the following formulas:

$$X' = A\rho + Bs \qquad iY' = A\rho - Bs$$

$$Z' = \lambda A\sigma - B/_{\lambda}\, r \qquad \pm H' = \lambda A\sigma + B/_{\lambda}\, r \qquad (17)$$

or the equivalent

$$\rho = \frac{1}{2A}\,(X' + iY') \qquad s = \frac{1}{2B}\,(X' - iY')$$

$$(18)$$

$$\sigma = \frac{1}{2\lambda A}(Z \pm H') \qquad r = -\frac{\lambda}{2B}(Z \pm H')$$

In which one may without disadvantage exclude the sphere-co-ordinates X'Y'Z'H''s accents, in that for our perception the space R's points are spheres, whose radius equals zero.

The formulas (17) and (18) show, that a straight line in r transforms as an unambiguously determined sphere in R, while to a given sphere correspond two lines in r:

$$(X, Y, Z, + H) \qquad (X, Y, Z, - H),$$

which are each other's reciprocal polarities relative to the linear complex:

$$H = 0 = \lambda A\sigma + B/_{\lambda}\, r, \qquad (12)$$

(17) and (18) evidently express, when H is defined as zero, the unambiguous association between the complexe (12)'s straight lines and the space R's point-spheres.

A plane - that is, a sphere, whose radius is infinitely large - projects as two straight lines (l_1 and l_2), which intersect the xy-planes endlessly distanced straight lines, and according to the above are l_1 and l_2's points the projection of those imaginary lines in the given plane, which go to the same's endlessly distanced circle-points.

Note in particular, that to a plane, touching the endlessly distanced imaginary circle, corresponds a line of the complex (H = 0) parallel to the xy plane.

23. *Two lines l_1 and λ_1, which intersect each other, transform as spheres, between which contact takes place.*

For l_1 and λ_1's polarities relative to (H = 0) also intersect each other, and in consequence the mentioned spheres have two mutual generatrices. But 2nd degree surfaces, whose intersection-curves consist of a conesection and two straight lines, touch each other in three points - the section-curves double-points. l_1 and λ_1's image-spheres thus have three contact points, of which two, however, imaginary and infinitely distanced, in ordinary parlance do not come into question.

Analytically our theorem is proved in the following way:

The condition for the intersection between the two lines:

$$r_1 z = x - l_1 \qquad r_2 z = x - \rho_2$$
$$s_1 z = y - \sigma_1 \qquad s_2 z = y - \sigma_2$$

is known to be expressed by the equation:

$$(r_1 - r_2)(\sigma_1 - \sigma_2) - (\rho_1 - \rho_2)(s_1 - s_2) = 0\,,$$

which by use of (18) gives:

$$(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 + (iH_1 - iH_2)^2 = 0\,,$$

which proves our statement.

Our theorem shows, that the collection of the straight lines, which intersect a given, transforms as all spheres, that touch a given, *and in consequence we know the special linear complex's projection.*

Conversely, corresponding to two spheres, which touch each other, there are two line-pairs, whose mutual relation is such that each line of the one pair intersects a line of the other.

24. The general linear complex's transformation. The general linear complex is produced by the equation:

$$(\rho\sigma - \pi\sigma) + mr + n\sigma + p\rho + qs + t = 0\,, \qquad (19)$$

which by use of (18) is found to be the equation of the corresponding "linear sphere-complex":

$$[X^2 + Y^2 + Z^2 - H^2] + MX + NY + PZ + QH + T = 0.$$[1]

Here M, N, P, Q, T signify constants that depend upon m, n, p, q, t, *while X, Y, Z, H are to be considered as - non-homogeneous - sphere co-ordinates.*

The last equation determines, as one easily sees, all spheres that intersect the image-sphere of the complexes (19) and (H = 0)'s linear mutual-congruence under constant angle.

If these complexes are simultaneous invariants equalling zero, or the two complexes, as Klein puts it, lie in involution, the constant angle is right.

To spheres, which intersect a given sphere under constant angle, correspond in the space r those straight lines of two linear complexes that are each other's reciprocal polarities relative to (H = 0).

In particular it should be noted that the spheres, which intersect a given one orthogonally, transform as the straight lines of a linear complex, lying in involution with (H = 0).

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1 This equation can be posed under the form:

$$(X-X_0)^2 + (Y-Y_0)^2 + (Z-Z_0)^2 + (iH-iH_0)^2 = C_0^2$$

in which we perceive X_0, Y_0, Z_0, H_0, C_0 as non-homogeneous co-ordinates of the linear complex. Mr Klein has drew my attention to the fact, that the sphere (X_0, Y_0, Z_0, H_0) is the image of the axes of the linear complex at hand.

Now given a linear complex, whose equation has the form:

$$ar + bs + c\rho + d\sigma + e = 0, \tag{20}$$

the corresponding relation between X, Y, Z, H is also linear, and thus the linear sphere-complex in question is generated by all *spheres, which intersect a given plane under a given angle.*

This one may also conclude from the fact that the complex (20) contains the xy-plane's infinitely distanced straight line, that thus the same's mutual-congruence with (H = 0) possesses directrices, which intersect this line.

When the complexes (20) and (H = 0) lie in involution, (20)'s lines transform as all spheres which intersect a given plane orthogonally, or, equivalently, as the spheres whose centres lie in a given plane.

The following four complexes:

$$X = 0 = A\rho + Bs \qquad Z = 0 = \lambda A\sigma - B/_{\lambda}\, r$$
$$iY = 0 = A\rho - Bs \qquad H = 0 = \lambda A\sigma + B/_{\lambda}\, r$$

lie, as one easily sees, pairwise in involution and furthermore contain, as mutual line, the xy-planes infinitely distanced line.

The special linear complex: (Const = 0), that is generated by all lines parallel with the xy-plane in association with the four general linear complexes (X = 0) (Y = 0) (Z = 0) (H = 0), thus forms a system, that is to be perceived as a degeneration of Mr. Klein's 6 fundamental-complexes. In analogy with the fact that, above we have introduced X, Y, Z, H as non-homogeneous co-ordinates of a geometry with four dimensions, the element of which is the sphere, these quantities can also be used as non-homogeneous line-co-ordinates.

It is of interest to note, that the linear complexes, whose equation is:

$$H = \lambda A\sigma + B/_{\lambda}\, r = \text{Const.},$$

and which according to the equation-form touch each other after a special linear congruence, whose directrices have joined themselves in the xy-plane's endlessly distanced line, transform as a set of sphere-complexes, which are characterised thereby, that all spheres of the same complex have equally large radii.

25. Various projections. A surface f and all its tangents in a given point project themselves as a surface F and all spheres that touch the same in a given point.

A line lying upon f as a sphere, that touches F along a curve.

When f is a linesurface, F is a sphere-envelope - a tube-surface.

If in particular f is a 2nd degree surface and as consequence thereof contains two systems of rectilinear generatrices, in two ways F may be perceived as a sphere-envelope, and it is significant that in this manner we*obtain the most general surface, which possess this property (the cyclide).*

A developable surface transforms itself in the envelope-surface of a set of spheres, of which two consecutive ones always touch each other - that is to say, in an imaginary linesurface, whose generatrices intersect the infinitely distanced imaginary circle. These line-surfaces are, one knows, just those that *Monge* characterises by their possessing only one system of rounded curves.

26. It is known that the immediate consequence of the Plücker understanding, that when ($l_1 = 0$) and ($l_2 = 0$) are the equations for two linear complexes,

$$l_1 + \mu \; l_2 = 0\,,$$

provided that μ signifies a parameter, represent a set of linear complexes, which contain a mutual linear congruence. Our projection-principle transforms this theorem into the following:

The spheres K, that intersect two given spheres S_1 and S_2 under given angles, V_1 and V_2 stand in the same relation to infinitely many spheres S. There are, corresponding to the said line-congruence's two directrices, two S, which are touched by all K.

The variable line complex: ($l_1 + \mu l_2 = 0$) intersects the complex ($H = 0$) along a linear congruence, whose directrices describe a 2nd degree surface - the average of the three complexes: $l_1 = 0$, $l_2 = 0$, $H = 0$, and in consequence the just mentioned spheres S envelop a cyclide, which, by the way, in this case is degenerated into a circle, after which all S intersect each other.

Here we also wish to draw attention to the fact that our sphere-projection allows the deduction of corresponding sphere-groups from interesting discontinuous line-groups, and vice versa. For example, from the well-known theory for the 3rd degree surface's 27 straight lines we derive the existence of groups of 27 spheres, of which each touches ten of the others.

On the other hand, for example, sphere-columns yield strangely discontinuous arrangements of a linear complex's lines.

§ 10.

Transformation of particulars concerning spheres in line-problems.

27. In this paragraph we wish to solve a few well-known problems concerning spheres, in that we consider the corresponding line-problems by our transformation-principle.

Problem I. How many spheres touch four given spheres?

The four spheres transform in four line-pairs $(l_1\ \lambda_1)(l_2\ \lambda_2)(l_3\ \lambda_3)(l_4\ \lambda_4)$, and the corresponding line-problem is thus to find the lines, which intersect four lines, chosen in such a way among the 8 stated, that one line is taken by each pair.

The lines l and λ can be arranged in 16 distinct groups of four:

$$l_1\, l_2\, l_3\, l_4 \qquad \lambda_1\, \lambda_2\, \lambda_3\, \lambda_4$$
$$l_1\, l_2\, l_3\, \lambda_4 \qquad \lambda_1\, \lambda_2\, \lambda_3\, l_4$$
$$\cdots\cdots\cdots\cdots$$

in such a way, that each group only contains one line of each pair. These 16 groups are, however, pairwise generated by lines, that are each other's reciprocal polarities relative to (H = 0), and in consequence also two associated groups' transversal-pairs $(t_1\ t_2)\ (\tau_1\ \tau_2)$ are each other's polarities relative to (H = 0). The four last-mentioned lines are thus projected as *two* spheres, and in consequence there exist 16 spheres, arranged in 8 pairs, that touch the four given.

Problem II. How many spheres intersect four given spheres under four given angles?

The spheres, which intersect a given sphere under the same angle, project themselves as those straight lines of two linear complexes, which are each other's reciprocal polarities relative to (H = 0). One thus has to consider four pairs of complexes $(l_1\ \lambda_1)(l_2\ \lambda_2)(l_3\ \lambda_3)(l_4\ \lambda_4)$, and the problem is to find the lines, that belong to four of these complexes, which are chosen in such a way, that one of each pair is taken.

Four linear complexes have two mutual-lines, and thus as solution one obtains, by following the same method as we used in the preceding problem's treatment, 16 spheres that are arranged in 8 pairs.

Our problem is simplified, when one or more of the given angles are right, insofar as a given sphere's orthogonal-spheres transform as the lines of *one* complex lying in involution with $(H = 0)$ (*n.24*). When all angles are right, the question is how many mutual-lines four with $(H = 0)$ in involution lying complexes have. There are two such lines, which are each other's reciprocal polarities relative to $(H = 0)$, *and in consequence there is only one sphere, that intersects the four given orthogonally.*

Problem III. To construct the spheres, that intersect five given spheres under the same angle.

Our transformation-principle turns this problem into the following: to find the linear complexes, which contain a line of each of five given line pairs $(l_1\ \lambda_1)......(l_5\ \lambda_5)$.

These 10 lines can be arranged in 32 different groups of five, in such a way, that each group contains *one* line of each pair:

$$(l_1\ l_2\ l_3\ l_4\ l_5)\ (\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

by which, however, note that pairwise these groups are each other's reciprocal polarities relative to $(H = 0)$. Each group gives a line-complex and, in all, 32 pairwisely conjugated linear complexes are thus obtained, which transform as 16 linear sphere-complexes. The 16 spheres, each of which is intersected under constant angle by the mentioned system's spheres, are our problem's solutions.

Two line-groups like:

$$l_1\ l_2\ \lambda_3\ \lambda_4\ l_5 \qquad \lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ l_5$$

contain four mutual lines, and thus the two corresponding linear complexes intersect each other after a linear congruence, whose directrices d_1 and d_2 are the mentioned four lines' transversals.

But the complex (H = 0) intersects that congruence along a 2nd degree surface, which is the image of a circle - the average-circle between two of the sought spheres, but likewise between d_1 and d_2's image-spheres. These last spheres can also be defined by the fact that they touch four of the five given spheres and thus one can, by the just stated construction, determine a number of circles upon an arbitrary one of the spheres searched for.

On each of the 16 spheres that intersect five given under the same angle, five circles can be constructed, provided that one can construct the spheres, which touch the five given.

§ 11.

Relation between rounded curves' and maintangent-curves' theory.

28. The transformation considered in the foregoing gains particular interest due to the following, in my opinion important theorem:

To a surface F's rounded curves given in R correspond in r line-surfaces which touch the image-surface f along maintangent-curves.

The surface f's tangents transform into spheres that touch F, and the idea is thus that, to *f's maintangents*, correspond *F's main-spheres.* This is also the case.

Because f is intersected by a maintangent in three coinciding points, which shows that three consecutive generatrices of the maintangent's image-sphere touch F. But such a sphere intersects F along a curve, which in both's point of contact has an apex, and this is just characteristic of main-spheres.

When it is now further considered that this apex's direction is tangent to a rounded curve, it is seen that two consecutive points of a maintangent-curve on f project as two lines, which touch F in consecutive points of the same rounded curve. To *f's maintangent-curves, perceived as point-creations, thus correspond imaginary linesurfaces which touch F along a rounded curve.*

But curves on f and F arrange themselves pairwise together as conjugated in such a way *(n. 17, e)* that the one's points are images of lines, which touch the other surface in points of the conjugated curve, and *thus our theorem is proved.*

The two ensuing examples can be regarded as a verification of this proposition.

A sphere in R is the image of a linear congruence, as whose focus-surface the two directrices are to be perceived. Now as is well-known any curve on a sphere is a rounded curve, and in reality the directrices also appear as maintangent-curves on any line-surface that belongs to a linear congruence. - A hyperboloid f in the space r gives in R a surface, which in two ways can be perceived as a sphere-envelope. Now the line-surfaces in the complex (H= 0), touching f after its maintangent-curves, that is to say, after its rectilinear generatrices, are themselves 2nd degree surfaces, and in consequence *the cyclide F's rounded curves are circles.*

As an interesting consequence of our theorem the following may be contemplated.

Kummer's surface of the first order and class has algebraic maintangent-curves of the 16th order, which generate the complete contact-average between the respective surface and linesurfaces of the 8th order.

Kummer's surface is namely the focus-surface for the general line-congruence of 2nd order and class, which projects - provided it belongs to (H = 0) - as a fourth degree surface which contains the infinitely distanced circle twice (n. 21, a).

But Mr. *Darboux* and *Moutard* [1] have shown, that the last mentioned surface's curvelines are curves of the 8th order, which intersect the infinitely distanced imaginary circle in 8 points, and thus these line transform as linesurfaces of the 8th order (n. 21, b).

Finally, if it is remembered, that these linesurfaces' generatrices are doubletangents to the Kummer surface, our theorem's correctness is realised.[2]

It is evident, that also the Kummer surface's degenerations, e.g.: *the wavesurface, the Plücker complex-surface, the Steiner surface of the 4th order and 3rd class*[3], a linesurface of 4th degree, the 3rd degree linesurface....have algebraic maintangent-curves.

29. Mr *Darboux* has shown that on an arbitrary surface in general a curveline located in the finite space can be determined - the touching-curve with the imaginary developable, which is circumscribed simultaneously around the given surface and the infinitely distanced imaginary circle.

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1 Comptes rendus. Year 1864.

2 Klein and Lie. Berliner Monatsbericht. 15 Decbr. 1870.

3 Clebsch has determined the Steiner surface's maintangent-curves.

As a consequence hereof it is in general possible to identify one maintangent-curve on the focussurface of a congruence belonging to a linear complex - the geometric locus of the points, for which the tangentplane likewise is the plane associated with the linear complex.

The infinitely small spheres, which touch F, namely consist of F's points in connection with the above stated imaginary developables, and in consequence the straight lines of the complex (H = 0), which touch the imagesurface f, divide into two systems - one system of doubletangents, and on the other hand the collection of lines that touch f in the points of a definite curve. But this curve is, as the projection of an imaginary linesurface that touches F along a rounded curve, one of f's maintangent-curves.

However, this determination of a maintangent-curve is rendered illusory, when not the congruence, but the focus-surface - or, more correctly, a reducible part of the same - is conditionally stated. For on a surface, as a rule only a finite number of points exist, whose tangentplane moreover is the plane which is associated with the said point by a given linear complex.

It is of interest to note, that a linesurface, whose generatrices belong to a linear complex, contains infinitely many points, for which the tangentplane in addition is the plane assigned by the linear complex. The collection of these points generates, by simple operations - differentiation and elimination - , a determinable maintangent-curve.

But Mr. *Clebsch* has shown, that when a maintangent-curve is known upon a linesurface, the others can be found by squaring.

The determination of maintangent-curves upon a linesurface belonging to a linear complex depends only on squaring.

In that we use our transformation-principle on the mentioned theorem of Mr Clebsch as well as on the deduced consequence, we obtain the following theorems:

When upon a tubesurface (sphere envelope) a rounded curve which is not circular is known, the others can be found by squaring.

Single-infinitely many spheres, that intersect a given sphere S under constant angle, envelope a tubesurface upon which a curveline can be defined, and the others thus determined by squaring.

That one can find a rounded curve upon the tubesurface mentioned in the last theorem, is apparent also from the fact that the tubesurface intersects S under constant angle. But this section-curve must be one of the tubesurface's rounded curves by the known lemma: When two surfaces intersect each other under constant angle, and the section-curve is a rounded line on the one surface, it must also be so on the other; but on a sphere all curves are rounded lines.

§ 12.

Correspondence between transformations of the two spaces.

30. Our projection can, according to *n. 16,* be expressed by five equations, which determine an arbitrary quantity of the two groups:

$$(x\,y\,z\,p\,q) \qquad (X\,Y\,Z\,P\,Q)\,,$$

as function of quantities of the other group. If now the one of the two spaces is subjected to an, e.g., transformation, by which surfaces that touch each other, are turned into similar surfaces, the corresponding transformation of the other space will possess the same property. The mentioned transformation of r can namely be expressed by five equations between x_1, y_1, z_1, p_1, q_1 and x_2, y_2, z_2, p_2, q_2 - the indices 1 and 2 refer to the space r's two states - and these relations are turned by aid of the transformation-equations between (x y z p q) and (X Y Z P Q) to relations between $(X_1, Y_1, Z_1, P_1, Q_1)$ and $(X_2, Y_2, Z_2, P_2, Q_2)$, which proves our proposition.

In that we restrict ourselves to linear transformations of r, we find between the corresponding transformations of R: *all movements (translation-movement, rotation-movement and the helicoidal movement), semblability-transformation, transformation by reciprocal radii, parallel transformation*[1] *- by that is understood transition from a surface to its parallel-surface - a reciprocal transformation studied by Mr. Bonnet*[2] etc., which all, corresponding to linear transformations of r, possess the property of turning rounded curves into rounded curves. We finally prove, *that to the general linear transformation of r correspond the most general transformation of R, by which rounded lines are covariant curves.*

31. When we now firstly consider such linear point-transformations of r, to which correspond linear point-transformations of R, it is evident, that we can only find such transformations of R, by which the endlessly distanced imaginary circle remains unchanged, and inversely it is also true that we obtain all of these.

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1 Bonnets "dilatation."

2 Comptes rendus. Many times in the 50-ies.

For as we know, such a linear point transformation of R on the one hand turns straight lines which intersect that circle into similar lines; on the other hand spheres into spheres, and thus the corresponding transformation of r is at the same time a point- and line-transformation, that is: a linear point transformation, which was to be proved.

The general linear transformation of R, that does not distort the infinitely distanced imaginary circle, contains 7 constants and can, as is well known, be composed by translation- and rotation-movements in combination with semblability-transformations. The corresponding transformation of r, that obviously also depends upon 7 constants, can be characterised so that it turns a linear complex (H = 0) and one determined by the same lines - the xy-planes' infinitely distanced line - into itself. One could also define this transformation so that it turns a special linear congruence into itself.

By analytical considerations one can in the following way determine the linear point-transformation of r corresponding to a transformation-movement of R. A translation-movement is expressed by the equations:

$$X_1 = X_2 + A;\quad Y_1 = Y_2 + B;\quad Z_1 = Z_2 + C;\quad H_1 = H_2,$$

which by using the formulas (17) give:

$$r_1 = r_2 + a;\quad s_1 = s_2 + b;\quad \rho_1 = \rho_2 + c;\quad \sigma_1 = \sigma_2 + d.$$

On insertion of these expressions in a straight line's equations:

$$r_1 z_1 = x_1 - \rho_1 \quad s_1 z_1 = y_1 - \sigma_1.$$

are obtained as definition of the mentioned transformation of r:

$$z_1 = z_2;\quad x_1 = x_2 + az_2 + c;\quad y_1 = y_2 + bz_2 + d.$$

Likewise it is easy to determine analytically the transformation of r corresponding the to a *semblability-transformation* of R. For the equations:

$$X_1 = mX_2;\quad Y_1 = mY_2;\quad Z_1 = mZ_2;\quad H_1 = mH_2$$

give, by using (17):

$$r_1 = mr_2;\quad \rho_1 = m\rho_2;\quad s_1 = ms_2;\quad \sigma_1 = m\sigma_2,$$

which define a linear transformation of r that can also be expressed by:

$$z_1 = z_2; \quad x_1 = mx_2; \quad y_1 = my_2 .$$

But these last relations define a linear point-transformation that can be defined so that *two straight lines retain their places.*

By geometric consideration we will show, that also rotation-movements of R metamorphose into transformations of the just stated kind. Let A be the rotation-axis and M and N the two points of the imaginary circle not distorted by the rotation. It is evident, that all imaginary lines, that intersect A, and that go through M or N, retain their position under the rotation, and in consequence the same is the case with these lines' imagepoints, which form two straight lines parallel with the xy-plane.

32. Transformation by reciprocal radii of the space R transforms points into points, spheres into spheres and finally straight lines of length equal to zero into similar lines; the corresponding transformation of r is thus a *linear point*-transformation, that turns the complex (H = 0) into itself. When one further notes that transformation by reciprocal radii lets a definite sphere's points and rectilinear generatrices maintain their position, it is realised, that the corresponding point-transformation does not distort two straight lines' points.

Mr *Klein*[1] has drawn our attention to the fact that the just mentioned transformation can be perceived as composed of two transformations relative to two linear complexes lying in involution, of which *in casu* (H = 0) is one, while the other corresponds to the collection of spheres which intersect the fundamental-sphere of the given transformation by reciprocal radii.

According to the above it is evident, that to a surface D, which through a transformation by reciprocal radii is turned into itself, corresponds in the space r one to (H = 0) belonging congruence, which is its own reciprocal polarity relative to a linear complex lying in involution with (H = 0). The focus-surface (f) of the said congruence is thus its own reciprocal polarity relative to both the stated linear complexes, and in consequence the collection of f's doubletangents generally decomposes into three congruences, of which the two relationally belong to (H = 0) and the complex lying in involution with the same.

33. One now considers all line-transformations of r, by which straight

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1 Zur Theorie . . . math. Annalen, Bd. II.

lines, that intersect each other are turned into similar lines[1], and on the other hand the corresponding transformations of R, which possess the property to turn spheres into spheres, spheres that touch each other in similar spheres.

By the stated line-transformation, the collection of a surface f_1's tangents is turned into all of another surface f_2's tangents, and especially f_1's main-tangents go over into f_2's maintangents - this irrespective of whether the line-transformation is a point-transformation or a point-plane-transformation.

By the corresponding transformation of R, the threefold infinity of spheres, that touch a given surface F_1 is turned into the collection of spheres, standing in the same relation to the other surface F_2, and especially F_1's main-spheres are transformed into F_2's main-spheres. A simple consequence hereof is that F_1's and F_2's arcuate-lines correspond to each other in the sense that when in an arbitrary relation:

$$\Phi(X_1\ Y_1\ Z_1\ P_1\ Q_1) = 0\,,$$

which is valid along one of F's rounded lines, are inserted $X_1\ Y_1\ Z_1\ P_1\ Q_1$'s values at $X_2\ Y_2\ Z_2\ P_2\ Q_2$, an equation is obtained, that is valid for one of F_2's rounded curves.

I will now show, that any transformation of R of the form:

$$X_1 = F_1\left(X_2\ Y_2\ Z_2\ \frac{dZ_2}{dX_2}\ \frac{dZ_2}{dY_2}\ \frac{d^2 Z_2}{dX_2}\ \ldots\ \frac{d^{m+n} Z_2}{dX_2^m \cdot dY_2^n}\right)$$

$$Y_1 = F_2\left(X_2\ Y_2\ Z_2\ \ldots\ldots\ldots\ldots\ \frac{d^{m+n} Z_2}{dX_2^m \cdot dY_2^n}\right)$$

$$Z_1 = F_3\left(X_2\ Y_2\ Z_2\ \ldots\ldots\ldots\ldots\ \frac{d^{m+n} Z_2}{dX_2^m \cdot dY_2^n}\right)$$

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1 Here are, as we know, two cases to be considered, insofar as lines, that go through a point, can either be transformed in similar lines, or in lines that lie in a plane.

which turns an arbitrary surface's rounded lines into rounded lines for the new surface, through my transformation corresponds to a linear transformation of r.

The proof can be straightforwardly reduced to demonstrating that when a transformation of r turns an arbitrary surface's maintangent-curves into maintangent-curves of the transformed surface, straight lines that intersect each other must be turned into similar lines by the same.

Firstly, the transformation in question must turn straight lines into straight lines, follows from that the straight line is the only curve, which is maintangent-curve on any surface that contains the same.

Further, to straight lines that intersect each other, must correspond lines of the same relative mode, can be deduced from the fact that the developable surface is the only linesurface, which possesses the property, that through each of its points runs only one maintangent-curve - that thus our transformation must turn developable surfaces into developable surfaces.

Our proposition is thus proved.

One may note that, corresponding to the two essentially different kinds of linear *transformations, exist two distinct classes of transformations, for which rounded curves are covariant curves.*

When one chooses among the stated transformations of R those which are point-transformations, *the most general point-transformation of R, by which rounded lines are covariant curves, is obtained,* a problem that *Liouville* first solved. That hereunder equivalence in the smallest parts is maintained, follows by the fact that infinitesimal spheres are transformed into infinitesimal spheres.

Parallel-transformation is known to turn rounded lines into rounded lines, and it is in reality easy to verify that the corresponding transformation of r is a linear point-transformation.

For the equations:

$$X_1 = X_2;\ \ Y_1 = Y_2;\ \ Z_1 = Z_2;\ \ H_1 = H_2 + A$$

transform (compare our considerations over translation-movement *n. 31*) into relations of the form:

$$z_1 = z_2;\ \ x_1 = x_2 + uz_2 + b;\ \ y_1 = y_2 + cz_2 + d.$$

34. Mr. *Bonnet* has many times considered a transformation, which he defines by the equations:

$$Z_2 = i\, Z_2 \sqrt{1 + p_2^2 + q_2^2} \ ; \ x_1 = x_2 + p_2 z_2 \ ; \ y_1 = y_2 + q_2 z_2 \ ,$$

whereby the two indices refer to the given and the transformed surfaces.

Mr. Bonnet shows that this transformation is reciprocal - in the sense, that twice applied it brings back the given surface, that it transforms curvelines into curvelines, that finally the following two relations:

$$\zeta_1 = iH_2 \, , \ H_1 = -i\zeta_2 \qquad (\alpha)$$

find place, provided that H_1 and H_2 signify curve-radii for corresponding points, that further ζ_1 and ζ_2 are z-ordinates for the corresponding curve-centres.

The Bonnet transformation is, as we will soon show, the image of a transformation of r relative to the linear complex:

$$Z + iH = 0.$$

Because, remembered that $(X = 0)$ $(Y = 0)$ $(Z = 0)$ $(H = 0)$ pairwise lie in involution, it is found, that the co-ordinates of the straight lines, that are each other's polarities relative to $[Z + iH = 0]$, fulfil the relations:

$$X_1 = X_2; \ Y_1 = Y_2; \ Z_1 = iH_2; \ H_1 = -iZ_2 \, . \qquad (\beta)$$

But these formulas determine a pairwise correspondence between all spheres of the space when X, Y, Z, H are interpreted as sphere-co-ordinates. This is just the same as the Bonnet transformation.

Because a surface F_1's mainspheres are transformed hereunder into a surface F_2's mainspheres, and hence we recover Bonnet's formulas (α). When one further considers F_1 generated by point-spheres, the equations (β) define F_2 as an envelope of spheres, whose centres lie in the plane $(z = 0)$, in that the equation $(H_1 = 0)$ draws $(Z_2 = 0)$ after itself as a consequence. In reality we are hereby carried to precisely the geometric construction described by Mr. Bonnet.